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Some Applications of Metacyclic *p*-Groups of Nilpotency Class Two

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Abstract: A group G is metacyclic if it contains a cyclic normal subgroup K such that G/K is also cyclic. Metacyclic p-groups classified by different authors. King classified metacyclic *p*-groups. Beuerle (2005) classified all finite metacyclic *p*-groups. A group is called capable if it is a central factor group. The purpose of this study is to compute the epicenter of finite nonabelian metacyclic *p*-groups of nilpotency class two, for some small order groups, using Groups, Algorithms and Programming (GAP) software. We also determine which of these groups are capable.

Keywords: *p*-groups, metacyclic groups, epicenter and capable groups

INTRODUCTION

A finite *p*-group G is called metacyclic if it has a cyclic normal subgroup K such that G/K is also cyclic. Finite metacyclic groups can be presented with two generators and three defining relations. Much attention has been given to some specific types of metacyclic groups. Zassenhaus discussed metacyclic groups with cyclic commutator quotient, as did Hall. Metacyclic *p*-groups of odd order have been classified by (Beyl *et al*., 1979). Significant progress was made when Sim (1994) proved that every finite metacyclic group can be decomposed naturally as a semidirect product of two Hall subgroups (Hempel, 2000).

For a group G the epicenter of G is defined as follows:

Definition 1: Beyl *et al*. (1979) the epicenter, $Z^*(G)$ of a group G is defined as:

$\cap\{\phi Z(E); (E, \phi)$ is a central extension of G}

It can be easily seen that the epicenter is a characteristic subgroup of G contained in its center.

The notion of the capability of a group was first studied by Baer (1938) who initiated a systemic investigation of the question which conditions a group G must fulfill in order to be the group of inner automorphisms of some group H, i.e.,

$$G \cong H/Z(H)$$

The term was coined by Hall and Senior (1964) who defined a capable group as equal to its central factor group. The capability for some groups has been studied by many authors including Baer (1938) who characterized finitely generated abelian groups which are capable in the following theorem:

Theorem 1: Baer (1938) Let A be a finitely generated abelian group written as $A = Z_{n_1} \oplus Z_{n_2} \oplus \ldots \oplus Z_{n_k}$ such that each integer n_{i+1} is divisible by n_i and $Z_0 = Z$, the infinite cyclic group. Then A is capable if and only if $k \geq 2$ and $n_{k-1} = n_k$.

Beyl *et al*. (1979) established a necessary and sufficient condition for a group to be a central quotient in terms of the epicenter as follows:

Theorem 2: Beyl *et al*. (1979) A group G is capable if and only if $Z^*(G) = 1$.

With the help of this characterization, Beyl *et al*. (1979) were able to determine the extra special *p*-groups which are capable and describe the capable finite metacyclic groups.

Beuerle (2005) gave a presentation of metacyclic *p*-groups in the form:

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$$G = \left\langle a, b; a^{p^m} = 1, b^{p^m} = a^k, bab^{-1} = a^r \right\rangle$$

where, $m, n \geq 0,\ r > 0,\ k \leq p^m,\ p^m, \left|k(r-1)\right|$

and $p^m | r^{p^n} - 1$

Metacyclic p-groups are divided by Beuerle (2005) into two categories as follows:

$$G_p(\alpha, \beta, \varepsilon, \delta, \pm) = \left\langle a, b; a^{p^\alpha} = 1, b^{p^\beta} = a^{p^{\alpha-\varepsilon}}, bab^{-1} = a^r \right\rangle$$

where, $r = p^{\alpha-\delta} + 1$ or $r = p^{\alpha-\delta} - 1$

We say that the group is of positive or negative type if $r = p^{\alpha-\delta} + 1$ or $r = p^{\alpha-\delta} - 1$, respectively.

For short, we write G_+ and G_- for $G(\alpha, \beta, \varepsilon, \delta, +)$ and $G(\alpha, \beta, \varepsilon, \delta, -)$, respectively.

The following theorem, due to King (1973), gives a standardized parametric presentation of metacyclic p-groups.

Theorem 3: Let G be a finite metacyclic p-group. Then there exist integers α, β, δ, ε with $\alpha, \beta > 0$ and δ, ε nonnegative, where $\delta \leq \min\{\alpha-1, \beta\}$ and $\delta+\varepsilon \leq \alpha$ such that for an odd prime p, $G_p(\alpha, \beta, \varepsilon, \delta, +)$. If $p = 2$, then in addition $\alpha-\delta > 1$ and $G_2(\alpha, \beta, \varepsilon, \delta, +)$ or $G_p(\alpha, \beta, \varepsilon, \delta, -)$, where in the second case $\varepsilon \leq 1$. Moreover, if G is of positive type, then $\delta > 0$ for all p.

The next proposition lists some of the basic properties and various subgroups of metacyclic p-groups:

Proposition 1: Beuerle (2005) Let G be a group of type $G_p(\alpha, \beta, \varepsilon, \delta, \pm)$ and $k \geq 1$. Then we have the following results for the order of G the center of G, the order of the center of G the derived subgroups of G the order of the derived subgroups of G and the exponent of G:

	G_+	G_-
$\lvert G_\pm \rvert$	$p^{\alpha+\beta}$	$p^{\alpha+\beta}$
$Z(G_\pm)$	$\left\langle a^{p^\delta}, b^{p^\delta} \right\rangle$	$\left\langle a^{2^{\alpha-1}}, b^{2^{\max\{1,\delta\}}} \right\rangle$
$\lvert Z(G_\pm) \rvert$	$p^{\alpha+\beta-2\delta}$	$2^{1+\beta-\max\{1,\delta\}}$
$\gamma_{k+1}(G_\pm)$	$\left\langle a^{p^{k(\alpha-\delta)}} \right\rangle$	$\left\langle a^{2^k} \right\rangle$
$\lvert \gamma_{k+1}(G_\pm) \rvert$	$p^{\max\{0,\, k\delta-(k-1)\alpha\}}$	$2^{\max\{0,\, \alpha-k\}}$
$Exp(G_+)$	$p^{\max\{\alpha,\, \beta+\varepsilon\}}$	$p^{\max\{\alpha,\, \beta+\varepsilon\}}$

Moreover, if $\beta > \max\{1, \delta\}$ then $Z(G_-)$ is cyclic if and only if $\varepsilon = 1$.

The nonabelian tensor square is a special case of the nonabelian tensor product which has its origin in K-theory as well as homotopy theory. It is defined as follows:

Definition 2: Brown *et al*. (1987) for a group G, the nonabelian tensor square, $G \otimes G$ is generated by the symbols $g \otimes h$, where $g, h \in G$, subject to the relations:

` $$gg' \otimes h = \left({}^{g}g' \otimes {}^{g}h\right)(g \otimes h)$$

and $$g \otimes hh' = (g \otimes h)\left({}^{h}g \otimes {}^{h}h'\right)$$

for all g, g′, h, h′∈G, where ${}^{h}g = hgh^{-1}$ denotes the conjugate of g by h.

A factor group of the tensor square is the exterior square, defined as follows:

Definition 3: Brown *et al*. (1987) for any group G the exterior square of G is defined as:

$$G \wedge G = {(G \otimes G)}/{\Delta(G)}$$

where,

$\nabla(G) = \langle x \otimes x : x \in G \rangle$ and $\nabla(G)$ is a central subgroup of $G \otimes G$.

In (Ellis, 1995) the tensor center of a group is defined as:

$$Z^{\otimes}(G) = \left\{ a \in G : a \otimes g = 1_{\otimes};\ \forall g \in G \right\}$$

Similar to the tensor center, the exterior center can be defined as:

$$Z^{\wedge}(G) = \left\{ a \in G : a \wedge g = 1_{\wedge};\ \forall g \in G \right\}$$

Here $1_{\otimes}$ and $1^{\wedge}$ denote the identities in $G \otimes G$ and $G \wedge G$ respectively. It can be easily shown that $Z^{\otimes}(G)$ and $Z^{\wedge}(G)$ are characteristic and central subgroups of G with $Z^{\otimes}(G) \subseteq Z^{\wedge}(G)$.

Though the criterion for capability stated in Theorem 2 is easily formulated, the implementation is another matter. As in all cases before, this still requires the cumbersome process of evaluating the factor groups. Ellis (1998) connects the epicenter with the exterior center, providing a much more efficient procedure to determine capability once the nonabelian tensor square is known and obtains the desired external characterization of the epicenter as follows:

Theorem 4: Ellis (1998) For any group G, the epicenter coincides with the exterior center, i.e.:

$$Z^*(G) = Z^\wedge(G)$$

The method of using tensor squares in the determination of capability was implemented by several authors. Beuerle and Kappe (2000) characterized infinite metacyclic groups which are capable.

Beuerle (2005) classified all finite nonabelian metacyclic *p*-groups. Using this classification, we compute the epicenter of finite nonabelian metacyclic 2-groups of nilpotency class at least three, for some small order groups, by using Groups, Algorithms and Programming (GAP) software. By using the criterion stated in Theorem 2, we determine which of them are capable.

Groups, Algorithms and Programming (GAP) software is used as a tool to verify the results found in determining the capability of the groups studied. GAP is a system for computational discrete algebra, with emphasis on computational group theory. GAP provides a programming language, a library of functions that implement algebraic algorithms written in the GAP language as well as libraries of algebraic objects such as for all non-isomorphic groups up to order 2000 (Rainbolt and Gallian, 2010).

Classification of metacyclic *p*-groups of nilpotency class two: In this section, the classification of all finite nonabelian metacyclic *p*-groups of nilpotency class two, done by Beuerle is stated.

Theorem 5: (Beuerle, 2005) Let G be a nonabelian metacyclic p-group of nilpotency class two. Then G is isomorphic to exactly one group in the following list:

1. $G \cong \left\langle a,\ b;\ a^{p^{\alpha}} = b^{p^{\beta}} = 1,\ \left[a, b\right] = a^{p^{\alpha-\delta}} \right\rangle$

where, $\alpha, \beta, \delta \in N$, $\alpha \geq 2\delta$, $\delta \leq \beta$

2. $G \cong Q = \langle a, b, a^4 = 1, b^2 = [a, b] = a^2 \rangle$

the group of quaternion of order 8.

EPICENTER COMPUTATIONS

In this section, we use GAP to develop appropriate coding for computing the epicenter for some small order groups of type (1), where p is odd and type (2). We provide GAP programmes to generate general codes and examples of groups of type (1) and type (2).

Type (1): First we use GAP programme to generate all finite nonabelian metacyclic *2*-groups of type (1) and construct some examples.

Generating type (1): GAP coding to generate all finite nonabelian metacyclic *2*-groups of type (1) is developed as follows:

```
type1: =
function (p, l, beta, n)
local F, m, a, b, G;
    for m in [1..beta] do
        if 2n<=l and m>=n and n>=1 then
        F: = FreeGroup (2);
        a: = F.1; b: = F.2
        G: = F/[a^(p^l), b^(p^m), (a*b*a^(-1)*b^(-1))^(-
        1)*a^(p^(l-n))];
        Print("p=",p," alpha=",l," beta=",m," delta =",n,"
        Order(G):",Order(G)," Order of Epicenter
        (G):",Order (Epicenter (G)),"\n");
        fi;
    od;
end;
```

We define the group with four parameters which are p, α, β and δ. The GAP code below is used to create the presentation of the group.

```
F: = FreeGroup (2);
a: = F.1; b: = F.2
G: = F/[a^(p^l), b^(p^m), (a*b*a^(-1)*b^(-1))^(-
1)*a^(p^(l-n))];
```

Constructing examples of type (1): Now, we define the parameters and put them in GAP. For example, we define $p = 3$, $\alpha = 2$, $\beta = 1, ..., 6$ and $\delta = 1$. First we need to read the file and call the function.

```
gap> Read("metacyclic_type3.g");
gap> type3 (3, 3, 4, 2);
p = 3 alpha = 2 beta = 1 delta = 1 Order(G):27 Order of
Epicenter(G):3
p = 3 alpha = 2 beta = 2 delta = 1 Order(G):81 Order of
Epicenter(G):1
p = 3 alpha = 2 beta = 3 delta = 1 Order(G):243 Order of
Epicenter(G):3
p = 3 alpha = 2 beta = 4 delta = 1 Order(G):729 Order of
Epicenter(G):9
p = 3 alpha = 2 beta = 5 delta = 1 Order(G):2187 Order of
Epicenter(G):27
p = 3 alpha = 2 beta = 6 delta = 1 Order(G):6561 Order of
Epicenter(G):81
```

Using the coding as before, we got the following results.

```
gap> type1 (3, 3, 6, 1);
p = 3 alpha = 3 beta = 1 delta = 1 Order(G):81 Order of
Epicenter(G):9
p = 3 alpha = 3 beta = 2 delta = 1 Order(G):243 Order of
Epicenter(G):3
p = 3 alpha = 3 beta = 3 delta = 1 Order(G):729 Order of
Epicenter(G):1
gap> type1 (3, 4, 5, 2);
p = 3 alpha = 4 beta = 2 delta = 2 Order(G):729 Order of
```

```
Epicenter(G):9
p = 3 alpha = 4 beta = 3 delta = 2 Order(G):2187 Order of Epicenter(G):3
gap> type1 (5, 2, 5, 1);
p = 5 alpha = 2 beta = 1 delta = 1 Order(G):125 Order of Epicenter(G):5
p = 5 alpha = 2 beta = 2 delta = 1 Order(G):625 Order of Epicenter(G):1
gap> type1 (7, 2, 5, 1);
p = 7 alpha = 2 beta = 1 delta = 1 Order(G):343 Order of Epicenter(G):7
gap> type1 (9, 2, 5, 1);
p = 9 alpha = 2 beta = 1 delta = 1 Order(G):729 Order of Epicenter(G):9
gap> type1 (11, 2, 5, 1);
p = 11 alpha = 2 beta = 1 delta = 1 Order(G):1331 Order of Epicenter(G):11
gap> type1 (13, 2, 5, 1);
p = 13 alpha = 2 beta = 1 delta = 1 Order(G):2197 Order of Epicenter(G):13
gap> LogTo();
gap> quit;
```

RESULTS ANALYSIS

Summary of results for type (1): For short, we write Ep (*G*) for the epicenter of *G*. A summary of GAP results for type (1) is given in Table 1.

Table 1: GAP results for type (1)

#	*p*	α	β	δ	$\lvert G \rvert$	Ep (*G*)
1	3	2	1	1	27	3
2	3	2	2	1	81	1
3	3	2	3	1	243	3
4	3	2	4	1	729	9
5	3	2	5	1	2187	27
6	3	2	6	1	6561	81
7	3	3	1	1	81	9
8	3	3	2	1	243	3
9	3	3	3	1	729	1
10	3	4	2	2	729	9
11	3	4	3	2	2187	3
12	5	2	1	1	125	5
13	5	2	2	1	625	1
14	7	2	1	1	343	7
15	11	2	1	1	1331	11
16	13	2	1	1	2197	13

Remark 1: Table 1 shows that the order of all groups computed satisfies $p^{\alpha+\beta}$ as stated in Proposition 1.

Type (2): With similar objective as for groups of type (1), we produce GAP algorithms for groups of type (2) and compute the order of the epicenter as follows:

```
gap> F: = FreeGroup (2);
<free group on the generators [ f1, f2 ]>
gap> a: = F.1; b: = F.2;
f1
f2
gap> Q: = F/[a^4,b^2*(a*b*a^(-1)*b^(-1))^(-1), b*a^(-1)*b^(-1)*a^(-1), b^2*a^(-2)];
<fp group on the generators [ f1, f2 ]>
gap> Order (Epicenter(Q));
2
gap> LogTo();
```

Capability determination: Table 1 shows that the groups of type (1) have trivial epicenter with α = β. GAP results shows that the group of type (2), the quaternion group of order 8, has nontrivial epicenter. Therefore, by Theorem 2, we conclude that groups of type (1) are capable with α = β and the rest are not capable. Similarly, we conclude that the quaternion group of order 8 is not capable.

It is worth to mention that Bacon and Kappe (2003) showed the same results for type (1), where *p* is odd. Also, the results for type (2) consistent with the results gotten by Ellis (1998) who showed that the quaternion group:

$$Q_{2n} = \left\langle a, b; a^n = 1, b^2 = \left[a, b\right] = a^2 \right\rangle$$

of order 2*n* is not capable.

Based on the above, we write the results in the following theorem:

Theorem 6: Let *G* be a metacyclic *p*-group of nilpotecny class two. Then *G* is capable if and only if *G* is of type (1) with α = β.

CONCLUSION

In this study we have computed the epicenter of finite nonabelian metacyclic *2*-groups of nilpotency class two, for some small order groups and determined their capability. We showed that the epicenter of groups of type (1) is trivial with α = β and the epicenter of the group of type (2), the quaternion group of order 8, is nontrivial. Also, we showed that the groups of type (1) are capable with α = β whereas the quaternion group of order 8 is not capable.

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