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Interval-valued Fuzzy Generalized Bi-ideals of Ordered Semigroups Redefined

[1]Asghar Khan, [2]Young Bae Jun, [3]Nor Haniza Sarmin and [3]Hidayat Ullah Khan

[1]Department of Mathematics, Abdul Wali Khan University Mardan, Khyber Pakhtoon Khwa, Pakistan
[2]Department of Mathematics Education and RINS, Gyeongsang National University, Jinju 660-701, Korea
[3]Department of Mathematical Sciences, Faculty of Science, Universiti Teknologi Malaysia 81310, UTM Johor Bahru, Johor, Malaysia

Abstract: In order to find more general forms of an interval valued $(\in,\in\vee q)$-fuzzy generalized bi-ideal in ordered semigroups, we have introduced the notion of an interval valued $(\in,\in\vee q_{\tilde{k}})$-fuzzy generalized bi-ideal in ordered semigroups and investigated several properties. We have dealt with characterizations of an interval valued $(\in,\in\vee q_{\tilde{k}})$-fuzzy generalized bi-ideal. In this article, we try to obtain a more general form than interval valued $(\in,\in\vee q)$-fuzzy generalized bi-ideals in ordered semigroups. The notion of an interval valued $(\in,\in\vee q_{\tilde{k}})$-fuzzy generalized bi-ideals is introduced and several properties are investigated. Characterizations of an interval valued $(\in,\in\vee q_{\tilde{k}})$-fuzzy generalized bi-ideal are established. A condition for an interval valued $(\in,\in\vee q_{\tilde{k}})$-fuzzy generalized bi-ideal to be an interval valued fuzzy generalized bi-ideal is provided. Using implication operators and the notion of implication-based an interval valued fuzzy generalized bi-ideals, characterizations of an interval valued fuzzy generalized bi-ideal and an interval valued $(\in,\in\vee q_{\tilde{k}})$-fuzzy generalized bi-ideal are considered.

Key words: Interval valued fuzzy generalized bi-ideals . Interval valued $(\in,\in\vee q)$-fuzzy generalized bi-ideal . Interval valued $(\in,\in\vee q_{\tilde{k}})$-fuzzy generalized bi-ideal . Interval valued fuzzifying generalized bi-ideal . $\tilde{t}$-implication-based interval valued fuzzy generalized bi-ideal

INTRODUCTION

The fundamental concept of a fuzzy set, introduced by L. A. Zadeh, provides a natural frame-work for generalizing several basic notions of algebra. The study of fuzzy sets in semigroups was introduced by Kuroki [1-3]. A systematic exposition of fuzzy semigroups was given by Mordeson *et al*. [4], where one can find theoretical results on fuzzy semigroups and their use in fuzzy coding, fuzzy finite state machines and fuzzy languages. The monograph by Mordeson and Malik [5] deals with the application of fuzzy approach to the concepts of automata and formal languages. Murali [6] proposed the definition of a fuzzy point belonging to a fuzzy subset under a natural equivalence on fuzzy subset. The concept of a fuzzy set in topological structure has been studied in [19]. The idea of quasi-coincidence of a fuzzy point with a fuzzy set, played a vital role to generate some different types of fuzzy subgroups. Bhakat and Das [7, 8] gave the concepts of (α,β)-fuzzy subgroups by using the belongs to relation $(\in)$ and quasi-coincident with relation (q) between a fuzzy point and a fuzzy subgroup and introduced the concept of an $(\in,\in\vee q)$-fuzzy subgroup. Many researchers have used the idea of generalized fuzzy sets and gave several results in different branches of algebra. In [9], Jun and Song initiated the study of (α,β)-fuzzy interior ideals of a semigroup. In [10], Kazanci and Yamak studied $(\in,\in\vee q)$-fuzzy bi-ideals of a semigroup. Shabir *et al*. [11] studied characterization of regular semigroups by (α,β)-fuzzy ideals. Jun *et al*. [12] discussed a generalization of an $(\in,\in\vee q)$-fuzzy ideals of a BCK/BCI-algebra. For further study on generalized fuzzy sets in ordered semigroups, we refer the reader to [13-15]. In mathematics, an ordered semigroup is a semigroup together with a partial order that is compatible with the semigroup operation. Ordered semigroups have many applications in the theory of sequential machines, formal languages, computer arithmetic and error-correcting codes. The concept of a fuzzy bi-ideal in ordered semigroups was first introduced by Kehayopulu and Tsingelis in [16],

Corresponding Author: Asghar Khan, Department of Mathematics, Abdul Wali Khan University Mardan, Khyber Pakhtoon Khwa, Pakistan

where some basic properties of fuzzy bi-ideals were discussed. A theory of fuzzy generalized sets on ordered semigroups can be developed. Using the idea of a quasi-coincidence of a fuzzy point with a fuzzy set, Khan *et al*. [17] introduced the concept of an interval valed (α,β)-fuzzy generalized bi-ideals in an ordered semigroup. They introduced a new sort of an interval valued fuzzy generalized bi-ideals, called interval valued (α,β)-fuzzy bi-ideals and studied interval valued $(\in,\in\vee q)$-fuzzy generalized bi-ideals. They provided different characterizations of interval valued fuzzy generalized bi-ideals of ordered semigroups in terms of interval valued $(\in,\in\vee q)$-fuzzy generalized bi-ideals and investigated different characterizations of interval valued fuzzy generalized bi-ideals of ordered semigroups in terms of interval valued $(\in,\in\vee q)$-fuzzy generalized bi-ideals. In this paper, we try to have a more general form of an interval valued $(\in,\in\vee q)$-fuzzy generalized bi-ideal of an ordered semigroup. We introduce the notion of an interval valued $(\in,\in\vee q_{\tilde{k}})$-fuzzy generalized bi-ideal of an ordered semigroup and give examples which are interval valued $(\in,\in\vee q_{\tilde{k}})$-fuzzy generalized bi-ideals but not interval valued $(\in,\in\vee q)$-fuzzy generalized bi-ideals. We discuss characterizations of interval valued $(\in,\in\vee q_{\tilde{k}})$-fuzzy generalized bi-ideals in ordered semigroups. We provide a condition for an interval valued $(\in,\in\vee q_{\tilde{k}})$-fuzzy generalized bi-ideal to be an interval valued fuzzy bi-ideal. We finally consider characterizations of an interval valued fuzzy generalized bi-ideal and an interval valued $(\in,\in\vee q_{\tilde{k}})$-fuzzy generalized bi-ideals by using implication operators and the notion of implication-based interval valued generalized fuzzy bi-ideals. The important achievement of the study with an interval valued $(\in,\in\vee q_{\tilde{k}})$-fuzzy generalized bi-ideal is that the notion of an interval valued $(\in,\in\vee q)$-fuzzy generalized bi-ideal is a special case of an interval valued $(\in,\in\vee q_{\tilde{k}})$-fuzzy generalized bi-ideal and thus several results in the paper [17] are corollaries of our results obtained in this paper.

PRELIMINARIES

By an ordered semigroup (or po-semigroup) we mean a structure $(S,\cdot,\leq)$ in which the following are satisfied:

(OS1) $(S,\cdot)$ is a semigroup
(OS2) $(S,\leq)$ is a poset
(OS3) $(\forall x,a,b\in S)\, a\leq b\Rightarrow a\cdot x\leq b\cdot x, x\cdot a\leq x\cdot b$

In what follows, xy is simply denoted by xy for all $x,y\in S$.

A nonempty subset A of an ordered semigroup S is called a subsemigroup of S if $A^2\subseteq A$.

A non-empty subset A of an ordered semigroup S is called a generalized bi-ideal of S if it satisfies

(b1) $(\forall a\in S, b\in A)\, a\leq b\Rightarrow a\in A$
(b2) $ASA\subseteq A$

Let $(S,\cdot,\leq)$ be an ordered semigroup. A nonempty subset A of S is called a bi-ideal of S if

(1) $(\forall a\in S, b\in A)\, a\leq b\rightarrow a\in A$
(2) $A^2\subseteq A$
(3) $ASA\subseteq A$

Obviously, every bi-ideal is a generalized bi-ideal of S, but the converse is not true.

Example: [21] Consider the ordered semigroup S = {a,b,c,d}

·	a	b	c	d
a	a	a	a	a
b	a	a	a	a
c	a	a	b	a
d	a	a	b	b

$\leq := \{(a,a),(b,b),(c,c),(d,d),(a,b)\}$

Its subsemigroups are: {a},{a,b},{a,b,c},{a,b,d} and {a,b,c,d}. All subsemigroups are bi-ideals.

Its generalized bi-ideals are: {a}, {a,b}, {a,c}, {a,d}, {a,b,c}, {a,b,d}, {a,c,d} and {a,b,c,d}. But {a,c}, {a,d} and {a,c,d} are not bi-ideals.

By an interval number $\tilde{a}$ we mean an interval $[a^-, a^+]$ where $0\leq a^-\leq a^+\leq 1$. The set of all interval numbers is denoted by D[0,1]. The interval [a,a] can be simply identified by the number $a\in[0,1]$. For the interval numbers, $\tilde{a}_i=[a_i^-,a_i^+]$, $\tilde{b}_i=[b_i^-,b_i^+]\in D[0,1], i\in I$, we define

$$(\forall i\in I)\quad \mathrm{rmax}\{\tilde{a}_i,\tilde{b}_i\}=[\max(a^-_i,b^-_i),\max(a^+_i,b^+_i)]$$

$$(\forall i\in I)\quad \mathrm{rmin}\{\tilde{a}_i,\tilde{b}_i\}=[\min(a^-_i,b^-_i),\min(a^+_i,b^+_i)]$$

$$\mathrm{rinf}\,\tilde{a}_i=[\wedge_{i\in I}a^-_i,\wedge_{i\in I}a^+],\ \mathrm{rsup}\,\tilde{a}_i=[\vee_{i\in I}a^-_i,\vee_{i\in I}a^+]$$

and put

$\tilde{a}_1 \le \tilde{a}_2 \Leftrightarrow a^-_1 \le a^-_2$ and $a^+_1 \le a^+_2$

$\tilde{a}_1 = \tilde{a}_2 \Leftrightarrow a^-_1 = a^-_2$ and $a^+_1 = a^+_2$

$\tilde{a}_1 < \tilde{a}_2 \Leftrightarrow a^-_1 \le a^-_2$ and $a^+_1 \ne a^+_2$

$k\tilde{a}_1 = [ka^-_i, ka^+_i]$, whenever $0 \le k \le 1$

Then, it is clear that $(D[0,1], \le, \vee, \wedge)$ forms a complete lattice with $0 = [0,0]$ as its least element and $1 = [1,1]$ as its greatest element.

The interval valued fuzzy subsets provide a more adequate description of uncertainty than the traditional fuzzy subsets; it is therefore important to use interval valued fuzzy subsets in applications. One of the main applications of fuzzy subsets is fuzzy control and one of the most computationally intensive part of fuzzy control is the defuzzification. Since a transition to interval valued fuzzy subsets usually increase the amount of computations, it is vitally important to design faster algorithms for the corresponding defuzzification.

An interval valued fuzzy subset $\tilde{F}: X \to D[0,1]$ of X is the set

$$\tilde{F} = \{x \in X \mid (x, [F^-(x), F^+(x)]) \in D[0,1]\}$$

where F^- and F^+ are two fuzzy subset such that $F^-(x) \le F^+(x)$ for all $x \in X$. Let $\tilde{F}$ be an interval valued fuzzy subset of X. Then for every $[0,0] < \tilde{t} \le [1,1]$, the crisp set $\tilde{F}_{\tilde{t}} = \{x \in X \mid \tilde{F} \ge \tilde{t}\}$ is called the level set of $\tilde{F}$.

We refer the reader to [17] for more details on operations on two interval-valued fuzzy sets of X.

Note that since every $a \in [0,1]$ is in correspondence with the interval $[a,a] \in D[0,1]$, hence a fuzzy set is a particular case of the interval-valued fuzzy sets.

For any $\tilde{F} = [F^-, F^+]$ and $\tilde{t} = [t^-, t^+]$, we define

$$\tilde{F}(x) + \tilde{t} = [F^-(x) + t^-,\ F^+(x) + t^+]$$

for all $x \in X$. In particular, if $F^-(x) + t^- > 1$ and $F^+(x) + t^+ > 1$, we write $\tilde{F}(x) + \tilde{t} > [1,1]$.

Definition: An interval valued fuzzy subset $\tilde{F}$ of an ordered semigroup S is called an interval valued fuzzy generalized bi-ideal of S if it satisfies:

(b3) $(\forall x, y \in S)\ \ x \le y \Rightarrow \tilde{F}(x) \ge \tilde{F}(y)$

(b4) $(\forall x, y, z \in S)\ \ \tilde{F}(xyz) \ge \mathrm{rmin}\{\tilde{F}(x), \tilde{F}(z)\}$

An interval valued fuzzy subset $\tilde{F}$ of an ordered semigroup S is called an interval valued fuzzy bi-ideal [17] of S if it satisfies:

(1) $(\forall x, y \in S)\ \ x \le y \Rightarrow \tilde{F}(x) \ge \tilde{F}(y)$

(2) $(\forall x, y \in S)\ \ \tilde{F}(xy) \ge \mathrm{rmin}\{\tilde{F}(x), \tilde{F}(y)\}$

(3) $(\forall x, y, z \in S)\ \ \tilde{F}(xyz) \ge \mathrm{rmin}\{\tilde{F}(x), \tilde{F}(z)\}$

Every interval valued fuzzy bi-ideal is an interval valued fuzzy generalized bi-ideal of S but the converse is not true in general.

Example: Consider the ordered semigroup S as given in Example 2.1 and define a fuzzy subset $\tilde{F}: S \to D[0,1]$. Then all interval valued fuzzy subsets $\tilde{F}$ of S which satisfies:

(i) $\tilde{F}(a) \ge \tilde{F}(x)$ for all $x \in S$

(ii) $\tilde{F}(b) \ge \tilde{F}(c)$ and

(iii) $\tilde{F}(b) \ge \tilde{F}(d)$ are interval valued fuzzy bi-ideals

All interval valued fuzzy subsets $\tilde{F}$ of S which satisfies:

$\tilde{F}(a) \ge \tilde{F}(x)$ for all $x \in S$ are interval valued fuzzy generalized bi-ideals.

The interval valued fuzzy generalized bi-ideals are not interval valued fuzzy bi-ideals. For example the interval valued fuzzy subset $\tilde{F}$ defined by

$$\tilde{F}(a) = [0.5, 0.5],\ \tilde{F}(b) = [0,0],\ \tilde{F}(c) = [0.2, 0.4],\ \tilde{F}(d) = [0,0]$$

is an interval valued fuzzy generalized bi-ideal of S but not an interval valued fuzzy bi-ideal of S, because

$$[0,0] = \tilde{F}(b) = \tilde{F}(cc) < \mathrm{rmin}\{\tilde{F}(c), \tilde{F}(c)\} = [0.2, 0.4]$$

An interval valued fuzzy subset $\tilde{F}$ of an ordered semigroup $(S, \cdot, \le)$ of the form

$$\tilde{F}(y) = \begin{cases} \tilde{t}(\ne [0,0]), & \text{if } y = x \\ [0,0], & \text{if } y \ne x \end{cases}$$

is called an interval valued fuzzy point with support x and interval value $\tilde{t} \in D[0,1]$ and is denoted by $U(x;\tilde{t})$. For an interval valued fuzzy subset v of S, we say that an interval valued fuzzy point $U(x;\tilde{t})$ is

(b5) contained in $\tilde{F}$, denoted by $U(x;\tilde{t}) \in \tilde{F}$, [22] if $\tilde{F}(x) \ge \tilde{t}$

(b6) quasi-coincident with $\tilde{F}$, denoted by $U(x;\tilde{t})q\tilde{F}$, [22] if $\tilde{F}(x) + \tilde{t} > [1,1]$

For an interval valued fuzzy point $U(x;\tilde{t})$ and an interval valued fuzzy subset $\tilde{F}$ of a set S, we say that

(b7) $U(x;\tilde{t}) \in \vee q\tilde{F}$ if $U(x;\tilde{t}) \in \tilde{F}$ or $U(x;\tilde{t})q\tilde{F}$

(b8) $U(x;\tilde{t})\overline{\alpha}\tilde{F}$ if $U(x;\tilde{t})\alpha\tilde{F}$ does not hold for $\alpha \in \{\in, q, \in\vee q\}$

INTERVAL VALUED $(\in, \in\vee q_{\tilde{k}})$-FUZZY GENERALIZED BI-IDEALS

In what follows, let S be an ordered semigroup and let $\tilde{k} = [k^-, k^+]$ denote an arbitrary element of D[0,1] unless otherwise specified. For an interval valued fuzzy point $U(x;\tilde{t})$ and an interval valued fuzzy subset $\tilde{F}$ of S, we say that

(c1) $U(x;\tilde{t})q_{\tilde{k}}\tilde{F}$ if $\tilde{F}(x) + \tilde{t} + \tilde{k} > [1,1]$ where $F^-(x) + t^- + k^- > 1$ and $F^+(x) + t^+ + k^+ > 1$.

(c2) $U(x;\tilde{t}) \in \vee q_{\tilde{k}}\tilde{F}$ if $U(x;\tilde{t}) \in \tilde{F}$ or $U(x;\tilde{t})q_{\tilde{k}}\tilde{F}$

(c3) $U(x;\tilde{t})\overline{\alpha}\tilde{F}$ if $U(x;\tilde{t})\alpha\tilde{F}$ does not hold for $\alpha \in \{q_{\tilde{k}}, \in\vee q_{\tilde{k}}\}$

We emphasis here that the interval valued fuzzy subset $\tilde{F}(x) = [F^-(x), F^+(x)]$ must satisfies the following condition:

$$\text{(E)}\quad [F^-(x),F^+(x)] \le \left[\frac{1-k^-}{2},\frac{1-k^+}{2}\right] \text{ or } \left[\frac{1-k^-}{2},\frac{1-k^+}{2}\right] < [F^-(x),F^+(x)] \text{ for all } x \in X$$

In what follows, we emphasize that all the interval valued fuzzy subsets of X must satisfy the condition (E) and any two elements of D[0,1] are comparable unless otherwise specified.

Theorem: Let $\tilde{F}$ be an interval valued fuzzy subset of S. Then the following are equivalent:

(1) $\left(\forall \tilde{t} \in D(\frac{1-k}{2},1]\right)$ $\left(\tilde{F}_{\tilde{t}} \neq \phi \Rightarrow \tilde{F}_{\tilde{t}} \text{ is generalized bi-ideal of S}\right)$

(2) $\tilde{F}$ satisfies the following assertions:

(2.1) $(\forall x, y \in S)$

$$\left(x \le y \Rightarrow \tilde{F}(y) \le \text{rmax}\{\tilde{F}(x),[\frac{1-k^-}{2},\frac{1-k^+}{2}]\}\right)$$

(2.2) $(\forall x, y, z \in S)$

$$\left(\text{rmin}\{\tilde{F}(x),\tilde{F}(z)\} \le \text{rmax}\{\tilde{F}(xyz),[\frac{1-k^-}{2},\frac{1-k^+}{2}]\}\right)$$

Proof: Assume that $\tilde{F}_{\tilde{t}}$ is a generalized bi-ideal of S or all $\tilde{t} \in D(\frac{1-k}{2},1]$ with $\tilde{F}_{\tilde{t}} \neq \phi$. If there exist $a,b \in S$ such that the condition (2.1) is not valid, that is, there exist $a,b \in S$ with $a \le b$ such that

$$\tilde{F}(b) > \text{rmax}\{\tilde{F}(a),[\frac{1-k^-}{2},\frac{1-k^+}{2}]\}$$

Then $\tilde{F}(b) \in D(\frac{1-k}{2},1]$ and $b \in \tilde{F}_{\tilde{F}(b)}$. But $\tilde{F}(a) < \tilde{F}(b)$ implies that $a \overline{\in} \tilde{F}_{\tilde{F}(b)}$, a contradiction. Hence (2.1) is valid. Suppose that (2.2) is false, that is, assume that there exist $a,b,c \in S$ such that

$$\tilde{r} := \text{rmin}\{\tilde{F}(a),\tilde{F}(c)\} > \text{rmax}\{\tilde{F}(abc),[\frac{1-k^-}{2},\frac{1-k^+}{2}]\}$$

Then $\tilde{r} \in D(\frac{1-k}{2},1]$ and $a,c \in \tilde{F}_{\tilde{r}}$. But $abc \overline{\in} \tilde{F}_{\tilde{r}}$ since $\tilde{F}(abc) < \tilde{r}$, which is impossible. Therefore

$$\text{rmin}\{\tilde{F}(x),\tilde{F}(z)\} \le \text{rmax}\{\tilde{F}(xyz),[\frac{1-k^-}{2},\frac{1-k^+}{2}]\}$$

for all $x,y,z \in S$

Conversely, assume that $\tilde{F}$ satisfies the three conditions (2.1) and (2.2). Suppose that $\tilde{F}_{\tilde{t}} \neq \phi$ for all $\tilde{t} \in D(\frac{1-k}{2},1]$. Let $x,y \in S$ be such that $x \le y$ and $y \in \tilde{F}_{\tilde{t}}$. Then $\tilde{F}(y) \ge \tilde{t}$ and so

$$\text{rmax}\{\tilde{F}(x),[\frac{1-k^-}{2},\frac{1-k^+}{2}]\} \ge \tilde{F}(y) \ge \tilde{t} > [\frac{1-k^-}{2},\frac{1-k^+}{2}]$$

Hence $\tilde{F}(x) \ge \tilde{t}$, that is $x \in \tilde{F}_{\tilde{t}}$. If $x,y,z \in \tilde{F}_{\tilde{t}}$, it follows from (2.2) that

$$\text{rmax}\{\tilde{F}(xyz),[\frac{1-k^-}{2},\frac{1-k^+}{2}]\} \ge \text{rmin}\{\tilde{F}(x),\tilde{F}(z)\} \ge \tilde{t}$$

so that $\tilde{F}(xyz) \ge \tilde{t}$ i.e., $x,y,z \in \tilde{F}_{\tilde{t}}$. Therefore $\tilde{F}_{\tilde{t}}$ is a generalized bi-ideal of S for all $\tilde{t} \in D(\frac{1-k}{2},1]$ with $\tilde{F}_{\tilde{t}} \neq \phi$.

If we take $\tilde{k}=[0,0]$ in Theorem 3.1, then we have the following corollary.

Corollary: [[17, 18], Theorem 3.3]

Let $\tilde{F}$ be an interval valued fuzzy subset of S. Then the following are equivalent:

(1) $\left(\forall \tilde{t} \in D(0.5,1]\right)$

$\left(\tilde{F}_{\tilde{t}} \neq \phi \Rightarrow \tilde{F}_{\tilde{t}} \text{ is generalized bi-ideal of S}\right)$

(2) $\tilde{F}$ satisfies the following assertions

(2.1) $(\forall x,y \in S)$

$\left(x \leq y \Rightarrow \tilde{F}(y) \leq \operatorname{rmax}\{\tilde{F}(x),[0.5,0.5]\}\right)$

(2.2) $(\forall x,y,z \in S)$

$\left(\operatorname{rmin}\{\tilde{F}(x),\tilde{F}(z)\} \leq \operatorname{rmax}\{\tilde{F}(xyz),[0.5,0.5]\}\right)$

Definition: An interval valued fuzzy subset $\tilde{F}$ of S is called an interval valued $(\in,\in \vee q_{\tilde{k}})$-fuzzy generalized bi-ideal of S if it satisfies the following conditions:

(c4) $U(y;\tilde{t}) \in \tilde{F} \Rightarrow U(x;\tilde{t}) \in \vee q_{\tilde{k}} \tilde{F}$ with $x \leq y$

(c5) $U(x;\tilde{t}_1) \in \tilde{F}, U(z;\tilde{t}_2) \in \tilde{F} \Rightarrow U(xyz;\operatorname{rmin}\{\tilde{t}_1,\tilde{t}_2\}) \in \vee q_{\tilde{k}} \tilde{F}$

for all x,y,z∈S and $\tilde{t},\tilde{t}_1,\tilde{t}_2 \in D(0,1]$.

An interval valued $(\in,\in \vee q_{\tilde{k}})$-fuzzy generalized bi-ideal of S with $\tilde{k}=[0,0]$ is called an interval valued $(\in,\in\vee q)$-fuzzy generalized bi-ideal [17, 18] of S.

Example: Consider the ordered semigroup S = {a,b,c,d,e} with the multiplication given in Table 1 and order relation "≤":

Table 1: Multiplication table for S

·	a	b	c	d	e
a	a	d	a	d	d
b	a	b	a	d	d
c	a	d	c	d	e
d	a	d	a	d	d
e	a	d	c	d	e

≤:= {(a,a),(a,c),(a,d),(a,e),(b,b),(b,d), (b,e),(c,c),(c,e),(d,d),(d,e),(e,e)}

(1) Define an interval valued fuzzy subset $\tilde{F}: S \to D[0,1]$ by

$$\tilde{F}=\begin{pmatrix} a & b & c & d & e \\ [0.4,0.6] & [0.35,0.45] & [0.35,0.45] & [0.25,0.35] & [0.15,0.30] \end{pmatrix}$$

Then $\tilde{F}$ is an interval valued $(\in,\in \vee q_{[0.041,0.04]})$-fuzzy generalized bi-ideal of S.

Theorem: An interval valued fuzzy subset $\tilde{F}$ of S is an interval valued $(\in,\in \vee q_{\tilde{k}})$-fuzzy generalized bi-ideal of S if and only if:

(1) $(\forall x,y \in S)\tilde{F}(x) \geq \operatorname{rmin}\{\tilde{F}(y),[\frac{1-k^-}{2},\frac{1-k^+}{2}]\}$ with $x \leq y$

(2) $(\forall x,y,z \in S)\tilde{F}(xyz) \geq \operatorname{rmin}\{\tilde{F}(x),\tilde{F}(z),[\frac{1-k^-}{2},\frac{1-k^+}{2}]\}$

Proof: Suppose that $\tilde{F}$ is an interval valued $(\in,\in \vee q_{\tilde{k}})$-fuzzy generalized bi-deal of S. Let x,y∈S such that x≤y. If $\tilde{F}(x)<\tilde{F}(y)$, then $\tilde{F}(x)<\tilde{t}\leq\tilde{F}(y)$ for some $\tilde{t}\in D(0,\frac{1-k}{2}]$. It follows that $U(y;\tilde{t})\in\tilde{F}$ but $U(x;\tilde{t})\overline{\in}\tilde{F}$. Since $\tilde{F}(x)+\tilde{t}<2\tilde{t}<[1,1]-\tilde{k}$, we get $U(x;\tilde{t})\overline{q}_{\tilde{k}}\tilde{F}$. Therefore $U(x;\tilde{t})\overline{\in \vee q_{\tilde{k}}}\tilde{F}$, which is a contradiction. Hence $\tilde{F}(x)\geq\tilde{F}(y)$. Now if

$$\tilde{F}(y)\geq[\frac{1-k^-}{2},\frac{1-k^+}{2}]$$

then

$$U(y;[\frac{1-k}{2},\frac{1-k}{2}])\in\tilde{F}$$

and so

$$U(x;[\frac{1-k}{2},\frac{1-k}{2}])\in\vee q_{\tilde{k}}\tilde{F}$$

which implies that

$$\tilde{F}(x)\geq[\frac{1-k^-}{2},\frac{1-k^+}{2}]$$

or

$$\tilde{F}(x)+[\frac{1-k^-}{2},\frac{1-k^+}{2}]>[1,1]-\tilde{k}$$

Hence

$$\tilde{F}(x)\geq[\frac{1-k^-}{2},\frac{1-k^+}{2}]$$

Otherwise

$$\tilde{F}(x)+[\frac{1-k^-}{2},\frac{1-k^+}{2}]<[\frac{1-k^-}{2},\frac{1-k^+}{2}]+[\frac{1-k^-}{2},\frac{1-k^+}{2}]$$
$$=[1,1]-[k^-,k^+]=[1,1]-\tilde{k}$$

a contradiction. Consequently,

$$\tilde{F}(x)\geq\operatorname{rmin}\{\tilde{F}(y),[\frac{1-k^-}{2},\frac{1-k^+}{2}]\}$$

for all $x,y \in S$ with $x \leq y$. Assume that $x,y,z \in S$ be such that

$$\text{rmin}\{\tilde{F}(x),\tilde{F}(z)\} < [\frac{1-k^-}{2},\frac{1-k^+}{2}]$$

We claim that

$$\tilde{F}(xyz) \geq \text{rmin}\{\tilde{F}(x),\tilde{F}(z)\}$$

If not, then

$$\tilde{F}(xyz) < \tilde{s} \leq \text{rmin}\{\tilde{F}(x),\tilde{F}(z)\}$$

for some

$$\tilde{s} \in D[0,\frac{1-k}{2}]$$

It follows that $U(x;\tilde{s}) \in \tilde{F}$ and $U(z;\tilde{s}) \in \tilde{F}$, but $U(xyz;\tilde{s}) \overline{\in} \tilde{F}$ and

$$\tilde{F}(xyz) + \tilde{s} < 2\tilde{s} < [1,1] - \tilde{k}$$

i.e., $U(xyz;\tilde{s})\overline{q}_{\tilde{k}}\tilde{F}$, a contradiction. Thus

$$\tilde{F}(xyz) \geq \text{rmin}\{\tilde{F}(x),\tilde{F}(z)\}$$

for all $x,y,z \in S$ with

$$\text{rmin}\{\tilde{F}(x),\tilde{F}(z)\} < [\frac{1-k^-}{2},\frac{1-k^+}{2}]$$

If

$$\text{rmin}\{\tilde{F}(x),\tilde{F}(z)\} \geq [\frac{1-k^-}{2},\frac{1-k^+}{2}]$$

then

$$U(x;[\frac{1-k^-}{2},\frac{1-k^+}{2}]) \in \tilde{F}$$

and

$$U(z;[\frac{1-k^-}{2},\frac{1-k^+}{2}]) \in \tilde{F}$$

Using (c5), we have

$$\begin{aligned} &U(xyz;[\frac{1-k^-}{2},\frac{1-k^+}{2}]) \\ &= U(xyz;\text{rmin}\{[\frac{1-k^-}{2},\frac{1-k^+}{2}],[\frac{1-k^-}{2},\frac{1-k^+}{2}]) \in \vee q_{\tilde{k}}\tilde{F} \end{aligned}$$

and so

$$\tilde{F}(xyz) \geq [\frac{1-k^-}{2},\frac{1-k^+}{2}]$$

or

$$\tilde{F}(xyz) + [\frac{1-k^-}{2},\frac{1-k^+}{2}] > [1,1] - \tilde{k}$$

$$\text{If } \tilde{F}(xyz) < [\frac{1-k^-}{2},\frac{1-k^+}{2}], \text{ then } \tilde{F}(xyz) + [\frac{1-k^-}{2},\frac{1-k^+}{2}] < [1,1] - \tilde{k}$$

which is a contradiction. Hence

$$\tilde{F}(xyz) \geq [\frac{1-k^-}{2},\frac{1-k^+}{2}]$$

Consequently,

$$\tilde{F}(xyz) \geq \text{rmin}\{\tilde{F}(x),\tilde{F}(z),[\frac{1-k^-}{2},\frac{1-k^+}{2}]\}$$

for all $x,y,z \in S$.

Conversely, let $\tilde{F}$ be an interval valued fuzzy subset of S that satisfies the three conditions (1), (2) and (3). Let $x,y \in S$ and $\tilde{t} \in D[0,\frac{1-k}{2}]$ be such that $x \leq y$ and $U(y;\tilde{t}) \in \tilde{F}$. Then $\tilde{F}(y) \geq \tilde{t}$ and so

$$\begin{aligned} \tilde{F}(x) &\geq \text{rmin}\{\tilde{F}(y),[\frac{1-k^-}{2},\frac{1-k^+}{2}]\} \\ &\geq \text{rmin}\{\tilde{t},[\frac{1-k^-}{2},\frac{1-k^+}{2}]\} \\ &= \begin{cases} \tilde{t}, & \text{if } \tilde{t} \leq [\frac{1-k^-}{2},\frac{1-k^+}{2}] \\ [\frac{1-k^-}{2},\frac{1-k^+}{2}], & \text{if } \tilde{t} > [\frac{1-k^-}{2},\frac{1-k^+}{2}] \end{cases} \end{aligned}$$

It follows that $U(x;\tilde{t}) \in \tilde{F}$ or

$$\tilde{F}(x) + \tilde{t} \geq [\frac{1-k^-}{2},\frac{1-k^+}{2}] + \tilde{t} > [1,1] - \tilde{k}$$

i.e., $U(x;\tilde{t})q_{\tilde{k}}\tilde{F}$. Hence $U(x;\tilde{t}) \in \vee q_{\tilde{k}}\tilde{F}$. Assume that $x,y,z \in S$ and $\tilde{t}_1, \tilde{t}_2 \in D(0,1]$ be such that $U(x;\tilde{t}_1) \in \tilde{F}$ and $U(z;\tilde{t}_2) \in \tilde{F}$. Then $\tilde{F}(x) \geq \tilde{t}_1$ and $\tilde{F}(z) \geq \tilde{t}_2$. It follows from (2) that

$$\begin{aligned} \tilde{F}(xyz) &\geq \text{rmin}\{\tilde{F}(x),\tilde{F}(z),[\frac{1-k^-}{2},\frac{1-k^+}{2}]\} \\ &\geq \text{rmin}\{\tilde{t}_1,\tilde{t}_2,[\frac{1-k^-}{2},\frac{1-k^+}{2}]\} \\ &= \begin{cases} \text{rmin}\{\tilde{t}_1,\tilde{t}_2\}, & \text{if } \text{rmin}\{\tilde{t}_1,\tilde{t}_2\} \leq [\frac{1-k^-}{2},\frac{1-k^+}{2}] \\ [\frac{1-k^-}{2},\frac{1-k^+}{2}], & \text{if } \text{rmin}\{\tilde{t}_1,\tilde{t}_2\} > [\frac{1-k^-}{2},\frac{1-k^+}{2}] \end{cases} \end{aligned}$$

so that $U(xyz;\text{rmin}\{\tilde{t}_1,\tilde{t}_2\}) \in \tilde{F}$ or

$$\begin{aligned} \tilde{F}(xyz) + \text{rmin}\{\tilde{t}_1,\tilde{t}_2\} &\geq [\frac{1-k^-}{2},\frac{1-k^+}{2}] + \text{rmin}\{\tilde{t}_1,\tilde{t}_2\} \\ &> [\frac{1-k^-}{2},\frac{1-k^+}{2}] + [\frac{1-k^-}{2},\frac{1-k^+}{2}] \\ &= [1,1] - \tilde{k} \end{aligned}$$

that is $U(xyz;r\min\{\tilde{t}_1,\tilde{t}_2\})q_{\tilde{k}}\tilde{F}$. Thus

$$U(xyz;r\min\{\tilde{t}_1,\tilde{t}_2\})\in\vee q_{\tilde{k}}\tilde{F}$$

If we take $\tilde{k}=[0,0]$ in the Theorem 3.5, then we have the following corollary.

Corollary: [[17], Theorem 4.3]

An interval valued fuzzy subset $\tilde{F}$ of S is an interval valued $(\in,\in\vee q)$-fuzzy generalized bi-ideal of S if and only if:

(1) $(\forall x,y\in S)\tilde{F}(x)\geq r\min\{\tilde{F}(y),[0.5,0.5]\}$ with $x\leq y$

(2) $(\forall x,y,z\in S)\tilde{F}(xyz)\geq r\min\{\tilde{F}(x),\tilde{F}(z),[0.5,0.5]\}$

Obviously, every interval valued fuzzy generalized bi-ideal of S is an interval valued $(\in,\in\vee q_{\tilde{k}})$-fuzzy generalized bi-ideal of S for some $\tilde{k}\in D(0,1]$. The following example shows that there exists $\tilde{k}\in D(0,1]$ such that

(i) $\tilde{F}$ is an interval valued $(\in,\in\vee q_{\tilde{k}})$-fuzzy generalized bi-ideal of S.

(ii) $\tilde{F}$ is not an interval valued fuzzy generalized bi-ideal of S.

Example: The interval valued $(\in,\in\vee q_{[0.041,0.04]})$-fuzzy generalized bi-ideal $\tilde{F}$ of S in Example 3.4 is not an interval valued fuzzy generalized bi-ideal of S since $\tilde{F}(b)=[0.35,0.45]$ and $\tilde{F}(c)=[0.35,0.45]$. But

$$\begin{aligned}[0.25,0.35]=\tilde{F}(d)&=\tilde{F}(cb)\leq r\min\{\tilde{F}(c),\tilde{F}(b)\}\\&=r\min\{[0.35,0.45],[0.35,0.45]\}=[0.35,0.45]\end{aligned}$$

We give a condition for an interval valued $(\in,\in\vee q_{\tilde{k}})$-fuzzy generalized bi-ideal to be an interval valued fuzzy generalized bi-ideal.

Theorem: Let $\tilde{F}$ be an interval valued $(\in,\in\vee q_{\tilde{k}})$-fuzzy generalized bi-ideal of S. If

$$\tilde{F}(x)<[\frac{1-k^-}{2},\frac{1-k^+}{2}]$$

for all $x\in S$. Then $\tilde{F}$ is an interval valued fuzzy generalized bi-ideal of S.

Proof: It is straightforward by Theorem 3.5.

Corollary: [[17], Theorem 4.9]

Let $\tilde{F}$ be an interval valued $(\in,\in\vee q)$-fuzzy generalized bi-ideal of S. If $\tilde{F}(x)<[0.5,0.5]$ for all $x\in S$. Then $\tilde{F}$ is an interval valued fuzzy generalized bi-ideal of S.

Proof: It follows from Theorem 3.8 by taking $\tilde{k}=[0,0]$.

Theorem: For an interval valued fuzzy subset $\tilde{F}$ of S, the following are equivalent:

(1) $\tilde{F}$ is an interval valued $(\in,\in\vee q_{\tilde{k}})$-fuzzy generalized bi-ideal of S.

(2) $(\forall\tilde{t}\in D(0,\frac{1-k}{2}])$

$\tilde{F}_{\tilde{t}}\neq\phi\Rightarrow\tilde{F}_{\tilde{t}}$ is a generalized bi-ideal of S.

Proof: Assume that $\tilde{F}$ is an interval valued $(\in,\in\vee q_{\tilde{k}})$-fuzzy generalized bi-ideal of S and let $\tilde{t}\in D(0,\frac{1-k}{2}]$ be such that $\tilde{F}_{\tilde{t}}\neq\phi$. Using Theorem 3.5 (1), we have

$$\tilde{F}(x)\geq r\min\{\tilde{F}(y),[\frac{1-k^-}{2},\frac{1-k^+}{2}]\}$$

for any $x,y\in S$ with $x\leq y$ and $y\in\tilde{F}_{\tilde{t}}$. It follows that

$$\tilde{F}(x)\geq r\min\{\tilde{t},[\frac{1-k^-}{2},\frac{1-k^+}{2}]\}=\tilde{t}$$

so that $x\in\tilde{F}_{\tilde{t}}$. Now let $x,z\in\tilde{F}_{\tilde{t}}$, then $\tilde{F}(x)\geq\tilde{t}$ and $\tilde{F}(z)\geq\tilde{t}$. Theorem 3.5 (2) implies that

$$\begin{aligned}\tilde{F}(xyz)&\geq r\min\{\tilde{F}(x),\tilde{F}(z),[\frac{1-k^-}{2},\frac{1-k^+}{2}]\}\\&\geq r\min\{\tilde{t},[\frac{1-k^-}{2},\frac{1-k^+}{2}]\}=\tilde{t}\end{aligned}$$

Therefore $xyz\in\tilde{F}_{\tilde{t}}$ and hence $\tilde{F}_{\tilde{t}}$ is a generalized bi-ideal of S, where $\tilde{t}\in D(0,\frac{1-k}{2}]$.

Conversely, let $\tilde{F}$ be an interval valued fuzzy subset of S such $\tilde{F}_{\tilde{t}}$ is non-empty and is a generalized bi-ideal of S for all $\tilde{t}\in D(0,\frac{1-k}{2}]$. If there exist $a,b\in S$ with $a\leq b$ and $b\in\tilde{F}_{\tilde{t}}$ such that

$$\tilde{F}(a)<r\min\{\tilde{F}(b),[\frac{1-k^-}{2},\frac{1-k^+}{2}]\}$$

then

$$\tilde{F}(a) < \tilde{t}_a \le \text{rmin}\{\tilde{F}(b),[\frac{1-k^-}{2},\frac{1-k^+}{2}]\}$$

for some $\tilde{t}_a \in D(0,\frac{1-k}{2}]$ and so $a \overline{\in} \tilde{F}_{\tilde{t}_a}$, a contradiction. Therefore,

$$\tilde{F}(a) \ge \text{rmin}\{\tilde{F}(b),[\frac{1-k^-}{2},\frac{1-k^+}{2}]\}$$

for all x,y∈S with x≤y. Suppose that

$$\tilde{F}(abc) < \text{rmin}\{\tilde{F}(a),\tilde{F}(c),[\frac{1-k^-}{2},\frac{1-k^+}{2}]\}$$

then

$$\tilde{F}(abc) < \tilde{t}_{abc} \le \text{rmin}\{\tilde{F}(a),\tilde{F}(c),[\frac{1-k^-}{2},\frac{1-k^+}{2}]\}$$

for some $\tilde{t}_{abc} \in D(0,\frac{1-k}{2}]$. It follows that $a \in \tilde{F}_{\tilde{t}_{abc}}$ and $c \in \tilde{F}_{\tilde{t}_{abc}}$ but $abc \overline{\in} \tilde{F}_{\tilde{t}_{abc}}$. This is impossible and thus

$$\tilde{F}(xyz) \ge \text{rmin}\{\tilde{F}(x),\tilde{F}(z),[\frac{1-k^-}{2},\frac{1-k^+}{2}]\}$$

for all x,y,z∈S. Using Theorem 3.5, we conclude that $\tilde{F}$ is an interval valued $(\in,\in \vee q_{\tilde{k}})$-fuzzy generalized bi-ideal of S.

Taking $\tilde{k} = [0,0]$ in Theorem 3.10 induces the following corollary.

Corollary: [[17], Theorem 4.10]

For an interval valued fuzzy subset $\tilde{F}$ of S, the following are equivalent:

(1) $\tilde{F}$ is an interval valued $(\in,\in\vee q)$-fuzzy generalized bi-ideal of S

(2) $(\forall \tilde{t} \in D(0,0.5])$

$\tilde{F}_{\tilde{t}} \ne \phi \Rightarrow \tilde{F}_{\tilde{t}}$ is a generalized bi-ideal of S

Theorem: For any generalized bi-ideal A of S, let $\tilde{F}$ be an interval valued fuzzy subset of S defined by

$$\tilde{F}(x) = \begin{cases} \tilde{t}_1, & \text{if} \quad x \in A \\ \tilde{t}_2, & \text{otherwise} \end{cases} \qquad (1)$$

where

$$\tilde{t}_1 \in D[\frac{1-k}{2},1] \text{ and } \tilde{t}_2 \in D(0,\frac{1-k}{2})$$

Then $\tilde{F}$ is an interval valued $(\in,\in \vee q_{\tilde{k}})$-fuzzy generalized bi-ideal of S.

Proof: Note that

$$\tilde{F}_{\tilde{r}} = \begin{cases} A, & \text{if} \quad \tilde{r} \in D(\tilde{t}_2,\frac{1-k}{2}) \\ S_2, & \text{if} \quad \tilde{r} \in D(0,\tilde{t}_2] \end{cases} \qquad (2)$$

which is a bi-ideal of S. It follows from Theorem 3.10 that $\tilde{F}$ is an interval valued $(\in,\in \vee q_{\tilde{k}})$-fuzzy generalized bi-ideal of S.

Corollary: [17]

For any generalized bi-ideal A of S, let $\tilde{F}$ be an interval valued fuzzy subset of S defined by

$$\tilde{F}(x) = \begin{cases} \tilde{t}_1, & \text{if} \quad x \in A \\ \tilde{t}_2, & \text{otherwise} \end{cases}$$

where $\tilde{t}_1 \in D[0.5,1]$ and $\tilde{t}_2 \in D(0,0.5)$. Then $\tilde{F}$ is an interval valued $(\in,\in\vee q)$-fuzzy generalized bi-ideal of S.

For any interval valued fuzzy subset $\tilde{F}$ of S and $\tilde{t} \in D(0,1]$, we consider four subsets:

$$Q(\tilde{F};\tilde{t}) := \{x \in S | U(x;\tilde{t}) q\tilde{F}\}$$

and

$$[\tilde{F}]_{\tilde{t}} := \{x \in S | U(x;\tilde{t}) \in \vee q\tilde{F}\}$$

$$Q^{\tilde{k}}(\tilde{F};\tilde{t}) := \{x \in S | U(x;\tilde{t}) q_{\tilde{k}}\tilde{F}\}$$

and

$$[\tilde{F}]^{\tilde{k}}_{\tilde{t}} := \{x \in S | U(x;\tilde{t}) \in \vee q_{\tilde{k}}\tilde{F}\}$$

It is clear that

$$[\tilde{F}]_{\tilde{t}} := \tilde{F}_{\tilde{t}} \cup Q(\tilde{F};\tilde{t}) \text{ and } [\tilde{F}]^{\tilde{k}}_{\tilde{t}} := \tilde{F}_{\tilde{t}} \cup Q^{\tilde{k}}$$

Theorem: If $\tilde{F}$ is an interval valued $(\in,\in \vee q_{\tilde{k}})$-fuzzy generalized bi-ideal of S, then

$$(\forall \tilde{t} \in D(\frac{1-k}{2},1]) \; Q^{\tilde{k}}(\tilde{F};\tilde{t}) \ne \phi \Rightarrow Q^{\tilde{k}}(\tilde{F};\tilde{t})$$

is generalize bi-ideal of S

Proof: Assume that $\tilde{F}$ is an interval valued $(\in,\in \vee q_{\tilde{k}})$-fuzzy generalized bi-ideal of S. Let $\tilde{t} \in D(\frac{1-k}{2},1]$ be such that $Q^{\tilde{k}}(\tilde{F};\tilde{t}) \neq \phi$. Let $y \in Q^{\tilde{k}}(\tilde{F};\tilde{t})$ and $x \in S$ be such that $x \leq y$. Then $\tilde{F}(y) + \tilde{t} > [1,1] - \tilde{k}$. By means of Theorem 3.5 (1), we have

$$\begin{aligned}\tilde{F}(x) &\geq \mathrm{rmin}\{\tilde{F}(y),[\frac{1-k^-}{2},\frac{1-k^+}{2}]\} \\ &= \begin{cases}[\frac{1-k^-}{2},\frac{1-k^+}{2}], & \text{if } \tilde{F}(y) \geq [\frac{1-k^-}{2},\frac{1-k^+}{2}] \\ \tilde{F}(y), & \text{if } \tilde{F}(y) < [\frac{1-k^-}{2},\frac{1-k^+}{2}]\end{cases} \\ &> [1,1] - \tilde{t} - \tilde{k}\end{aligned}$$

and so $x \in Q^{\tilde{k}}(\tilde{F};\tilde{t})$. Let $x,z \in Q^{\tilde{k}}(\tilde{F};\tilde{t})$, then $\tilde{F}(x) + \tilde{t} > [1,1] - \tilde{k}$ and $\tilde{F}(z) + \tilde{t} > [1,1] - \tilde{k}$. It follows from Theorem 3.5 (2) that

$$\begin{aligned}\tilde{F}(xyz) &\geq \mathrm{rmin}\{\tilde{F}(x),\tilde{F}(z),[\frac{1-k^-}{2},\frac{1-k^+}{2}]\} \\ &= \begin{cases}\mathrm{rmin}\{\tilde{F}(x),\tilde{F}(z)\}, & \text{if } \mathrm{rmin}\{\tilde{F}(x),\tilde{F}(z)\} < [\frac{1-k^-}{2},\frac{1-k^+}{2}] \\ [\frac{1-k^-}{2},\frac{1-k^+}{2}], & \text{if } \mathrm{rmin}\{\tilde{F}(x),\tilde{F}(z)\} \geq [\frac{1-k^-}{2},\frac{1-k^+}{2}]\end{cases} \\ &> [1,1] - \tilde{t} - \tilde{k}\end{aligned}$$

and hence $xyz \in Q^{\tilde{k}}(\tilde{F};\tilde{t})$. Therefore $Q^{\tilde{k}}(\tilde{F};\tilde{t})$ is a generalized bi-ideal of S.

Corollary: If $\tilde{F}$ is an interval valued $(\in,\in\vee q)$-fuzzy generalized bi-ideal of S, then

$$(\forall \tilde{t} \in D(0.5,1])\ Q^{\tilde{k}}(\tilde{F};\tilde{t}) \neq \phi \Rightarrow Q^{\tilde{k}}(\tilde{F};\tilde{t})$$

is generalize bi-ideal of S

Corollary: Let $\tilde{F}$ be an interval valued $(\in,\in\vee q_{\tilde{k}})$-fuzzy generalized bi-ideal of S, if $0 \leq \tilde{k} < \tilde{r} < [1,1]$, then

$$(\forall \tilde{t} \in D(\frac{1-r}{2},1])\ Q^{\tilde{k}}(\tilde{F};\tilde{t}) \neq \phi \Rightarrow Q^{\tilde{k}}(\tilde{F};\tilde{t})$$

is generalize bi-ideal of S

Proof: It is straightforward by Theorem 3.10 and 3.14.

Theorem: For any interval valued fuzzy subset $\tilde{F}$ of S, the following are equivalent:

(1) $\tilde{F}$ is an interval valued $(\in,\in\vee q_{\tilde{k}})$-fuzzy generalized bi-ideal of S.
(2) $(\forall \tilde{t} \in D(0,1])$ $[\tilde{F}]_{\tilde{t}}^{\tilde{k}} \neq \phi \Rightarrow [\tilde{F}]_{\tilde{t}}^{\tilde{k}}$ is generalize bi-ideal of S

Proof: Assume that $\tilde{F}$ is an interval valued $(\in,\in\vee q_{\tilde{k}})$-fuzzy generalized bi-ideal of S and let $\tilde{t} \in D(0,1]$ such that $[\tilde{F}]_{\tilde{t}}^{\tilde{k}} \neq \phi$. Let $y \in [\tilde{F}]_{\tilde{t}}^{\tilde{k}}$ and $x \in S$ be such that $x \leq y$. Then $y \in \tilde{F}_{\tilde{t}}$ or $y \in Q^{\tilde{k}}(\tilde{F};\tilde{t})$, i.e., $\tilde{F}(y) \geq \tilde{t}$ or $\tilde{F}(y) + \tilde{t} > [1,1] - \tilde{k}$. Using Theorem 3.5 (1), we get

$$\tilde{F}(x) \geq \mathrm{rmin}\{\tilde{F}(y),[\frac{1-k^-}{2},\frac{1-k^+}{2}]\} \tag{3}$$

We consider two cases:

$$\tilde{F}(y) \leq [\frac{1-k^-}{2},\frac{1-k^+}{2}] \text{ and } \tilde{F}(y) > [\frac{1-k^-}{2},\frac{1-k^+}{2}]$$

The first case implies from (3) that $\tilde{F}(x) \geq \tilde{F}(y)$. Thus if $\tilde{F}(y) \geq \tilde{t}$ then $\tilde{F}(x) \geq \tilde{t}$ and so $x \in \tilde{F}_{\tilde{t}} \subseteq [\tilde{F}]_{\tilde{t}}^{\tilde{k}}$. If

$$\tilde{F}(y) + \tilde{t} > [1,1] - \tilde{k}$$

then

$$\tilde{F}(x) + \tilde{t} \geq \tilde{F}(y) + \tilde{t} > [1,1] - \tilde{k}$$

which implies that $U(x;\tilde{t})q_{\tilde{k}}\tilde{F}$ i.e.,

$$x \in Q^{\tilde{k}}(\tilde{F};\tilde{t}) \subseteq [\tilde{F}]_{\tilde{t}}^{\tilde{k}}$$

Combining the second case and (3) induces

$$\tilde{F}(x) \geq [\frac{1-k^-}{2},\frac{1-k^+}{2}]$$

If $\tilde{t} \leq [\frac{1-k^-}{2},\frac{1-k^+}{2}]$, then $\tilde{F}(x) \geq \tilde{t}$ and hence $x \in \tilde{F}_{\tilde{t}} \subseteq [\tilde{F}]_{\tilde{t}}^{\tilde{k}}$. If $\tilde{t} > [\frac{1-k^-}{2},\frac{1-k^+}{2}]$, then

$$\tilde{F}(x) + \tilde{t} > [\frac{1-k^-}{2},\frac{1-k^+}{2}] + [\frac{1-k^-}{2},\frac{1-k^+}{2}] = [1,1] - \tilde{k}$$

which implies that $x \in Q^{\tilde{k}}(\tilde{F};\tilde{t}) \subseteq [\tilde{F}]_{\tilde{t}}^{\tilde{k}}$. Therefore $[\tilde{F}]_{\tilde{t}}^{\tilde{k}}$ satisfies the condition (b1). Let $x,z \in [\tilde{F}]_{\tilde{t}}^{\tilde{k}}$. Then $x \in \tilde{F}_{\tilde{t}}$ or $x \in Q^{\tilde{k}}(\tilde{F};\tilde{t})$ and $z \in \tilde{F}_{\tilde{t}}$ or $z \in Q^{\tilde{k}}(\tilde{F};\tilde{t})$, i.e., $\tilde{F}(x) \geq \tilde{t}$ or $\tilde{F}(x) + \tilde{t} > [1,1] - \tilde{k}$ and $\tilde{F}(z) \geq \tilde{t}$ or $\tilde{F}(z) + \tilde{t} > [1,1] - \tilde{k}$. We consider the following four cases.

(i) If $\tilde{F}(x) \geq \tilde{t}$ and $\tilde{F}(z) \geq \tilde{t}$
(ii) If $\tilde{F}(x) \geq \tilde{t}$ and $\tilde{F}(z) + \tilde{t} > [1,1] - \tilde{k}$
(iii) If $\tilde{F}(x) + \tilde{t} > [1,1] - \tilde{k}$ and $\tilde{F}(z) \geq \tilde{t}$
(iv) If $\tilde{F}(x) + \tilde{t} > [1,1] - \tilde{k}$ and $\tilde{F}(z) + \tilde{t} > [1,1] - \tilde{k}$

For the case (i), Theorem 3.5 (2) implies that

$$\tilde{F}(xyz) \geq \mathrm{rmin}\{\tilde{F}(x),\tilde{F}(z),[\frac{1-k^-}{2},\frac{1-k^+}{2}]\} \geq \mathrm{rmin}\{\tilde{t},[\frac{1-k^-}{2},\frac{1-k^+}{2}]\} = \begin{cases} [\frac{1-k^-}{2},\frac{1-k^+}{2}], & \text{if } \tilde{t} > [\frac{1-k^-}{2},\frac{1-k^+}{2}] \\ \tilde{t}, & \text{if } \tilde{t} \leq [\frac{1-k^-}{2},\frac{1-k^+}{2}] \end{cases}$$

so that $xyz \in \tilde{F}_{\tilde{t}}$ or

$$\tilde{F}(xyz) + \tilde{t} + \tilde{k} > [\frac{1-k^-}{2},\frac{1-k^+}{2}] + [\frac{1-k^-}{2},\frac{1-k^+}{2}] + \tilde{k} = [1,1]$$

that is $xyz \in Q^{\tilde{k}}(\tilde{F};\tilde{t})$. Hence $xyz \in [\tilde{F}]_{\tilde{t}}^{\tilde{k}}$. For the second case assume that $\tilde{t} > [\frac{1-k^-}{2},\frac{1-k^+}{2}]$, then

$$[1,1] - \tilde{t} - \tilde{k} < [1,1] - \tilde{t} < [\frac{1-k^-}{2},\frac{1-k^+}{2}]$$

and so

$$\tilde{F}(xyz) \geq \mathrm{rmin}\{\tilde{F}(x),\tilde{F}(z),[\frac{1-k^-}{2},\frac{1-k^+}{2}]\} = \begin{cases} \mathrm{rmin}\{\tilde{F}(z),[\frac{1-k^-}{2},\frac{1-k^+}{2}]\} > [1,1] - \tilde{t} - \tilde{k}, & \text{if } \mathrm{rmin}\{\tilde{F}(z),[\frac{1-k^-}{2},\frac{1-k^+}{2}]\} \leq \tilde{F}(x) \\ \tilde{F}(x) \geq \tilde{t}, & \text{if } \mathrm{rmin}\{\tilde{F}(z),[\frac{1-k^-}{2},\frac{1-k^+}{2}]\} > \tilde{F}(x) \end{cases}$$

Thus $xyz \in \tilde{F}_{\tilde{t}} \cup Q^{\tilde{k}}(\tilde{F};\tilde{t}) = [\tilde{F}]_{\tilde{t}}^{\tilde{k}}$ Suppose that $\tilde{t} \leq [\frac{1-k^-}{2},\frac{1-k^+}{2}]$ then

$$\tilde{F}(xyz) \geq \mathrm{rmin}\{\tilde{F}(x),\tilde{F}(z),[\frac{1-k^-}{2},\frac{1-k^+}{2}]\} = \begin{cases} \mathrm{rmin}\{\tilde{F}(x),[\frac{1-k^-}{2},\frac{1-k^+}{2}]\} \geq \tilde{t}, & \text{if } \mathrm{rmin}\{\tilde{F}(x),[\frac{1-k^-}{2},\frac{1-k^+}{2}]\} \leq \tilde{F}(z) \\ \tilde{F}(z) > [1,1] - \tilde{t} - \tilde{k}, & \text{if } \mathrm{rmin}\{\tilde{F}(x),[\frac{1-k^-}{2},\frac{1-k^+}{2}]\} > \tilde{F}(z) \end{cases}$$

and thus

$$xyz \in \tilde{F}_{\tilde{t}} \cup Q^{\tilde{k}}(\tilde{F};\tilde{t}) = [\tilde{F}]_{\tilde{t}}^{\tilde{k}}$$

We have similar result for the case (iii). For the final case, if $\tilde{t} > [\frac{1-k^-}{2},\frac{1-k^+}{2}]$ then

$$[1,1] - \tilde{t} - \tilde{k} < [\frac{1-k^-}{2},\frac{1-k^+}{2}]$$

Hence

$$\tilde{F}(xyz) \geq \mathrm{rmin}\{\tilde{F}(x),\tilde{F}(z),[\frac{1-k^-}{2},\frac{1-k^+}{2}]\} = \Big\{ [\frac{1-k^-}{2},\frac{1-k^+}{2}] > [1,1] - \tilde{t} - \tilde{k}$$

whenever $\mathrm{rmin}\{\tilde{F}(x),\tilde{F}(z)\} < [\frac{1-k^-}{2},\frac{1-k^+}{2}]$ Thus

$$xyz \in Q^{\tilde{k}}(\tilde{F};\tilde{t}) \subseteq [\tilde{F}]_{\tilde{t}}^{\tilde{k}}. \text{ If } \tilde{t} \leq [\frac{1-k^-}{2},\frac{1-k^+}{2}]$$

then

$$\tilde{F}(xyz) \geq \mathrm{rmin}\{\tilde{F}(x),\tilde{F}(z),[\frac{1-k^-}{2},\frac{1-k^+}{2}]\} = \begin{cases} [\frac{1-k^-}{2},\frac{1-k^+}{2}] \geq \tilde{t}, & \text{if } \mathrm{rmin}\{\tilde{F}(x),\tilde{F}(z)\} \geq [\frac{1-k^-}{2},\frac{1-k^+}{2}] \\ \mathrm{rmin}\{\tilde{F}(x),\tilde{F}(z)\} > [1,1] - \tilde{t} - \tilde{k}, & \text{if } \mathrm{rmin}\{\tilde{F}(x),\tilde{F}(z)\} < [\frac{1-k^-}{2},\frac{1-k^+}{2}] \end{cases}$$

which implies that

$$xyz \in \tilde{F}_{\tilde{t}} \cup Q^{\tilde{k}}(\tilde{F};\tilde{t}) = [\tilde{F}]_{\tilde{t}}^{\tilde{k}}$$

Conversely, suppose that (2) is valid. If there exist $a,b\in S$ such that $a\leq b$ and

$$\tilde{F}(a) < \mathrm{rmin}\{\tilde{F}(b),[\frac{1-k^-}{2},\frac{1-k^+}{2}]\}$$

then

$$\tilde{F}(a) < \tilde{t}_a \leq \mathrm{rmin}\{\tilde{F}(b),[\frac{1-k^-}{2},\frac{1-k^+}{2}]\}$$

for some $\tilde{t}_a \in D(0,\frac{1-k}{2}]$. It follows that $b\in \tilde{F}_{\tilde{t}_a} \subseteq [\tilde{F}]^{\tilde{k}}_{\tilde{t}_a}$ but $a \,\overline{\in}\, \tilde{F}_{\tilde{t}_a}$. Also, we have $\tilde{F}(a)+\tilde{t}_a < 2\tilde{t}_a \leq [1,1]-\tilde{k}$ and so $U(a;\tilde{t}_a)\,\overline{q}_{\tilde{k}}\,\tilde{F}$ i.e., $a \,\overline{\in}\, Q^{\tilde{k}}(\tilde{F};\tilde{t}_a)$. Therefore $a \,\overline{\in}\, [\tilde{F}]^{\tilde{k}}_{\tilde{t}_a}$ a contradiction. Hence

$$\tilde{F}(x) \succ \mathrm{rmin}\{\tilde{F}(y),[\frac{1-k^-}{2},\frac{1-k^+}{2}]\}$$

for all $x,y\in S$ with $x\leq y$. Assume that there exist $a,b,c\in S$ such that

$$\tilde{F}(abc) < \tilde{t} \leq \mathrm{rmin}\{\tilde{F}(a),\tilde{F}(c),[\frac{1-k^-}{2},\frac{1-k^+}{2}]\}$$

for some $\tilde{t}\in D(0,\frac{1-k}{2}]$. It follows that $a\in \tilde{F}_{\tilde{t}} \subseteq [\tilde{F}]^{\tilde{k}}_{\tilde{t}}$ and $c\in \tilde{F}_{\tilde{t}} \subseteq [\tilde{F}]^{\tilde{k}}_{\tilde{t}}$, so from (b2) $abc\in [\tilde{F}]^{\tilde{k}}_{\tilde{t}}$. Thus $\tilde{F}(abc)\geq \tilde{t}$ or $\tilde{F}(abc)+\tilde{t} > [1,1]-\tilde{k}$, a contradiction. Therefore

$$\tilde{F}(xyz) \geq \mathrm{rmin}\{\tilde{F}(x),\tilde{F}(z),[\frac{1-k^-}{2},\frac{1-k^+}{2}]\}$$

for all $x,y,z\in S$. Using Theorem 3.5, we conclude that $\tilde{F}$ is an $(\in,\in\vee q_{\tilde{k}})$-fuzzy generalized bi-ideal of S.

Corollary: For any interval valued fuzzy subset $\tilde{F}$ of S, the following are equivalent:

(1) $\tilde{F}$ is an interval valued $(\in,\in\vee q)$-fuzzy generalized bi-ideal of S.

(2) $(\forall \tilde{t}\in D(0,1])\ [\tilde{F}]_{\tilde{t}} \neq \phi \Rightarrow [\tilde{F}]_{\tilde{t}}$. is generalize bi-ideal of S

An interval valued fuzzy subset $\tilde{F}$ of S is said to be proper if $\mathrm{Im}(\tilde{F})$ has at least two elements. Two interval valued fuzzy subsets are said to be equivalent if they have same family of level subsets. Otherwise, they are said to be non-equivalent.

Theorem: Let $\tilde{F}$ be an interval valued $(\in,\in\vee q_{\tilde{k}})$-fuzzy generalized bi-ideal of S such that

$$\#\{\tilde{F}(x)|\tilde{F}(x) < [\frac{1-k^-}{2},\frac{1-k^-}{2}]\} \geq 2$$

Then there exist two proper non-equivalent interval valued $(\in,\in\vee q_{\tilde{k}})$-fuzzy generalized bi-ideals of S such that $\tilde{F}$ can be expressed as the union of them.

Proof: Let

$$\{\tilde{F}(x)|\tilde{F}(x) < [\frac{1-k^-}{2},\frac{1-k^-}{2}]\} = \{\tilde{t}_1,\tilde{t}_2,\tilde{t}_3,\ldots,\tilde{t}_n\}$$

where $\tilde{t}_1 > \tilde{t}_2 > \tilde{t}_3 > \ldots > \tilde{t}_n$ and $n\geq 2$. Then the chain of interval valued $(\in,\in\vee q_{\tilde{k}})$-level generalized bi-ideals of $\tilde{F}$ is

$$[\tilde{F}]^{\tilde{k}}_{[\frac{1-k^-}{2},\frac{1-k^+}{2}]} \subseteq [\tilde{F}]^{\tilde{k}}_{\tilde{t}_1} \subseteq [\tilde{F}]^{\tilde{k}}_{\tilde{t}_2} \subseteq \ldots \subseteq [\tilde{F}]^{\tilde{k}}_{\tilde{t}_n} = S$$

Let $\tilde{A}$ and $\tilde{B}$ be interval valued fuzzy subsets of S defined by

$$\tilde{A}(x) = \begin{cases} \tilde{t}_1, & \text{if } x\in [\tilde{A}]^{\tilde{k}}_{\tilde{t}_1} \\ \tilde{t}_2, & \text{if } x\in [\tilde{A}]^{\tilde{k}}_{\tilde{t}_2} \setminus [\tilde{A}]^{\tilde{k}}_{\tilde{t}_1} \\ \ldots \\ \tilde{t}_n, & \text{if } x\in [\tilde{A}]^{\tilde{k}}_{\tilde{t}_n} \setminus [\tilde{A}]^{\tilde{k}}_{\tilde{t}_{n-1}} \end{cases}$$

and

$$\tilde{B}(x) = \begin{cases} \tilde{B}(x), & \text{if } x\in [\tilde{B}]^{\tilde{k}}_{[\frac{1-k^-}{2},\frac{1-k^+}{2}]} \\ \tilde{k}, & \text{if } x\in [\tilde{B}]^{\tilde{k}}_{\tilde{t}_2} \setminus [\tilde{B}]^{\tilde{k}}_{[\frac{1-k^-}{2},\frac{1-k^+}{2}]} \\ \tilde{t}_3, & \text{if } x\in [\tilde{B}]^{\tilde{k}}_{\tilde{t}_2} \setminus [\tilde{B}]^{\tilde{k}}_{\tilde{t}_1} \\ \ldots \\ \tilde{t}_n, & \text{if } x\in [\tilde{B}]^{\tilde{k}}_{\tilde{t}_n} \setminus [\tilde{B}]^{\tilde{k}}_{\tilde{t}_{n-1}} \end{cases}$$

respectively, where $\tilde{t}_3 < \tilde{k} < \tilde{t}_2$. Then $\tilde{A}$ and $\tilde{B}$ are interval valued $(\in,\in\vee q_{\tilde{k}})$-fuzzy generalized bi-ideals of S and $\tilde{A},\tilde{B}\leq \tilde{F}$. The chains of interval valued $(\in,\in\vee q_{\tilde{k}})$-level generalized bi-ideals of $\tilde{A}$ and $\tilde{B}$ are, respectively, given by

$$[\tilde{A}]^{\tilde{k}}_{\tilde{t}_1} \subseteq [\tilde{A}]^{\tilde{k}}_{\tilde{t}_2} \subseteq \ldots \subseteq [\tilde{A}]^{\tilde{k}}_{\tilde{t}_n}$$

and

$$[\tilde{B}]^{\tilde{k}}_{[\frac{1-k^-}{2},\frac{1-k^+}{2}]} \subseteq [\tilde{B}]^{\tilde{k}}_{\tilde{t}_2} \subseteq \ldots \subseteq [\tilde{B}]^{\tilde{k}}_{\tilde{t}_n}$$

Therefore $\tilde{A}$ and $\tilde{B}$ are non-equivalent and clearly $\tilde{F} = \tilde{A} \cup \tilde{B}$. This completes the proof.

IMPLICATION-BASED INTERVAL VALUED FUZZY GENERALIZED BI-IDEALS

Fuzzy logic is an extension of set theoretic multivalued logic in which the truth values are linguistic variables or terms of the linguistic variable truth. Some operators, for example $\vee, \wedge, \neg, \rightarrow$ in fuzzy logic are also defined by using truth tables and the extension principle can be applied to derive definitions of the operators. In fuzzy logic, the truth value of fuzzy proposition Φ is denoted by $[\Phi]$. For a universe U of discourse, we display the fuzzy logical and corresponding set-theoretical notations used in this paper

$$[x \in \tilde{F}] = \tilde{F}(x) \quad (4.1)$$

$$[\Phi \wedge \Psi] = \min\{[\Phi],[\Psi]\} \quad (4.2)$$

$$[\Phi \rightarrow \Psi] = \min\{1, 1-[\Phi]+[\Psi]\} \quad (4.3)$$

$$[\forall \Phi(x)] = \inf_{x \in U}[\Phi(x)] \quad (4.4)$$

$$?\Phi \text{ if and only if } [\Phi] = 1 \text{ for all valuation} \quad (4.5)$$

The truth valuation rules given in (4.3) are those in the Lukasiewicz system of continuous-valued logic. Of course, various implication operators have been defined. We show only a selection of them in the following.

(a) Gaines-Rescher implication operator (I_{GR}):

$$I_{GR}(a,b) = \begin{cases} 1, & \text{if } a \le b \\ 0, & \text{otherwise} \end{cases}$$

(b) Godel implication operator (I_G):

$$I_G(a,b) = \begin{cases} 1, & \text{if } a \le b \\ b, & \text{otherwise} \end{cases}$$

(c) The contraposition of Godel implication operator (I_{cG}):

$$I_{cG}(a,b) = \begin{cases} 1, & \text{if } a \le b \\ 1-a, & \text{otherwise.} \end{cases}$$

Ying [22] introduced the concept of fuzzifying topology. We can expand his/her idea to ordered semigroups and we define an interval valued fuzzifying bi-ideal as follows.

Definition: An interval valued fuzzy subset $\tilde{F}$ of S is called an interval valued fuzzifying generalized bi-ideal of S if it satisfies the following conditions:

(d1) $(\forall x,y \in S)(x \le y \Rightarrow (?[y \in \hat{F}] \rightarrow [x \in \hat{F}]))$

(d2) $(\forall x,y,z \in S)(?\,\mathrm{rmin}\{[x \in \tilde{F}],[z \in \tilde{F}]\} \rightarrow [xyz \in \tilde{F}])$

Obviously, conditions (d1) and (d2) are equivalent to (b4) and (b5), respectively. Therefore an interval valued fuzzifying generalized bi-ideal is an ordinary interval valued fuzzy bi-ideal. In [20], the concept of t-tautology is introduced, i.e.,

$$?\Phi \text{ if and only if } [\Phi] \ge t \text{ for all valuations} \quad (4.6)$$

Definition: An interval valued fuzzy subset $\tilde{F}$ of S and $\tilde{t} \in D(0,1]$ is called a $\tilde{t}$-implication-based generalized bi-ideal of S if it satisfies:

(d4) $(\forall x,y \in S)(x \le y \rightarrow ?_t([y \in \tilde{F}] \rightarrow [x \in \tilde{F}]))$
(d5) $(\forall x,y,z \in S)(?_t\,\mathrm{rmin}\{[x \in \tilde{F}],[z \in \tilde{F}]\} \rightarrow [xyz \in \tilde{F}])$

Let I be an implication operator. Clearly, $\tilde{F}$ is a $\tilde{t}$-implication-based interval valued fuzzy generalized bi-ideal of S if and only if it satisfies:

(d7) $(\forall x,y \in S)\ x \le y \Rightarrow I(\tilde{F}(x),\tilde{F}(y)) \ge \tilde{t}$

(d8) $(\forall x,y \in S)\ I(\mathrm{rmin}\{\tilde{F}(x),\tilde{F}(z),\tilde{F}(xyz)\}) \ge \tilde{t}$

Theorem: For any interval valued fuzzy subset $\tilde{F}$ of S, we have,

(1) If $I = I_{GR}$ then $\tilde{F}$ is a [0.5, 0.5]-implication-based interval valued fuzzy generalized bi-ideal of S if and only if $\tilde{F}$ is an interval valued fuzzy generalized bi-ideal of S.
(2) If $I = I_G$, then $\tilde{F}$ is a $[\frac{1-k^-}{2},\frac{1-k^+}{2}]$-implication-based interval valued fuzzy generalized bi-ideal of S if and only if $\tilde{F}$ is an interval valued $(\in,\in \vee q_{\tilde{k}})$-fuzzy generalized bi-ideal of S.
(3) If $I = I_{GR}$, then $\tilde{F}$ is a $[\frac{1-k^-}{2},\frac{1-k^+}{2}]$-implication-based interval valued generalized fuzzy bi-ideal of S if and only if $\tilde{F}$ satisfies the following conditions:

(3.1) $x \leq y \Rightarrow \text{rmax}\{\tilde{F}(x),[\frac{1-k^-}{2},\frac{1-k^+}{2}]\} \geq \text{rmin}\{\tilde{F}(y),[1,1]\}$

(3.2) $\text{rmax}\{\tilde{F}(xyz),[\frac{1-k^-}{2},\frac{1-k^+}{2}]\} \geq \text{rmin}\{\tilde{F}(x),\tilde{F}(z),[1,1]\}$

for all $x,y,z\in S$

Proof

(1) Straightforward.

(2) Assume that $\tilde{F}$ is a $[\frac{1-k^-}{2},\frac{1-k^+}{2}]$-implication-based interval valued fuzzy generalized bi-ideal of S. Then

(i) $(\forall x,y\in S)\ \ x \leq y \Rightarrow I_G(\tilde{F}(x),\tilde{F}(y)) \geq [\frac{1-k^-}{2},\frac{1-k^+}{2}]$

(ii) $(\forall x,y,z\in S)\ \ I_G(\text{rmin}\{\tilde{F}(x),\tilde{F}(z),\tilde{F}(xyz)\}) \geq [\frac{1-k^-}{2},\frac{1-k^+}{2}]$

Let $x,y\in S$ be such that $x\leq y$. Using (i) we have $\tilde{F}(y) \geq \tilde{F}(x)$ or $\tilde{F}(x) > \tilde{F}(y) \geq [\frac{1-k^-}{2},\frac{1-k^+}{2}]$. Hence then

$$\tilde{F}(x) \geq \text{rmin}\{\tilde{F}(y),[\tfrac{1-k^-}{2},\tfrac{1-k^+}{2}]\}$$

Case (ii) implies that

$$\tilde{F}(xyz) \geq \text{rmin}\{\tilde{F}(x),\tilde{F}(z)\}$$

or

$$\text{rmin}\{\tilde{F}(x),\tilde{F}(z)\} > \tilde{F}(xyz) \geq [\tfrac{1-k^-}{2},\tfrac{1-k^+}{2}]$$

Thus

$$\tilde{F}(xyz) \geq \text{rmin}\{\tilde{F}(x),\tilde{F}(z),[\tfrac{1-k^-}{2},\tfrac{1-k^+}{2}]\}$$

Using Theorem 3.5, we conclude that $\tilde{F}$ is an interval valued $(\in,\in \vee q_{\tilde{k}})$-fuzzy generalized bi-ideal of S.

Conversely, suppose that $\tilde{F}$ is an interval valued $(\in,\in \vee q_{\tilde{k}})$-fuzzy generalized bi-ideal of S. Let $x,y\in S$ such that $x\leq y$. Using Theorem 3.5 (1), we have

$$I_G(\tilde{F}(x),\tilde{F}(y)) = \begin{cases} [1,1] \geq [\frac{1-k^-}{2},\frac{1-k^+}{2}], \\ \text{if } \text{rmin}\{\tilde{F}(x),[\frac{1-k^-}{2},\frac{1-k^+}{2}]\} = \tilde{F}(x) \\ \\ \tilde{F}(y) \geq [\frac{1-k^-}{2},\frac{1-k^+}{2}], \\ \text{if } \text{rmin}\{\tilde{F}(x),[\frac{1-k^-}{2},\frac{1-k^+}{2}]\} = [\frac{1-k^-}{2},\frac{1-k^+}{2}] \end{cases}$$

From Theorem 3.5 (2), if

$$\text{rmin}\{\tilde{F}(x),\tilde{F}(z),[\tfrac{1-k^-}{2},\tfrac{1-k^+}{2}]\} = \text{rmin}\{\tilde{F}(x),\tilde{F}(z)\}$$

then

$$\tilde{F}(xyz) \geq \text{rmin}\{\tilde{F}(x),\tilde{F}(z)\}$$

and so

$$I_G(\text{rmin}\{\tilde{F}(x),\tilde{F}(z),\tilde{F}(xyz)) = [1,1] \geq [\tfrac{1-k^-}{2},\tfrac{1-k^+}{2}]$$

If

$$\text{rmin}\{\tilde{F}(x),\tilde{F}(z),[\tfrac{1-k^-}{2},\tfrac{1-k^+}{2}]\} = [\tfrac{1-k^-}{2},\tfrac{1-k^+}{2}]$$

then

$$\tilde{F}(xyz) \geq [\tfrac{1-k^-}{2},\tfrac{1-k^+}{2}]$$

and thus

$$I_G(\text{rmin}\{\tilde{F}(x),\tilde{F}(z),\tilde{F}(xyz)) \geq [\tfrac{1-k^-}{2},\tfrac{1-k^+}{2}].$$

Consequently, $\tilde{F}$ is a $[\frac{1-k^-}{2},\frac{1-k^+}{2}]$-implication-based interval valued fuzzy generalized bi-ideal of S.

(3) Suppose that $\tilde{F}$ satisfies (3.1) and (3.2). Let $x,y\in S$ be such that $x\leq y$. In (3.1), if

$$\text{rmin}\{\tilde{F}(y),[1,1]\} = [1,1]$$

then

$$\text{rmax}\{\tilde{F}(x),[\tfrac{1-k^-}{2},\tfrac{1-k^+}{2}]\} = [1,1]$$

and hence $\tilde{F}(x) = [1,1] \geq \tilde{F}(y)$. Therefore

$$I_{cG}(\tilde{F}(x),\tilde{F}(y)) = [1,1] \geq [\tfrac{1-k^-}{2},\tfrac{1-k^+}{2}]$$

If $\tilde{F}(y) < [1,1]$, then

$$\text{rmax}\{\tilde{F}(x),[\tfrac{1-k^-}{2},\tfrac{1-k^+}{2}]\} \geq \tilde{F}(y) \tag{4.7}$$

If $\tilde{F}(x) > [\frac{1-k^-}{2},\frac{1-k^+}{2}]$ in (4.7), then $\tilde{F}(x) \geq \tilde{F}(y)$ and thus

$$I_{cG}(\tilde{F}(x),\tilde{F}(y)) = [1,1] \geq [\tfrac{1-k^-}{2},\tfrac{1-k^+}{2}]$$

If $\tilde{F}(x) \leq [\frac{1-k^-}{2},\frac{1-k^+}{2}]$ in (4.7), then $\tilde{F}(y) \leq [\frac{1-k^-}{2},\frac{1-k^+}{2}]$. Hence

$$I_{cG}(\tilde{F}(x),\tilde{F}(y)) = \begin{cases} [1,1] \geq [\frac{1-k^-}{2},\frac{1-k^+}{2}], & \text{if } \ \tilde{F}(x) \geq \tilde{F}(y) \\ [1,1] - \tilde{F}(y) \geq [\frac{1-k^-}{2},\frac{1-k^+}{2}], & \text{otherwise} \end{cases}$$

In (3.2), if

$$\text{rmin}\{\tilde{F}(x),\tilde{F}(z),[1,1]\} = [1,1]$$

then

$$\text{rmax}\{\tilde{F}(xyz),[\tfrac{1-k^-}{2},\tfrac{1-k^+}{2}]\} = [1,1]$$

and so

$$\tilde{F}(xyz) = [1,1] \geq \text{rmin}\{\tilde{F}(x),\tilde{F}(z)\}$$

Therefore

$$I_{cG}(\text{rmin}\{\tilde{F}(x),\tilde{F}(z)\},\tilde{F}(xyz)) = [1,1] \geq [\tfrac{1-k^-}{2},\tfrac{1-k^+}{2}]$$

If

$$\text{rmin}\{\tilde{F}(x),\tilde{F}(z),[1,1]\} = \text{rmin}\{\tilde{F}(x),\tilde{F}(z)\}$$

then

$$\mathrm{rmax}\{\tilde{F}(xyz),[\tfrac{1-k^-}{2},\tfrac{1-k^+}{2}]\} \ge \mathrm{rmin}\{\tilde{F}(x),\tilde{F}(z)\} \qquad (4.8)$$

Thus, if

$$\mathrm{rmax}\{\tilde{F}(xyz),[\tfrac{1-k^-}{2},\tfrac{1-k^+}{2}]\} = [\tfrac{1-k^-}{2},\tfrac{1-k^+}{2}]$$

in (4.8), then

$$\tilde{F}(xyz) \le [\tfrac{1-k^-}{2},\tfrac{1-k^+}{2}]$$

and

$$\mathrm{rmin}\{\tilde{F}(x),\tilde{F}(z)\} \le [\tfrac{1-k^-}{2},\tfrac{1-k^+}{2}]$$

Therefore

$$I_{cG}(\mathrm{rmin}\{\tilde{F}(x),\tilde{F}(z)\},\tilde{F}(xyz)) = [1,1] \ge [\tfrac{1-k^-}{2},\tfrac{1-k^+}{2}]$$

whenever

$$\tilde{F}(xyz) \ge \mathrm{rmin}\{\tilde{F}(x),\tilde{F}(z)\}$$

and

$$I_{cG}(\mathrm{rmin}\{\tilde{F}(x),\tilde{F}(z)\},\tilde{F}(xyz)) = [1,1] - \mathrm{rmin}\{\tilde{F}(x),\tilde{F}(z)\} \ge [\tfrac{1-k^-}{2},\tfrac{1-k^+}{2}]$$

whenever

$$\tilde{F}(xyz) < \mathrm{rmin}\{\tilde{F}(x),\tilde{F}(z)\}$$

Now, if

$$\mathrm{rmax}\{\tilde{F}(xyz),[\tfrac{1-k^-}{2},\tfrac{1-k^+}{2}]\} = \tilde{F}(xyz)$$

in (4.8), then

$$\tilde{F}(xyz) \ge \mathrm{rmin}\{\tilde{F}(x),\tilde{F}(z)\}$$

and so

$$I_{cG}(\mathrm{rmin}\{\tilde{F}(x),\tilde{F}(z)\},\tilde{F}(xyz)) = [1,1] \ge [\tfrac{1-k^-}{2},\tfrac{1-k^+}{2}]$$

Consequently, $\tilde{F}$ is a $[\tfrac{1-k^-}{2},\tfrac{1-k^+}{2}]$-implication-based interval valued fuzzy generalized bi-ideal of S.

Conversely assume that $\tilde{F}$ is a $[\tfrac{1-k^-}{2},\tfrac{1-k^+}{2}]$-implication-based interval valued fuzzy generalized bi-ideal of S. Then

(i) $(\forall x,y\in S)\quad x\le y \Rightarrow IcG(\tilde{F}(x),\tilde{F}(y)) \ge [\dfrac{1-k^-}{2},\dfrac{1-k^-}{2}]$

(ii) $(\forall x,y,z\in S)$

$$IcG(\mathrm{rmin}\{\tilde{F}(x),\tilde{F}(z)\},\tilde{F}(xyz)) \ge [\frac{1-k^-}{2},\frac{1-k^-}{2}]$$

Let $x,y\in S$ be such that $x\le y$, (iv) implies that

$$IcG(\tilde{F}(x),\tilde{F}(y)\} = [1,1]$$

or

$$[1,1] - \tilde{F}(y) \ge [\frac{1-k^-}{2},\frac{1-k^-}{2}]$$

so that $\tilde{F}(y) \ge \tilde{F}(x)$ or

$$\tilde{F}(y) \le [\frac{1-k^-}{2},\frac{1-k^-}{2}]$$

Therefore

$$\mathrm{rmax}\{\tilde{F}(x),[\tfrac{1-k^-}{2},\tfrac{1-k^+}{2}]\} \ge \tilde{F}(y) = \mathrm{rmin}\{\tilde{F}(y),[1,1]\}$$

Finally, from (v), we have

$$IcG(\mathrm{rmin}\{\tilde{F}(x),\tilde{F}(z)\},\tilde{F}(xyz) = [1,1]$$

i.e.,

$$\mathrm{rmin}\{\tilde{F}(x),\tilde{F}(z)\} \le \tilde{F}(xyz)$$

or

$$[1,1] - \mathrm{rmin}\{\tilde{F}(x),\tilde{F}(z)\} \ge [\frac{1-k^-}{2},\frac{1-k^-}{2}]$$

Hence

$$\mathrm{rmax}\{\tilde{F}(z),[\tfrac{1-k^-}{2},\tfrac{1-k^+}{2}]\} \ge \mathrm{rmin}\{\tilde{F}(x),\tilde{F}(z)\} = \mathrm{rmin}\{\tilde{F}(x),\tilde{F}(z),[1,1]\}$$

for all $x,y,z\in S$. This completes the proof.

Corollary: If $I = I_G$, then any interval valued fuzzy subset $\tilde{F}$ of S is a [0.5, 0.5]-implication-based interval valued fuzzy generalized bi-ideal of S if and only if $\tilde{F}$ is an interval valued $(\in,\in\vee q)$-fuzzy generalized bi-ideal of S.

Corollary: If $I = I_{cG}$, then any interval valued fuzzy subset $\tilde{F}$ of S is a [0.5, 0.5]-implication-based interval valued generalized fuzzy bi-ideal of S if and only if $\tilde{F}$ satisfies the following condition:

(3.3) $x\le y \Rightarrow \mathrm{rmax}\{\tilde{F}(x),[0.5,0.5]\} \ge \mathrm{rmin}\{\tilde{F}(y),[1,1]\}$

(3.4) $\mathrm{rmax}\{\tilde{F}(xyz),[0.5,0.5]\} \ge \mathrm{rmin}\{\tilde{F}(x),\tilde{F}(z),[1,1]\}$ for all $x,y,z\in S$.

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