More General Forms of Interval Valued Fuzzy Filters of Ordered Semigroups

Bijan Davvaz, Asghar Khan, Nor Haniza Sarmin, and Hidayatullah Khan

Abstract

In mathematics, an ordered semigroup is a semigroup together with a partial order that is compatible with the semigroup operation. Ordered semigroups have many applications in the theory of sequential machines, formal languages, computer arithmetics, and error-correcting codes. In this article, we try to obtain a more general form than interval valued $(\in, \in \lor q)$ -fuzzy left (right) filters in ordered semigroups. The notion of an interval valued $(\in, \in \lor q_{\tilde{\iota}})$ -fuzzy left (right) filter is introduced, and several properties are investigated. Characterizations of an interval valued $(\in, \in \lor q_{\tilde{i}})$ -fuzzy left (right) filter are established. A condition for an interval valued $(\in, \in \lor q_{\tilde{\iota}})$ -fuzzy left (right) filter to be an interval valued fuzzy left (right) filter is provided. Using implication operators and the notion of implication-based interval valued fuzzy left (resp. right) filters, characterizations of an interval valued fuzzy left (resp. right) filter and an interval valued $(\in, \in \lor q_{\tilde{L}})$ -fuzzy left (resp. right) filter are considered.

Keywords: semigroup; filter; ordered semigroup; fuzzy set; interval valued fuzzy set.

1. Introduction

Interval-valued fuzzy sets were proposed as a natural extension of fuzzy sets. Interval valued fuzzy sets were introduced independently by Zadeh [29], Grattan-Guiness [9], John [11], in the same year, where the value of the membership functions are intervals of numbers instead of the numbers. The fundamental concept of a

fuzzy set, introduced by Zadeh [30], provides a natural frame-work for generalizing several basic notions of algebra. The study of fuzzy sets in semigroups was introduced by Kuroki [18, 19, 20]. Also see [1, 4, 8, 10, 15, 24, 26]. A systematic exposition of fuzzy semigroups was given by Mordeson et al. [21], where one can find theoretical results on fuzzy semigroups and their use in fuzzy coding, fuzzy finite state machines and fuzzy languages. The monograph by Mordeson and Malik [22] deals with the application of fuzzy approach to the concepts of automata and formal languages. Murali [23] proposed the definition of a fuzzy point belonging to a fuzzy subset under a natural equivalence on fuzzy subset. The idea of quasi-coincidence of a fuzzy point with a fuzzy set, played a vital role to generate some different types of fuzzy subgroups. Bhakat and Das [2, 3] gave the concepts of (α, β) -fuzzy subgroups by using the belongs to relation (\in) and quasi-coincident with relation (q) between a fuzzy point and a fuzzy subgroup, and introduced the concept of an $(\in, \in \lor q)$ -fuzzy subgroup. Many researchers used the idea of generalized fuzzy sets and gave several results in different branches of algebras. In [12], Jun and Song initiated the study of (α,β) -fuzzy interior ideals of a semigroup. In [14], Kazanci and Yamak studied $(\in, \in \lor q)$ -fuzzy bi-ideals of a semigroup. In [7], Davvaz and Mozafar studied the notion of $(\in, \in \lor q)$ -fuzzy Lie subalgebras and ideals. Also see [6, 27, 31]. Shabir et al. [25], studied characterization of regular semigroups by (α, β) -fuzzy ideals. Jun et. al [13], discussed a generalization of an $(\in, \in \lor q)$ -fuzzy ideals of a *BCK/BCI* -algebra. In mathematics, an ordered semigroup is a semigroup together with a partial order that is compatible with the semigroup operation. Ordered semigroups have many applications in the theory of sequential machines, formal languages, computer arithmetics, design of fast adders and error-correcting codes. The concept of a fuzzy filter in ordered semigroups was first introduced by Kehayopulu and Tsingelis in [13], where some basic properties of fuzzy filters and prime fuzzy ideals were discussed. A theory of interval valued fuzzy generalized sets on ordered semigroups can be developed. Using the idea of a quasi-coincidence of an interval valued fuzzy point with an interval valued fuzzy set, Khan et al. [16]

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introduced the concept of an interval valued (α, β) -fuzzy bi-ideals in an ordered semigroup. They introduced a new sort of interval valued fuzzy bi-ideals, called interval valued (α, β) -fuzzy bi-ideals, and studied interval valued $(\in, \in \lor q)$ -fuzzy bi-ideals. They provided different characterizations of bi-ideals of ordered semigroups in terms of $(\in, \in \lor q)$ -fuzzy filters, and extended their study in $(\in, \in \lor q)$ -fuzzy left (right) and $(\in, \in \lor q)$ -fuzzy bi-filters and investigated different characterizations of left (right) and bi-filters of ordered semigroups in terms of $(\in, \in \lor q)$ -fuzzy left (right) filters and $(\in, \in \lor q)$ -fuzzy bi-filters. They also introduced the concepts of fuzzy left (right and bi)-filters with thresholds. In this paper, we try to have more general form of an interval valued $(\in, \in \lor q)$ -fuzzy left (right) filter of an ordered semigroup. We introduced the notion of an interval valued $(\in, \in \lor q_{\tilde{k}})$ -fuzzy left (right) filter of an ordered semigroup, and gave examples which are interval valued $(\in, \in \lor q_{\tilde{k}})$ -fuzzy left (right) filter but not interval valued $(\in, \in \lor q)$ -fuzzy left (right) filter. We discuss characterizations of interval valued $(\in, \in \lor q_{\tilde{i}})$ -fuzzy left (right) filters in ordered semigroups. We provided a condition for an interval valued $(\in, \in \lor q_{\tilde{k}})$ -fuzzy left (right) filter to be an interval valued fuzzy left (right) filter. We finally considered characterizations of an interval valued fuzzy left (resp. right) filter and an interval valued $(\in, \in \lor q_{\tilde{k}})$ -fuzzy left (resp. right) filter by using implication operators and the notion of implication-based interval valued fuzzy left (resp. right) filters. The important achievement of the study with an interval valued $(\in, \in \lor q_{\tilde{k}})$ -fuzzy left (resp. right) filter is that the notion of an interval valued $(\in, \in \lor q)$ -fuzzy left (resp. right) filter is a special case of an interval valued $(\in, \in \lor q_i)$ -fuzzy left (resp. right) filter, and thus several results in the paper [17] are corollaries of our results obtained in this paper.

2. Preliminaries

By an *ordered semigroup* (or *po-semigroup*) we mean a structure (S, \cdot, \leq) in which the following are satisfied:

- (OS1) (S,\cdot) is a semigroup,
- (OS2) (S, \leq) is a poset,
- $(OS3) \quad (\forall x, a, b \in S) \quad (a \le b \Longrightarrow a \cdot x \le b \cdot x, x \cdot a \le x \cdot b).$

Remark 1 [5]: We know in the concepts of S-language and S-automation over a finite non-empty set X, we

need an ordered semigroup. Indeed, let X be a non-empty finite set and X^* be the free semigroup generated by X with identity Λ . Let (S, \cdot, \leq) be an ordered semigroup. A function f from X^* into S is called an S-language over X. An S-automation over X is a 4-tuple A = (R, p, h, g), where R is a finite non-empty set, p is a function from $R \times X \times R$ into S, and h and g are functions from R into S. Therefore, the results obtained in this paper are fundamental for the advanced study on fuzzy ordered semigroups and fuzzy automata.

In what follows, $x \cdot y$ is simply denoted by xy for all $x, y \in S$.

A nonempty subset A of an ordered semigroup S is called a *subsemigroup* of S if $A^2 \subseteq A$. A non-empty subset F of an ordered semigroup S is called a *left* (resp. *right*) *filter* of S if it satisfies

- (b1) $(\forall a \in S) (\forall b \in F) (a \le b \Longrightarrow b \in F),$
- $(b2) \quad (\forall a, b \in S) \quad (a, b \in F \Longrightarrow ab \in F),$
- (b3) $(\forall a, b \in S)$ $(ab \in F \Rightarrow a \in F(\operatorname{resp} b \in F))$

If F is both a left filter and a right filter of S, we say that F is a *filter* of S.

A fuzzy subset μ of an ordered semigroup S is called a fuzzy left (resp. right) filter [4] of S if it satisfies:

- $(b4)(\forall x, y \in S) \ (x \le y \Longrightarrow \mu(x) \le \mu(y)),$
- $(b5)(\forall x, y \in S) \quad (\mu(xy) \ge \min\{\mu(x), \mu(y)\}),$
- $(b6)(\forall x, y \in S) \quad (\mu(x)(\operatorname{resp},\mu(y)) \ge \mu(xy))$

By an *interval number* \tilde{a} we mean an interval $[a^-, a^+]$ where $0 \le a^- \le a^+ \le 1$. The set of all interval nubers is denoted by D[0,1]. The interval [a,a] can be simply identified by the number $a \in [0,1]$. For the interval numbers $\tilde{a}_i = [a_i^-, a_i^+]$, $\tilde{b}_i = [b_i^-, b_i^+] \in D[0,1]$, $i \in I$, we define for every $i \in I$ ($\max\{\tilde{a}_i, \tilde{b}_i\} = [\max(a_i^-, b_i^-), \max(a_i^+, b_i^+)]$), ($\min\{\tilde{a}_i, \tilde{b}_i\} = [\min(a_i^-, b_i^-), \min(a_i^+, b_i^+)]$),

$$\operatorname{rinf}_{\tilde{a}_{i}} = \left[\bigwedge_{i \in I} a_{i}^{-}, \bigwedge_{i \in I} a_{i}^{+} \right],$$
$$\operatorname{rsup}_{\tilde{a}_{i}} = \left[\bigvee_{i \in I} a_{i}^{-}, \bigvee_{i \in I} a_{i}^{+} \right]$$

and put

- $\tilde{a}_1 \leq \tilde{a}_2 \Leftrightarrow a_1^- \leq a_2^-$ and $a_1^+ \leq a_2^+$,
- $\tilde{a}_1 = \tilde{a}_2 \Leftrightarrow a_1^- = a_2^-$ and $a_1^+ = a_2^+$,
- $\tilde{a}_1 < \tilde{a}_2 \Leftrightarrow \tilde{a}_1 \le \tilde{a}_2$ and $\tilde{a}_1 \ne \tilde{a}_2$,
- $k\tilde{a}_i = [ka_i^-, ka_i^+]$, whenever $0 \le k \le 1$.

Then, it is clear that $(D[0,1],\leq,\vee,\wedge)$ forms a complete lattice with $\tilde{0} = [0,0]$ as its least element and $\tilde{1} = [1,1]$ as its greatest element.

The interval valued fuzzy subsets provide a more adequate description of uncertainty than the traditional fuzzy subsets; it is therefore important to use interval valued fuzzy subsets in applications. One of the main applications of fuzzy subsets is fuzzy control, and one of the most computationally intensive part of fuzzy control is the "defuzzification". Since a transition to interval valued fuzzy subsets usually increase the amount of computations, it is vitally important to design faster algorithms for the corresponding defuzzification.

An interval valued fuzzy subset $\tilde{F}: X \to D[0,1]$ of X is the set

$$\tilde{F} = \{x \in X \mid (x, [F^{-}(x), F^{+}(x)]) \in D[0, 1]\},\$$

where F^- and F^+ are two fuzzy subset such that $F^-(x) \le F^+(x)$ for all $x \in X$. Let \tilde{F} be an interval valued fuzzy subset of X. Then, for every $[0,0] < \tilde{t} \le [1,1]$, the crisp set

$$U(\tilde{F};\tilde{t}) = \{x \in X \mid \tilde{F}(x) \ge \tilde{t}\}$$

is called the *level set* of \tilde{F}

Note that since every $a \in [0,1]$ is in correspondence with the interval $[a,a] \in D[0,1]$, hence a fuzzy set is a particular case of the interval valued fuzzy sets.

For any $\tilde{F} = [F^-, F^+]$ and $\tilde{t} = [t^-, t^+]$, we define $\tilde{F}(x) + \tilde{t} = [F^-(x) + t^-, F^+(x) + t^+]$, for all $x \in X$. In particular, if $F^-(x) + t^- > 1$ and $F^+(x) + t^+ > 1$, we write $\tilde{F}(x) + \tilde{t} > [1,1]$.

An interval valued fuzzy subset \tilde{F} of a set S of the form

$$\tilde{F}(y) := \begin{cases} \tilde{t} \in D(0,1] & \text{if } y = x, \\ [0,0] & \text{if } y \neq x, \end{cases}$$

is called an *interval valued fuzzy point* with support x and value \tilde{t} and is denoted by $x_{\tilde{t}}$.

For an interval valued fuzzy subset \tilde{F} of a set S, we say that an interval valued fuzzy point $x_{\tilde{i}}$ is

(b7) contained in \tilde{F} denoted by $x_{\tilde{t}} \in \tilde{F}$, if $\tilde{F}(x) \ge \tilde{t}$. (b8) quasi-coincident with \tilde{F} denoted by $x_{\tilde{t}} \in \tilde{F}$ if $\tilde{F}(x) + \tilde{t} > [1,1]$.

For an interval valued fuzzy point x_i and an interval valued fuzzy subset \tilde{F} of a set S, we say that (b9) $x_i \in \bigvee q \tilde{F}$ if $x_i \in \tilde{F}$ or $x_i q \tilde{F}$

(b10) $x_{\tilde{i}} \alpha \tilde{F}$ if $x_{i} \alpha \tilde{F}$ does not hold for $\alpha \in \{ \in, q, \in \lor q \}$.

3. Generalizations of interval valued $(\in, \in \lor q)$ -fuzzy filters

In what follows, let *S* be an ordered semigroup and let $\tilde{k} = [k^-, k^+]$ denote an arbitrary element of D[0,1)unless otherwise specified. For an interval valued fuzzy point x and an interval valued fuzzy subset \tilde{F} of *S*,

we say that

(c1)
$$x_{\tilde{t}} = q_{\tilde{k}}F$$
 if $F(x)+t+k > [1,1]$, where
 $F^{-}+t^{-}+k^{-} > 1$ and $F^{+}+t^{+}+k^{+} > 1$.
(c2) $x_{\tilde{t}} \in \lor q_{\tilde{k}}\tilde{F}$ if $x_{\tilde{t}} \in \tilde{F}$ or $x_{\tilde{t}}q_{\tilde{k}}\tilde{F}$
(c3) $x_{\tilde{t}}\overline{\alpha}\tilde{F}$ if $x_{\tilde{t}}\alpha\tilde{F}$ does not hold for
 $\alpha \in \{q_{\tilde{t}}, \in \lor q_{\tilde{t}}\}$.

3.1 Theorem: Let \tilde{F} be an interval valued fuzzy subset of S. Then, the following are equivalent:

(1)
$$\left(\forall \tilde{t} \in D\left(\frac{1-k}{2}, 1\right]\right) \left(U\left(\tilde{F}; \tilde{t}\right) \neq \emptyset \Rightarrow U\left(\tilde{F}; \tilde{t}\right) \right)$$

is a left (right) filter of S

(2) \tilde{F} satisfies the following assertions:

$$(2.1)(\forall x, y \in S) \begin{cases} x \leq y \Rightarrow \tilde{F}(x) \\ \leq r \max\left\{\tilde{F}(y), \left[\frac{1-k^{-}}{2}, \frac{1-k^{+}}{2}\right]\right\} \end{cases}, \\ (2.2) \quad (\forall x, y \in S) \begin{cases} r \min\left\{\tilde{F}(x), \tilde{F}(y)\right\} \\ \leq r \max\left\{\frac{\tilde{F}(xy), \\ \left[\frac{1-k^{-}}{2}, \frac{1-k^{+}}{2}\right]\right\} \end{cases}, \\ (2.3)(\forall x, y \in S)(\tilde{F}(xy) \leq \left[r \max\left\{\frac{\tilde{F}(x), \\ \left[\frac{1-k^{-}}{2}, \frac{1-k^{+}}{2}\right]\right\}\right], \\ (\operatorname{resp.} r \max\left\{\tilde{F}(y), \left[\frac{1-k^{-}}{2}, \frac{1-k^{+}}{2}\right]\right\} \right]. \end{cases}$$

Proof: Assume that $U(\tilde{F};\tilde{t})$ is a filter of *S* for all $\tilde{t} \in D\left(\frac{1-k}{2},1\right]$ with $U(\tilde{F};\tilde{t}) \neq \emptyset$. If there exist $a,b \in S$ such that the condition (2.1) is not valid, that is, there exist $a,b \in S$ with $a \leq b$ such that

$$\tilde{F}(a) > r \max\left\{\tilde{F}(b), \left[\frac{1-k^{-}}{2}, \frac{1-k^{+}}{2}\right]\right\}.$$

Then, $\tilde{F}(a) \in D(\frac{1-k}{2}, 1]$ and $a \in U(\tilde{F}; \tilde{F}(a))$. But $\tilde{F}(y) < \tilde{F}(a)$ implies that $b \notin U(\tilde{F}; \tilde{F}(a))$, a

contradiction. Hence, (2.1) is valid. Suppose that (2.2) is false, that is

$$\tilde{s} := r \min\left\{\tilde{F}(a)\tilde{F}(c)\right\}$$

> $r \max\left\{\tilde{F}(ac), \left[\frac{1-k^{-}}{2}, \frac{1-k^{+}}{2}\right]\right\}$

for some a, c. Then, $\tilde{s} \in D\left(\frac{1-k}{2}, 1\right)$ and $a, c \in U(\tilde{L}, \tilde{s})$. But $c \in U(\tilde{L}, \tilde{s})$ gives $\tilde{L}(c)$. That is a

 $U(\tilde{F};\tilde{s})$. But $ac \notin U(\tilde{F};\tilde{s})$ since $\tilde{F}(ac) < \tilde{s}$. That is a contradiction, and so (2.2) holds. Assume that there exist $a, b \in S$ such that

$$\tilde{F}(ab) > r \max\left\{\tilde{F}(a), \left[\frac{1-k^{-}}{2}, \frac{1-k^{+}}{2}\right]\right\}$$

Then, $\tilde{F}(ab) \in D\left(\frac{1-k}{2}, 1\right)$ and $ab \in U\left(\tilde{F}; \tilde{F}(ab)\right)$. But $a \notin U\left(\tilde{F}; \tilde{F}(ab)\right)$, which is impossible. Therefore, $\tilde{F}(ab) \leq r \max\left\{\tilde{F}(a), \left[\frac{1-k^{-}}{2}, \frac{1-k^{+}}{2}\right]\right\}$ for all $a, b \in S$.

Conversely, assume that \tilde{F} satisfies the three conditions (2.1), (2.2) and (2.3). Suppose that $U(\tilde{F};\tilde{t}) \neq \emptyset$ for all $\tilde{t} \in D\left(\frac{1-k}{2},1\right]$. Let $x, y \in S$ be such that $x \leq y$ and $x \in U(\tilde{F};\tilde{t})$. Then, $\tilde{F}(x) \geq \tilde{t}$ and so

$$\max\left\{\tilde{F}(y), \left\lfloor \frac{1-k^{-}}{2}, \frac{1-k^{+}}{2} \right\rfloor\right\}$$
$$\geq \tilde{F}(x) \geq \tilde{t} > \left\lfloor \frac{1-k^{-}}{2}, \frac{1-k^{+}}{2} \right\rfloor.$$

Hence, $\tilde{F}(y) \ge \tilde{t}$, that is $y \in U(\tilde{F}; \tilde{t})$. $x, y \in U(\tilde{F}; \tilde{t})$, it follows from (2.2) that

$$r \max\left\{\tilde{F}(xy), \left[\frac{1-k^{-}}{2}, \frac{1-k^{+}}{2}\right]\right\} \ge$$
$$r \min\left\{\tilde{F}(x), \tilde{F}(y)\right\} \ge \tilde{t} > \left[\frac{1-k^{-}}{2}, \frac{1-k^{+}}{2}\right]$$

so that $\tilde{F}(xy) \ge \tilde{t}$ i.e., $xy \in U(\tilde{F}; \tilde{t})$. Let $x, y \in S$ be such that $xy \in U(\tilde{F}; \tilde{t})$. Then, $\tilde{F}(xy) \ge \tilde{t}$ and thus

$$r \max\left\{\tilde{F}(x), \left[\frac{1-k^{-}}{2}, \frac{1-k^{+}}{2}\right]\right\}$$
$$\geq \tilde{F}(xy) \geq \tilde{t} > \left[\frac{1-k^{-}}{2}, \frac{1-k^{+}}{2}\right].$$

Hence $\tilde{F}(x) \ge \tilde{t}$ i.e., $x \in U(\tilde{F}; \tilde{t})$. Therefore, $U(\tilde{F}; \tilde{t})$

is a filter of *S* for all $\tilde{t} \in D\left(\frac{1-k}{2}, 1\right]$ with $U\left(\tilde{F}; \tilde{t}\right) \neq \emptyset$. Similarly, we can obtain the desired result

for the right case. If we take $\tilde{k} = [0,0]$ in Theorem 3.1, then we have the following corollary.

3.2 Corollary [14, Theorem 3.3]: Let \tilde{F} be an interval valued fuzzy subset of S. Then, the following are equivalent:

(1)
$$(\forall \tilde{t} \in D(0.5,1]) \begin{pmatrix} U(\tilde{F};\tilde{t}) \neq \emptyset \\ \Rightarrow U(\tilde{F};\tilde{t}) \text{ is a left (right) filter of } S \end{pmatrix}$$
.

(2) \tilde{E} satisfies the following assertions:

(2.1)
$$(\forall x, y \in S) (x \le y \Rightarrow \tilde{F}(x) \le r \max{\{\tilde{F}(y), [0.5, 0.5]\}})$$

(2.2) $(\forall x, y \in S)$
 $(r \min{\{\tilde{F}(x), \tilde{F}(y)\}} \le r \max{\{\tilde{F}(xy), [0.5, 0.5]\}}),$
(2.3) $(\forall x, y \in S) \tilde{F}(xy) \le (r \max{\{\tilde{F}(x), [0.5, 0.5]\}})$
 $(\text{resp.} r \max{\{\tilde{F}(y), [0.5, 0.5]\}}).$

3.3 Definition: An interval valued fuzzy subset \tilde{F} of S is called an $(\in, \in \lor q_{\tilde{k}})$ -fuzzy left (resp. right) filter of S if it satisfies the following conditions:

(c4)
$$(x \le y, x_{\tilde{t}} \in F \Rightarrow y_{\tilde{t}} \in \lor q_{\tilde{k}}F),$$

(c5) $\begin{pmatrix} x_{\tilde{t}_{1}}, \in \tilde{F}, y_{\tilde{t}_{2}} \in \tilde{F} \\ \Rightarrow (xy)_{r\min(\tilde{t}_{1}, \tilde{t}_{2})} \in \lor q_{\tilde{k}}\tilde{F} \end{pmatrix},$
(c6) $((xy)_{\tilde{t}} \in \tilde{F} \Rightarrow x_{\tilde{t}} \in \lor q_{\tilde{k}}\tilde{F}, (\text{resp. } y_{\tilde{t}} \in \lor q_{\tilde{k}}\tilde{F})$
for all $x, y \in S$ and $\tilde{t}, \tilde{t}, \tilde{t}, \in D(0, 1].$

An $(\in, \in \lor q_{\tilde{k}})$ -fuzzy left (resp. right) filter of *S* with If $\tilde{k} = [0,0]$ is called an $(\in, \in \lor q)$ -fuzzy left (resp. right) filter of *S*.

3.4 Example: Consider the ordered semigroup $S = \{a, b, c, d, e, f\}$ with the multiplication given in Table 1 and order relation " \leq ":

Table 1. Multiplication table for S.

	•	а	b	С	d	е	f	
	а	а	b	b	d	e	f	
	b	b	b	b	b	b	b	
	С	b	b	b	b	b	b	
	d	d	b	b	d	е	f	
	е	е	f	f	е	е	f	
	f	f	f	f	f	f	f	
$\leq := \{(a,a,), \overline{(b,b)}, (c,c), (d,d), (e,e), (f,f), (a,d), (a,d), (e,e), (f,f), (a,d), (a,d$								
$(a, e), (d, e), (b, f), (c, f), (c, e), (f, e)\}.$								

Let \tilde{F} be an interval valued fuzzy subset defined by

$$\tilde{F}(x) = \begin{cases} 0.9, 0.99] & \text{if } x = a \\ 0.2, 0.3] & \text{if } x \in \{b, c, f\} \\ 0.8, 0.9] & \text{if } x = d \\ 0.3, 0.4] & \text{if } x = e. \end{cases}$$

Then, \tilde{F} is an interval valued $(\in, \in \lor q_{[0,4,0,5]})$ -fuzzy filter of S.

3.5 Theorem: An interval valued fuzzy subset \tilde{F} of S is an interval valued $(\in, \in \lor q_{\tilde{k}})$ -fuzzy left (resp. right) filter of *S* if and only if:

$$(1)_{\left(\forall x, y \in S\right)} \left[\tilde{F}(y) \ge r \min \begin{cases} F(x), \\ \left[\frac{1-k^{-}}{2}, \frac{1-k^{+}}{2}\right] \end{cases} \text{ with } x \le y \end{cases}.$$

$$(2)_{\left(\forall x, y \in S\right)} \left[\tilde{F}(xy) \ge r \min \begin{cases} \tilde{F}(x), \tilde{F}(y), \\ \left[\frac{1-k^{-}}{2}, \frac{1-k^{+}}{2}\right] \end{cases} \right].$$

$$(3)_{\left(\forall x, y \in S\right)} \left[\tilde{F}(x)(\text{resp. } \tilde{F}(y)) \ge r \min \begin{cases} \tilde{F}(xy), \\ \left[\frac{1-k^{-}}{2}, \frac{1-k^{+}}{2}\right] \end{cases} \right].$$

Proof: Suppose that \tilde{F} is an interval valued $(\in, \in \lor q_{\tilde{\iota}})$ -fuzzy filter of S. Let $x, y \in S$ such that $x \le y$. If $\tilde{F}(y) < \tilde{F}(x)$, then $\tilde{F}(y) < \tilde{t} \le \tilde{F}(x)$ for some $\tilde{t} \in D\left(0, \frac{1-k}{2}\right)$. It follows that $x_{\tilde{t}} \in \tilde{F}$, but $y_{\tilde{t}} \in \tilde{F}$. Since $\tilde{F}(y) + \tilde{t} < 2\tilde{t} < [1,1] - \tilde{k}$, we get $y_{\tilde{t}}q_{\tilde{k}}\tilde{F}$. Therefore, $y_i \overline{\in \lor q_i} \tilde{F}$, which is a contradiction. Hence, $\tilde{F}(y) \ge \tilde{F}(x)$. Now, if $\tilde{F}(x) \ge \left\lfloor \frac{1-k^-}{2}, \frac{1-k^+}{2} \right\rfloor$, then $x_{\left[\frac{1-k^{-}}{2},\frac{1-k^{+}}{2}\right]} \in \tilde{F}$ and so $y_{\left[\frac{1-k^{-}}{2},\frac{1-k^{+}}{2}\right]} \in \lor q_{\tilde{k}}\tilde{F}$ which that $\tilde{F}(y) \ge \left\lceil \frac{1-k^-}{2}, \frac{1-k^+}{2} \right\rceil$

implies

$$\frac{1-k^+}{2}$$
 or

$$\tilde{F}(y) + \left[\frac{1-k^{-}}{2}, \frac{1-k^{+}}{2}\right] > [1,1] - \tilde{k}.$$
Hence, $\tilde{F}(y) \ge \left[\frac{1-k^{-}}{2}, \frac{1-k^{+}}{2}\right].$ Otherwise,
 $\tilde{F}(y) + \left[\frac{1-k^{-}}{2}, \frac{1-k^{+}}{2}\right] < \left[\frac{1-k^{-}}{2}, \frac{1-k^{+}}{2}\right] + \left[\frac{1-k^{-}}{2}, \frac{1-k^{+}}{2}\right]$

$$= [1,1] - \tilde{k}, \text{ a contradiction.}$$

Consequently, $\tilde{F}(y) \ge r \min \left\{ \begin{bmatrix} \tilde{F}(x), \\ \left\lceil \frac{1-k^{-}}{2}, \frac{1-k^{+}}{2} \right\rceil \right\}$ for all $\tilde{t}_{a} \in D\left(0, \frac{1-k}{2}\right)$ and $(ab)_{\tilde{t}_{a}} \in \tilde{F}$, but $a_{\tilde{t}_{a}} \in \tilde{F}$. Also,

 $r\min\left\{\tilde{F}(x),\tilde{F}(y)\right\} < \left[\frac{1-k^{-}}{2},\frac{1-k^{+}}{2}\right]$. We claim that $\tilde{F}(xy) \ge r \min \{ \tilde{F}(x), \tilde{F}(y) \}$. If not, then $\tilde{F}(xy) < \tilde{t} \le$ $r\min\left\{\tilde{F}(x),\tilde{F}(y)\right\}$ for some $\tilde{t}\in D\left(0,\frac{1-k}{2}\right)$. It follows that $x_{\tilde{i}} \in \tilde{F}$ and $y_{\tilde{i}} \in \tilde{F}$, but $(xy)_{\tilde{i}} \in \tilde{F}$ and i.e., $(xy)_{\tilde{i}} \overline{q_{\tilde{k}}} \tilde{F}$. This is a contradiction. Thus, $\tilde{F}(xy) \ge r \min\{\tilde{F}(x), \tilde{F}(y)\}$ for all $x, y \in S$ with $r\min\left\{\tilde{F}(x),\tilde{F}(y)\right\} < \left[\frac{1-k^{-}}{2},\frac{1-k^{+}}{2}\right]. \text{ If } r\min\left\{\tilde{F}(x),\tilde{F}(y)\right\} \ge 1$ $\left[\frac{1-k^{-}}{2},\frac{1-k^{+}}{2}\right] \quad , \quad \text{then} \quad x_{\left[\frac{1-k^{-}}{2},\frac{1-k^{+}}{2}\right]} \in \tilde{F} \quad \text{and}$ SO $y_{\left[\frac{1-k^{-}}{2},\frac{1-k^{+}}{2}\right]} \in \tilde{F}$. Using (c5), we have $(xy)_{\left[\frac{1-k^{-}}{2},\frac{1-k^{+}}{2}\right]} = xy_{r\min\left\{\left[\frac{1-k^{-}}{2},\frac{1-k^{+}}{2}\right],\left[\frac{1-k^{-}}{2},\frac{1-k^{+}}{2}\right]\right\}} \in \lor q_{\tilde{k}}\tilde{F},$

and so

$$\tilde{F}(xy) \ge \left[\frac{1-k^-}{2}, \frac{1-k^+}{2}\right]$$

or

$$\tilde{F}(xy) + \left[\frac{1-k^{-}}{2}, \frac{1-k^{+}}{2}\right] > [1,1] - \tilde{k}.$$

If $\tilde{F}(xy) < \left[\frac{1-k^{-}}{2}, \frac{1-k^{+}}{2}\right]$, then
 $\tilde{F}(xy) + \left[\frac{1-k^{-}}{2}, \frac{1-k^{+}}{2}\right] < \left[\frac{1-k^{-}}{2}, \frac{1-k^{+}}{2}\right] + \left[\frac{1-k^{-}}{2}, \frac{1-k^{+}}{2}\right]$
 $= [1,1] - \tilde{k},$

which is a contradiction. Hence,
$$\tilde{F}(xy) \ge \left[\frac{1-k^-}{2}, \frac{1-k^+}{2}\right]$$
.
Consequently, $\tilde{F}(xy) \ge r \min\left\{\tilde{F}(x), \tilde{F}(y), \left[\frac{1-k^-}{2}, \frac{1-k^+}{2}\right]\right\}$
for all $x, y \in S$. Assume that there exist $a, b \in S$ such that $\tilde{F}(a) < r \min\left\{\tilde{F}(ab), \left[\frac{1-k^-}{2}, \frac{1-k^+}{2}\right]\right\}$. Taking $\tilde{t}_a := \frac{1}{2} \left[\tilde{F}(a) + r \min\left\{\frac{\tilde{F}(ab),}{\left[\frac{1-k^-}{2}, \frac{1-k^+}{2}\right]\right\}\right]$ implies that

 $x, y \in S$ with $x \leq y$. Let $x, y \in S$ be such that $\tilde{F}(a) + \tilde{t}_a < 2\tilde{t}_a < [1,1] - \tilde{k}$, that is, $a_{t_a} \overline{q_k} \tilde{F}$. Hence,

 $a_{\tilde{t}_{a}} \in \nabla q_{\tilde{k}} \tilde{F}, \quad \text{a contradiction. Therefore,} \quad \tilde{F}(x) \geq \quad \text{iff} \\ r \min\left\{\tilde{F}(xy), \left[\frac{1-k^{-}}{2}, \frac{1-k^{+}}{2}\right]\right\} \quad \text{for all} \quad x, y \in S.$ (2)

Conversely, let \tilde{F} be an interval valued fuzzy subset of *S* that satisfies the three conditions (1), (2) and (3). Let $x, y \in S$ and $\tilde{t} \in D(0,1]$ be such that $x \leq y$ and $x_{\tilde{t}} \in \tilde{F}$. Then, $\tilde{F}(x) \geq \tilde{t}$, and so

$$\tilde{F}(y) \ge r \min\left\{\tilde{F}(x), \left[\frac{1-k^-}{2}, \frac{1-k^+}{2}\right]\right\} \ge r \min\left\{\tilde{t}, \left[\frac{1-k^-}{2}, \frac{1-k^+}{2}\right]\right\}.$$

It follows that $y_{\tilde{t}} \in \tilde{F}$ or $\tilde{F}(y) + \tilde{t} \ge \tilde{F} + \tilde{t} > [1,1] - \tilde{k}$, i.e., $y_{\tilde{t}} q_{\tilde{k}} \tilde{F}$.

Hence, $y_{\tilde{t}} \in \bigvee q_{\tilde{k}} \tilde{F}$. Let $x, y \in S$ and $\tilde{t}_1, \tilde{t}_2 \in D(0,1]$ be such that $x_{\tilde{t}_1} \in \tilde{F}$ and $y_{\tilde{t}_2} \in \tilde{F}$. Then, $\tilde{F}(x) \ge \tilde{t}_1$ and $\tilde{F}(x) \ge \tilde{t}_2$. It follows from (2) that

$$\tilde{F}(xy) \ge r \min\left\{ \begin{bmatrix} \tilde{F}(x), \tilde{F}(y), \\ \left[\frac{1-k^{-}}{2}, \frac{1-k^{+}}{2}\right] \right\} \ge r \min\left\{ \tilde{t}_{1}, \tilde{t}_{2}, \left[\frac{1-k^{-}}{2}, \frac{1-k^{+}}{2}\right] \right\}$$

so that $(xy)_{r\min\{\tilde{t}_1,\tilde{t}_2\}} \in \tilde{F}$ or

$$\widetilde{F}(xy) + r \min\{\widetilde{t}_{1}, \widetilde{t}_{2}\} \ge \left[\frac{1-k^{-}}{2}, \frac{1-k^{+}}{2}\right] + r \min\{\widetilde{t}_{1}, \widetilde{t}_{2}\}$$
$$> \left[\frac{1-k^{-}}{2}, \frac{1-k^{+}}{2}\right] + \left[\frac{1-k^{-}}{2}, \frac{1-k^{+}}{2}\right] = [1,1] - \widetilde{k},$$

that is $(xy)_{r\min\{\tilde{i}_1,\tilde{i}_2\}} q_{\tilde{k}} \tilde{F}$. Therefore,

 $(xy)_{\min\{\tilde{t}_1, \tilde{t}_2\}} \in \bigvee q_{\tilde{k}} \tilde{F}$. Let $x, y \in S$ and $\tilde{t} \in D(0,1]$ be such that $(xy)_{\tilde{t}} \in \tilde{F}$. Then, $\tilde{F}(xy) \ge \tilde{t}$ which implies from (3) that

$$\tilde{F}(x) \ge r \min\left\{\tilde{F}(xy), \left[\frac{1-k^{-}}{2}, \frac{1-k^{+}}{2}\right]\right\} \ge r \min\left\{\tilde{t}, \left[\frac{1-k^{-}}{2}, \frac{1-k^{+}}{2}\right]\right\}$$

Thus, $x_{\tilde{t}} \in \tilde{F}$ or

$$\tilde{F}(x) + \tilde{t} \ge \left[\frac{1-k^{-}}{2}, \frac{1-k^{+}}{2}\right] + \tilde{t} > [1,1] - \tilde{k},$$

i.e., $x_{\tilde{t}} q_{\tilde{k}} \tilde{F}$. Hence, $x_{\tilde{t}} \in \bigvee q_{\tilde{k}} \tilde{F}$. Therefore, \tilde{F} is a left $(\in, \in \lor q_{\tilde{k}})$ -fuzzy filter of *S*. Similarly, we can obtain the desired result for the right case.

If we take $\tilde{k} = [0,0]$ in Theorem 3.5, then we have the following corollary.

3.6 Corollary [14, Theorem 4.3]: An interval valued fuzzy subset \tilde{F} of S is an interval valued $(\in, \in \lor q)$ -fuzzy left (resp. right) filter of S if and only

$$\begin{array}{l} \text{II:}\\ (1) \left(\forall x, y \in S \right) \quad \left(x \leq y \Longrightarrow \tilde{F}(y) \geq r \min \left\{ \tilde{F}(x), [0.5, 0.5] \right\} \right). \\ (2) \left(\forall x, y \in S \right) \quad \left(\tilde{F}(xy) \geq r \min \left\{ \tilde{F}(x), \tilde{F}(y), [0.5, 0.5] \right\} \right). \\ (3) \left(\forall x, y \in S \right) \left(\tilde{F}(x) \left(\text{resp. } \tilde{F}(y) \right) \geq r \min \left\{ \tilde{F}(xy), [0.5, 0.5] \right\} \right). \end{array}$$

Obviously, every fuzzy left (resp. right) filter of *S* is an $(\in, \in \lor q_{\tilde{k}})$ -fuzzy left (resp. right) filter of *S* for some $\tilde{k} \in D(0,1]$. The following example shows that there exists $\tilde{k} \in D(0,1]$ such that

(i) \tilde{F} is an $(\in, \in \lor q_{\tilde{k}})$ -fuzzy left (resp. right) filter of *S*.

(ii) \tilde{F} is not a fuzzy left (resp. right) filter of S.

3.7 *Example:* The interval valued $(\in, \in \lor q_{[0.4,0.5]})$ -fuzzy left (right) filter \tilde{F} of *S* in Example 3.4 is not an interval valued fuzzy left (right) filter of *S* since $a \le e$ and

$$\tilde{F}(a) = [0.9, 0.99] > [0.3, 0.4] = \tilde{F}(e)$$

We give a condition for an interval valued $(\in, \in \lor q_{\tilde{k}})$ -fuzzy left (resp. right) filter to be an interval valued fuzzy left (resp. right) filter.

3.8 *Theorem:* Let
$$\tilde{F}$$
 be an interval valued $(\in, \in \lor q_{\tilde{k}})$ -fuzzy left (resp. right) filter of *S*. If $\tilde{F}(x) < \left[\frac{1-k^{-}}{2}, \frac{1-k^{+}}{2}\right]$ for all $x \in S$, then \tilde{F} is an

interval valued fuzzy left (resp. right) filter of *S*. *Proof:* It is straightforward by Theorem 3.5.

3.9 Corollary [14, Theorem 4.9]: Let \tilde{F} be an interval valued $(\in, \in \lor q_{\tilde{k}})$ -fuzzy left (resp. right) filter of S. If $\tilde{F}(x) < [0.5, 0.5]$ for all $x \in S$. Then, \tilde{F} is an interval valued fuzzy left (resp. right) filter of S.

Proof: It follows from Theorem 3.8 by taking $\tilde{k} = [0,0]$. 3.10 Theorem: Let *S* be an ordered semigroup. If $[0,0] \le \tilde{k} < \tilde{r} < [1,1]$. Then, every interval valued $(\in, \in \lor q_{\tilde{k}})$ -fuzzy left (resp. right) filter is an interval valued $(\in, \in \lor q_{\tilde{k}})$ -fuzzy left (resp. right) filter.

Proof. It is straightforward.

Let $S = \{a, b, c, d, e, f\}$ be an ordered semigroup which is considered in Example 3.4. Let \tilde{F} be an interval valued fuzzy subset of *S* defined by

$$\tilde{F}(x) := \begin{cases} 0.7, 0.8 & \text{if } x = f, \\ 0.4, 0.5 & \text{if } x \in \{a, d, e\}, \\ 0.2, 0.3 & \text{if } x \in \{b, c\}. \end{cases}$$

Then, \tilde{F} is an interval valued $(\in, \in \lor q_{\tilde{k}})$ -fuzzy left

filter of S with $\tilde{k} \ge [0.2, 0.2]$. Note that $f \le e$ and

$$\tilde{F}(e) = [0.4, 0.5] < [0.43, 0.53] = \left[\frac{1-k^{-}}{2}, \frac{1-k^{+}}{2}\right]$$
$$= r \min\left\{\tilde{F}(f), \left[\frac{1-k^{-}}{2}, \frac{1-k^{+}}{2}\right]\right\}$$

where $\tilde{k} = [0.14, -0.6]$. Thus, \tilde{F} does not satisfy the first condition of Theorem 3.5, and so \tilde{F} is not an interval valued $(\in, \in \lor q_{\tilde{\iota}})$ -fuzzy left filter of S for $\tilde{k} = [0.14, -0.6]$. This shows that the converse of Theorem 3.10 is not true.

3.11 Theorem: For an interval valued fuzzy subset \tilde{F} of *S*, the following are equivalent:

(1) \tilde{F} is an interval valued $(\in, \in \lor q_{\tilde{i}})$ -fuzzy left (resp. right) filter of S.

(2)
$$\left(\forall \tilde{t} \in D\left(\frac{1-k}{2},1\right]\right) \left(\forall \tilde{t} \in D\left(\frac{1-k}{2},1\right] \right) \left(\exists v \in V\left(\tilde{F};\tilde{t}\right) \\ \exists v \in V\left(\tilde{F};\tilde{t}\right)$$

Proof: Assume that \tilde{F} is an interval valued $(\in, \in \lor q_k)$ -fuzzy left filter of *S* and let $\tilde{t} \in D\left(0, \frac{1-k}{2}\right)$ be such that $U(\tilde{F};\tilde{t})\neq\emptyset$. Using Theorem 3.5 (1), we have

$$\tilde{F}(y) \ge r \min\left\{\tilde{F}(x), \left[\frac{1-k^{-}}{2}, \frac{1-k^{+}}{2}\right]\right\}$$

for any $x, y \in S$ with $x \leq y$ and $x \in U(\tilde{F}; \tilde{t})$. It follows that

$$\tilde{F}(y) \ge r \min\left\{\tilde{t}, \left[\frac{1-k^-}{2}, \frac{1-k^+}{2}\right]\right\} = \tilde{t}$$

so that $y \in U(\tilde{F}; \tilde{t})$. Let $x, y \in U(\tilde{F}; \tilde{t})$. Then, $\tilde{F}(x) \ge \tilde{t}$ and $\tilde{F}(y) \ge \tilde{t}$. Theorem 3.5(2) induces that

$$\widetilde{F}(xy) \ge r \min\left\{\widetilde{F}(x), \widetilde{F}(y), \left[\frac{1-k^{-}}{2}, \frac{1-k^{+}}{2}\right]\right\}$$
$$\ge r \min\left\{\widetilde{t}, \left[\frac{1-k^{-}}{2}, \frac{1-k^{+}}{2}\right]\right\} = \widetilde{t}.$$

Thus, $xy \in U(\tilde{F}; \tilde{t})$. Now, let $xy \in U(\tilde{F}; \tilde{t})$. Then, $\tilde{F}(xy) \geq \tilde{t}$. Theorem 3.5 (3)

$$\widetilde{F}(x) \ge r \min\left\{\widetilde{F}(xy), \left[\frac{1-k^-}{2}, \frac{1-k^+}{2}\right]\right\}$$
$$\ge r \min\left\{\widetilde{t}, \left[\frac{1-k^-}{2}, \frac{1-k^+}{2}\right]\right\} = \widetilde{t}.$$

Thus, $x \in U(\tilde{F}; \tilde{t})$ and therefore $x \in U(\tilde{F}; \tilde{t})$ for $\tilde{t} \in D\left(0, \frac{1-k}{2}\right)$, is a filter of S.

Conversely, let \tilde{F} be an interval valued fuzzy subset of S such $x \in U(\tilde{F}; \tilde{t})$ is non-empty and is a left filter of S for all $\tilde{t} \in D\left(0, \frac{1-k}{2}\right)$. If there exist $a, b \in S$ with $a \leq b$ and

$$\tilde{F}(b) < r \min\left\{\tilde{F}(a), \left[\frac{1-k^{-}}{2}, \frac{1-k^{+}}{2}\right]\right\}$$

then $\tilde{F}(b) < \tilde{t}_b \le r \min\left\{\tilde{F}(a), \left|\frac{1-k^-}{2}, \frac{1-k^+}{2}\right|\right\}$

for some $\tilde{t}_b \in D\left(0, \frac{1-k}{2}\right)$ and so $b \notin U\left(\tilde{F}; \tilde{t}_b\right)$. This is Therefore, $\tilde{F}(y) \ge r \min$ contradiction. а $\left\{\tilde{F}(x), \left|\frac{1-k^{-}}{2}, \frac{1-k^{+}}{2}\right|\right\} \text{ for all } x, y \in S \text{ with } x \leq y.$

Assume that

$$\tilde{F}(ab) < r \min\left\{\tilde{F}(a), \tilde{F}(b), \left[\frac{1-k^{-}}{2}, \frac{1-k^{+}}{2}\right]\right\}.$$
Then, $\tilde{F}(ab) < \tilde{t} \le r \min\left\{\tilde{F}(a), \tilde{F}(b), \left[\frac{1-k^{-}}{2}, \frac{1-k^{+}}{2}\right]\right\}$
for some $\tilde{t} \in D\left(0, \frac{1-k}{2}\right]$. It follows that $a \in U\left(\tilde{F}; \tilde{t}\right)$
and $b \in U\left(\tilde{F}; \tilde{t}\right)$, but $ab \notin U\left(\tilde{F}; \tilde{t}\right)$. This is impossible,

and thus

$$\tilde{F}(xy) \ge r \min\left\{\tilde{F}(x), \tilde{F}(y), \left[\frac{1-k^{-}}{2}, \frac{1-k^{+}}{2}\right]\right\}$$

for all $x, y \in S$. Suppose that

$$\tilde{F}(a) < r \min\left\{\tilde{F}(ab), \left[\frac{1-k^{-}}{2}, \frac{1-k^{+}}{2}\right]\right\}$$

for some $a, b \in S$. Then, there exists $\tilde{t}_a \in D\left(0, \frac{1-k}{2}\right)$ such that

$$\tilde{F}(a) < \tilde{t}_a \le r \min\left\{\tilde{F}(ab), \left[\frac{1-k^-}{2}, \frac{1-k^+}{2}\right]\right\}$$

Hence, $(ab)_{\tilde{t}_{a}} \in \tilde{F}$ and $a_{\tilde{t}_{a}} \in \tilde{F}$.

Also, $\tilde{F}(a) + \tilde{t}_a < 2\tilde{t}_a \le [1,1] - \tilde{k}$, i.e., $a_{\tilde{t}_a} \overline{q_{\tilde{k}}} \tilde{F}$. Thus, $a_{\tilde{t}_a} \overline{\in \lor q_{\tilde{k}}} \tilde{F}$, a contradiction. Therefore,

$$\tilde{F}(x) \ge r \min\left\{\tilde{F}(xy), \left[\frac{1-k^{-}}{2}, \frac{1-k^{+}}{2}\right]\right\}$$

for all $x, y \in S$. Using Theorem 3.5, we conclude that \tilde{F} is an interval valued $(\in, \in \lor q_{\tilde{\iota}})$ -fuzzy left filter of S. For the right case, it is checked by the similar way.

Taking
$$\tilde{k} = [0,0]$$
 in Theorem 3.11 induces the following corollary.

3.12 Corollary [14, Theorem 4.10]: For an interval valued fuzzy subset \tilde{F} of S, the following are equivalent:

(1) \tilde{F} is an interval valued $(\in, \in \lor q)$ -fuzzy left (resp. right) filter of S.

(2)
$$(\forall \tilde{t} \in D(0, 0.5]) \begin{pmatrix} U(\tilde{F}; \tilde{t}) \neq \emptyset \Rightarrow U(\tilde{F}; \tilde{t}) \\ \text{is a left (right) filter of } S \end{pmatrix}$$
.

3.13 Theorem: For any left (resp. right) filter F of S, let \tilde{F} be an interval valued fuzzy subset of S defined by

$$\tilde{F}(x) = \begin{cases} \tilde{t}_1 & \text{if } x \in F \\ \tilde{t}_2 & \text{otherwise,} \end{cases}$$
(3.1)

where $\tilde{t}_1 \in [\tilde{k}, [1, 1]]$ and $\tilde{t}_2 \in ([0, 0], \tilde{k})$. Then, \tilde{F} is an interval valued $(\in, \in \lor q_{\tilde{i}})$ -fuzzy left (resp. right) filter of S.

Proof: Note that

$$U\left(\tilde{F};\tilde{r}\right) = \begin{cases} F & \text{if } \tilde{r} \in \left(\tilde{t}_{2},\tilde{k}\right] \\ S & \text{if } \tilde{r} \in \left(\left[0,0\right],\tilde{t}_{2}\right], \end{cases}$$
(3.2)

which is a left (resp. right) filter of S. It follows from Theorem 3.11 that \tilde{F} is an interval valued $(\in, \in \lor q_{\tilde{i}})$ -fuzzy left (resp. right) filter of S.

3.14 Corollary [14]: For any left (resp. right) filter F of S let \tilde{F} be an interval valued fuzzy subset of S defined by

$$\tilde{F}(x) = \begin{cases} \tilde{t}_1 & \text{if } x \in F \\ \tilde{t}_2 & \text{otherwise} \end{cases}$$

Then, \tilde{F} is an $(\in, \in \lor q)$ -fuzzy left (resp. right) filter of $Q^{\tilde{k}}(\tilde{F};\tilde{t})$. Then, $\tilde{F}(xy) + \tilde{t} > [1,1] - \tilde{k}$ and so *S* .

For any interval valued fuzzy subset \tilde{F} of S and $\tilde{t} \in D(0,1]$, we consider four subsets:

$$Q(\tilde{F};\tilde{t}) := \left\{ x \in S \mid x_{\tilde{t}} q \tilde{F} \right\} \text{ and}$$
$$\left[\tilde{F} \right]_{\tilde{t}} := \left\{ x \in S \mid x_{\tilde{t}} \in \lor q \tilde{F} \right\}.$$

$$Q^{\tilde{k}}\left(\tilde{F};\tilde{t}\right) := \left\{ x \in S \mid x_{\tilde{t}} q_{\tilde{k}} \tilde{F} \right\} \text{ and}$$
$$\left[\tilde{F}\right]_{\tilde{t}}^{\tilde{k}} := \left\{ x \in S \mid x_{\tilde{t}} \in \lor q_{\tilde{k}} \tilde{F} \right\}.$$
is clear that
$$\left[\tilde{F}\right]_{\tilde{t}} := U\left(\tilde{F};\tilde{t}\right) \bigcup Q^{\tilde{k}}\left(\tilde{F};\tilde{t}\right) \text{ and}$$

 $\left[\tilde{F}\right]_{\tilde{k}}^{k} := \left[\tilde{F}\right]_{\tilde{k}} \left[\bigcup Q^{\tilde{k}}\left(\tilde{F};\tilde{t}\right)\right].$

It

3.15 Theorem: If \tilde{F} is an interval valued $(\in, \in \lor q_{\tilde{i}})$ -fuzzy left (resp. right) filter of *S*, then

$$\left(\forall \tilde{t} \in D\left(\frac{1-k}{2},1\right]\right) \left(Q^{\tilde{k}}\left(\tilde{F};\tilde{t}\right) \neq \emptyset \Rightarrow Q^{\tilde{k}}\left(\tilde{F};\tilde{t}\right) \\ \text{is a left (resp. right) filter of } S\right)$$

Proof: Assume that \tilde{F} is an interval valued $(\in, \in \lor q_{\tilde{k}})$ -fuzzy left filter of *S*. Let $\tilde{t} \in D\left(\frac{1-k}{2}, 1\right)$

be such that $Q^{\tilde{k}}(\tilde{F};\tilde{t}) \neq \emptyset$. Let $x \in Q^{\tilde{k}}(\tilde{F};\tilde{t})$ and $y \in S$ be such that $x \leq y$. Then, $\tilde{F}(x) + \tilde{t} > [1,1] - \tilde{k}$. By means of Theorem 3.5 (1), we have

$$\tilde{F}(y) \ge r \min\left\{\tilde{F}(x), \left\lfloor \frac{1-k^-}{2}, \frac{1-k^+}{2} \right\rfloor\right\}$$
$$= \begin{cases} \left\lfloor \frac{1-k^-}{2}, \frac{1-k^+}{2} \right\rfloor & \text{if } \tilde{F}(x) \ge \left\lfloor \frac{1-k^-}{2}, \frac{1-k^+}{2} \right\rfloor, \\ \tilde{F}(x) & \text{if } \tilde{F}(x) < \left\lfloor \frac{1-k^-}{2}, \frac{1-k^+}{2} \right\rfloor, \end{cases}$$
$$\ge \lceil 1, 1 \rceil - \tilde{t} - \tilde{k}, \end{cases}$$

and so $y \in Q^{\tilde{k}}(\tilde{F};\tilde{t})$. Let $x, y \in Q^{\tilde{k}}(\tilde{F};\tilde{t})$. Then, $\tilde{F}(x) + \tilde{t} \ge [1,1] - \tilde{k}$ and then $\tilde{F}(y) + \tilde{t} \ge [1,1] - \tilde{k}$. It follows from Theorem 3.5 (2) that

$$\tilde{F}(xy) \ge r \min\left\{\tilde{F}(x), \tilde{F}(y), \left[\frac{1-k^{-}}{2}, \frac{1-k^{+}}{2}\right]\right\}$$

$$= \begin{cases} r \min\left\{\tilde{F}(x), \tilde{F}(y)\right\} & \text{if } r \min\left\{\tilde{F}(x), \tilde{F}(y)\right\} < \left[\frac{1-k^{-}}{2}, \frac{1-k^{+}}{2}\right], \\ \left[\frac{1-k^{-}}{2}, \frac{1-k^{+}}{2}\right] & \text{if } r \min\left\{\tilde{F}(x), \tilde{F}(y)\right\} \ge \left[\frac{1-k^{-}}{2}, \frac{1-k^{+}}{2}\right], \\ > [1,1] - \tilde{t} - \tilde{k}, \end{cases}$$

where $\tilde{t}_1 \in [[0.5, 0.5], [1,1]]$ and $\tilde{t}_2 \in ([0,0], [0.5, 0.5])$. and so $xy \in Q^{\tilde{k}}(\tilde{F}; \tilde{t})$. Let $x, y \in S$ such that $xy \in S$

$$\tilde{F}(x) \ge r \min\left\{\tilde{F}(xy), \left[\frac{1-k^{-}}{2}, \frac{1-k^{+}}{2}\right]\right\}$$
$$= \begin{cases} \left[\frac{1-k^{-}}{2}, \frac{1-k^{+}}{2}\right] & \text{if } \tilde{F}(xy) \ge \left[\frac{1-k^{-}}{2}, \frac{1-k^{+}}{2}\right], \\ \tilde{F}(x) & \text{if } \tilde{F}(xy) < \left[\frac{1-k^{-}}{2}, \frac{1-k^{+}}{2}\right], \end{cases}$$

$$> [1,1] - \tilde{t} - \tilde{k},$$

Hence, $x \in Q^{\tilde{k}}(\tilde{F};\tilde{t})$. Therefore, $Q^{\tilde{k}}(\tilde{F};\tilde{t})$ is a left filter of *S*. Similarly, we can obtain the desired result for the right case.

3.16 Corollary: If \tilde{F} is an interval valued $(\in, \in \lor q)$ -fuzzy left (resp. right) filter of *S*, then

$$\left(\forall t \in (0.5,1] \right) \begin{pmatrix} Q^{\tilde{k}} \left(\tilde{F}; \tilde{t} \right) \neq \emptyset \\ \Rightarrow Q^{\tilde{k}} \left(\tilde{F}; \tilde{t} \right) \text{ is a left (resp. right) filter of } S \end{pmatrix}$$

3.17 Corollary: Let \tilde{F} be an interval valued $(\in, \in \lor q_{\tilde{k}})$ -fuzzy left (resp. right) filter of *S*, if $[0,0] \le \tilde{k} < \tilde{r} < [1,1]$, then

$$\left(\forall \tilde{t} \in D\left(\frac{1-r}{2},1\right)\right) \begin{pmatrix} Q^{\tilde{k}}\left(\tilde{F};\tilde{t}\right) \neq \emptyset \\ \Rightarrow Q^{\tilde{k}}\left(\tilde{F};\tilde{t}\right) \text{ is a left (resp. right) filter of } S \end{pmatrix}.$$

Proof. It is straightforward by Theorem 3.10 and 3.15. 3.18 *Theorem:* For any interval valued fuzzy subset \tilde{F} of *S*, the following are equivalent:

(1) \tilde{F} is an interval valued $(\in, \in \lor q_{\tilde{k}})$ -fuzzy left (resp. right) filter of *S*.

$$(2) (\forall \tilde{t} \in D(0,1]) \begin{pmatrix} \left[\tilde{F}\right]_{\tilde{t}}^{\tilde{k}} \neq \emptyset \\ \Rightarrow \left[\tilde{F}\right]_{\tilde{t}}^{\tilde{k}} \\ \text{ is a left (resp. right)} \\ \text{ filter of } S \end{pmatrix}.$$

Proof: Assume that \tilde{F} is an interval valued $(\in, \in \lor q_k)$ -fuzzy left filter of S and let $\tilde{t} \in D(0,1]$ such that $[\tilde{F}]_{\tilde{t}}^{\tilde{k}} \neq \emptyset$. Let $x \in [\tilde{F}]_{\tilde{t}}^{\tilde{k}}$ and $y \in S$ be such that $x \leq y$. Then, $x \in U(\tilde{F}; \tilde{t})$ or $x \in Q^{\tilde{k}}(\tilde{F}; \tilde{t})$, i.e., $\tilde{F}(x) \geq \tilde{t}$ or $\tilde{F}(x) + \tilde{t} > [1,1] - \tilde{k}$. Using Theorem 3.5 (1), we get

$$\tilde{F}(y) \ge r \min\left\{\tilde{F}(x), \left[\frac{1-k^{-}}{2}, \frac{1-k^{+}}{2}\right]\right\}.$$
(3.3)

We consider two cases: $\tilde{F}(x) \leq \left[\frac{1-k^{-}}{2}, \frac{1-k^{+}}{2}\right]$ and $\tilde{F}(x) > \left[\frac{1-k^{-}}{2}, \frac{1-k^{+}}{2}\right]$. The first case implies from (3.4) that $\tilde{F}(y) \geq \tilde{F}(x)$. Thus, if $\tilde{F}(x) \geq \tilde{t}$, then $\tilde{F}(y) \geq \tilde{t}$ and so $y \in U(\tilde{F}; \tilde{t}) \subseteq [\tilde{F}]_{\tilde{t}}^{\tilde{k}}$. If $\tilde{F}(x) + \tilde{t} > [1,1] - \tilde{k}$, then $\tilde{F}(y) + \tilde{t} \geq \tilde{F}(x) + \tilde{t} > [1,1] - \tilde{k}$ which implies that $y_{\tilde{t}} q_{\tilde{k}} \tilde{F}$, i.e., $y \in Q^{\tilde{k}}(\tilde{F}; \tilde{t}) \subseteq [\tilde{F}]_{\tilde{t}}^{\tilde{k}}$. Combining the second case and (3.4) induces

$$\tilde{F}(y) \ge \left[\frac{1-k^-}{2}, \frac{1-k^+}{2}\right]$$
. If $\tilde{t} \le \left[\frac{1-k^-}{2}, \frac{1-k^+}{2}\right]$ and

$$\tilde{F}(y) \ge \tilde{t} \quad \text{and} \quad \text{hence} \quad y \in U\left(\tilde{F}; \tilde{t}\right) \subseteq \left[\tilde{F}\right]_{\tilde{t}}^{k}. \quad \text{If}$$
$$\tilde{t} > \left[\frac{1-k^{-}}{2}, \frac{1-k^{+}}{2}\right], \quad \text{then}$$
$$\tilde{F}(y) + \tilde{t} > \left[\frac{1-k^{-}}{2}, \frac{1-k^{+}}{2}\right] + \left[\frac{1-k^{-}}{2}, \frac{1-k^{+}}{2}\right] = [1,1] - \tilde{k},$$

which implies that $y \in Q^{\tilde{k}}(\tilde{F};\tilde{t}) \subseteq [\tilde{F}]_{\tilde{t}}^{\tilde{k}}$. Therefore, $[\tilde{F}]_{\tilde{t}}^{\tilde{k}}$ satisfies the condition (b1). Let $x, y \in [\tilde{F}]_{\tilde{t}}^{\tilde{k}}$. Then, $x \in U(\tilde{F};\tilde{t})$ or $x_{\tilde{t}} \neq_{\tilde{k}} \tilde{F}$ and $y \in U(\tilde{F};\tilde{t})$ or $y_{\tilde{t}} q_{\tilde{k}} \tilde{F}$ that is, $\tilde{F}(x) \ge \tilde{t}$ or $\tilde{F}(x) + \tilde{t} > [1,1] - \tilde{k}$ and $\tilde{F}(y) \ge \tilde{t}$ or $\tilde{F}(y) + \tilde{t} > [1,1] - \tilde{k}$. We consider the following four cases.

(i) If
$$\tilde{F}(x) \ge \tilde{t}$$
 and $\tilde{F}(y) \ge \tilde{t}$.
(ii) If $\tilde{F}(x) \ge \tilde{t}$ and $\tilde{F}(y) + \tilde{t} > [1,1] - \tilde{k}$

(*iii*) If
$$\tilde{F}(x) + \tilde{t} > [1,1] - \tilde{k}$$
 and $\tilde{F}(y) \ge \tilde{t}$.

(*iv*) If $\tilde{F}(x) + \tilde{t} \ge [1,1] - \tilde{k}$ and $\tilde{F}(y) + \tilde{t} \ge [1,1] - \tilde{k}$. For the case (*i*), Theorem 3.5 (2) implies that

$$\tilde{F}(xy) \ge r \min \left\{ \tilde{F}(x), \tilde{F}(y), \left\lfloor \frac{1-k^{-}}{2}, \frac{1-k^{+}}{2} \right\rfloor \right\} \\ \ge r \min \left\{ \tilde{t}, \left\lfloor \frac{1-k^{-}}{2}, \frac{1-k^{+}}{2} \right\rfloor \right\} \\ = \begin{cases} \left\lfloor \frac{1-k^{-}}{2}, \frac{1-k^{+}}{2} \right\rfloor & \text{if } \tilde{t} > \left\lfloor \frac{1-k^{-}}{2}, \frac{1-k^{+}}{2} \right\rfloor, \\ \tilde{t} & \text{if } \tilde{t} \le \left\lfloor \frac{1-k^{-}}{2}, \frac{1-k^{+}}{2} \right\rfloor, \end{cases}$$

so that $xy \in U(\tilde{F}; \tilde{t})$ or

$$\tilde{F}(xy) + \tilde{t} + \tilde{k}$$

$$> \left[\frac{1-k^{-}}{2}, \frac{1-k^{+}}{2}\right] + \left[\frac{1-k^{-}}{2}, \frac{1-k^{+}}{2}\right] + \tilde{k}$$

$$= [1,1],$$

that is $xy \in Q^{\tilde{k}}(\tilde{F};\tilde{t})$ Hence, $xy \in [\tilde{F}]_{\tilde{t}}^{\tilde{k}}$. For the second case assume that $\tilde{t} > [\frac{1-k^-}{2}, \frac{1-k^+}{2}]$. Then,

$$\begin{bmatrix} 1 & , 1 \end{bmatrix} - \tilde{t} - \tilde{k}$$

$$\leq \begin{bmatrix} 1 & , 1 \end{bmatrix} - \tilde{t}$$

$$< \begin{bmatrix} \frac{1 - k^{-}}{2} & , \frac{1 - k^{+}}{2} \end{bmatrix}$$

and so

$$\begin{split} \tilde{F}(xy) &\geq r \min\left\{\tilde{F}(x), \tilde{F}(y), \left[\frac{1-k^{-}}{2}, \frac{1-k^{+}}{2}\right]\right\} \\ &= \begin{cases} r \min\left\{\tilde{F}(y), \left[\frac{1-k^{-}}{2}, \frac{1-k^{+}}{2}\right]\right\} > [1,1] - \tilde{t} - \tilde{k} & \text{if } r \min\left\{\tilde{F}(y), \left[\frac{1-k^{-}}{2}, \frac{1-k^{+}}{2}\right]\right\} \\ &\leq \tilde{F}(x), \\ \\ \tilde{F}(x) &\geq \tilde{t} & \text{if } r \min\left\{\tilde{F}(y), \left[\frac{1-k^{-}}{2}, \frac{1-k^{+}}{2}\right]\right\} \\ &> \tilde{F}(x). \end{split}$$

Thus, $xy \in U(\tilde{F}; \tilde{t}) \bigcup Q^{\tilde{k}}(\tilde{F}; \tilde{t}) = [\tilde{F}]_{\tilde{t}}^{\tilde{k}}$. Suppose that $\tilde{t} \leq [\underline{1-k^{-}}, \underline{1-k^{+}}]$. Then,

$$\tilde{F}(xy) \ge r \min\left\{\tilde{F}(x), \tilde{F}(y), \left[\frac{1-k^{-}}{2}, \frac{1-k^{+}}{2}\right]\right\}$$

$$= \begin{cases} r \min\left\{\tilde{F}(x), \left[\frac{1-k^{-}}{2}, \frac{1-k^{+}}{2}\right]\right\} \ge \tilde{t} & \text{if } r \min\left\{\tilde{F}(x), \left[\frac{1-k^{-}}{2}, \frac{1-k^{+}}{2}\right]\right\} \\ \le \tilde{F}(y), \\ \tilde{F}(y) \ge [1,1] - \tilde{t} - \tilde{k} & \text{if } r \min\left\{\tilde{F}(x), \left[\frac{1-k^{-}}{2}, \frac{1-k^{+}}{2}\right]\right\} \\ \ge \tilde{F}(y). \end{cases}$$

and thus $xy \in U(\tilde{F};\tilde{t}) \bigcup Q^{\tilde{k}}(\tilde{F};\tilde{t}) = [\tilde{F}]_{\tilde{t}}^{\tilde{k}}$. We have similar result for the case *(iii)*. For the final case, if

$$\tilde{t} > \left\lfloor \frac{1-k^{-}}{2}, \frac{1-k^{+}}{2} \right\rfloor, \text{ then}$$

$$[1,1] - \tilde{t} - \tilde{k} < \left[\frac{1-k^{-}}{2}, \frac{1-k^{+}}{2} \right]. \text{ Hence,}$$

$$\tilde{F}(xy) \ge r \min \left\{ \tilde{F}(x), \tilde{F}(y), \left[\frac{1-k^{-}}{2}, \frac{1-k^{+}}{2} \right] \right\}$$

$$= \left[\frac{1-k^{-}}{2}, \frac{1-k^{+}}{2} \right] > [1,1] - \tilde{t} - \tilde{k}$$

whenever $r\min\left\{\tilde{F}(x),\tilde{F}(y)\right\} < \left\lfloor \frac{1-k^-}{2}, \frac{1-k^+}{2} \right\rfloor$. Thus,

$$xy \in Q^{\tilde{k}}\left(\tilde{F}; \tilde{t}\right) \subseteq \left[\tilde{F}\right]_{\tilde{t}}^{\tilde{k}}$$
. If $\tilde{t} \leq \left[\frac{1-k^{-}}{2}, \frac{1-k^{+}}{2}\right]$, then

 $\tilde{F}(xy) \ge r \min\left\{\tilde{F}(x), \tilde{F}(y), \left[\frac{1-k^{-}}{2}, \frac{1-k^{+}}{2}\right]\right\}$ $= \begin{cases} \left[\frac{1-k^{-}}{2}, \frac{1-k^{+}}{2}\right] \ge \tilde{t} \\ r \min\left\{\tilde{F}(x), \tilde{F}(y)\right\} \\ \ge \left[\frac{1-k^{-}}{2}, \frac{1-k^{+}}{2}\right], \\ if r \min\left\{\tilde{F}(x), \tilde{F}(y)\right\} \\ \ge [1,1]-\tilde{t}-\tilde{k} \end{cases} \quad if r \min\left\{\tilde{F}(x), \tilde{F}(y)\right\}$

which implies that

$$xy \in U\left(\tilde{F};\tilde{t}\right) \bigcup Q^{\tilde{k}}\left(\tilde{F};\tilde{t}\right) = \left[\tilde{F}\right]_{\tilde{t}}^{k}.$$

Let $x, y \in S$ be such that $xy \in [\tilde{F}]_{\tilde{t}}^{\tilde{k}}$. Then, $\tilde{F}(xy) \geq \tilde{t}$ or $\tilde{F}(xy) + \tilde{t} > [1,1] - \tilde{k}$. It follows from Theorem 3.5 (3) that

$$\tilde{F}(x) \ge r \min\left\{\tilde{F}(xy), \left[\frac{1-k^{-}}{2}, \frac{1-k^{+}}{2}\right]\right\}.$$
 (3.4)

We consider two cases:

$$\widetilde{F}(xy) \leq \left[\frac{1-k^-}{2}, \frac{1-k^+}{2}\right]$$

and $\widetilde{F}(xy) > \left[\frac{1-k^-}{2}, \frac{1-k^+}{2}\right].$

The first case implies from (3.5) that $\tilde{F}(x) \ge \tilde{F}(xy)$. Thus, if $\tilde{F}(xy) \ge \tilde{t}$, then $\tilde{F}(x) \ge \tilde{t}$ and so $x \in U(\tilde{F};\tilde{t}) \subseteq [\tilde{F}]_{\tilde{t}}^{\tilde{k}}$. If $\tilde{F}(xy) + \tilde{t} > [1,1] - \tilde{k}$, then $\tilde{F}(x) + \tilde{t} \ge \tilde{F}(xy) + \tilde{t} > [1,1] - \tilde{k}$ which implies that $x_{\tilde{t}} \neq_{\tilde{k}} \tilde{F}$, i.e., $x \in Q^{\tilde{k}}(\tilde{F};\tilde{t}) \subseteq [\tilde{F}]_{\tilde{t}}^{\tilde{k}}$. Combining the second case and (3.5) induces $\tilde{F}(xy) \ge [\frac{1-k^{-}}{2}, \frac{1-k^{+}}{2}]$.

If
$$\tilde{t} \leq \left[\frac{1-k^-}{2}, \frac{1-k^+}{2}\right]$$
, then $\tilde{F}(x) \geq \tilde{t}$ and hence
 $x \in U(\tilde{F}; \tilde{t}) \subseteq \left[\tilde{F}\right]_{\tilde{t}}^{\tilde{k}}$. If $\tilde{t} > \left[\frac{1-k^-}{2}, \frac{1-k^+}{2}\right]$, then
 $\tilde{F}(x) + \tilde{t} > \left[\frac{1-k^-}{2}, \frac{1-k^+}{2}\right] + \left[\frac{1-k^-}{2}, \frac{1-k^+}{2}\right] = [1,1] - \tilde{k}$,

which implies that $x \in Q^{\tilde{k}}(\tilde{F};\tilde{t}) \subseteq [\tilde{F}]_{\tilde{t}}^{k}$. Therefore, $[\tilde{F}]_{\tilde{t}}^{\tilde{k}}$ satisfies the condition (b3). Therefore, $[\tilde{F}]_{\tilde{t}}^{\tilde{k}}$ is a left filter of *S*.

Conversely, suppose that (2) is valid. If there exist $a, b \in S$ such that $a \leq b$ and $\tilde{F}(b) \leq r \min\left\{\tilde{F}(a), \left\lceil \frac{1-k^{-}}{2}, \frac{1-k^{+}}{2} \right\rceil \right\}$, then

$$\tilde{F}(b) < \tilde{t}_b \le r \min\left\{\tilde{F}(a), \left[\frac{1-k^-}{2}, \frac{1-k^+}{2}\right]\right\}$$
 for some

 $\tilde{t}_{b} \in D\left(0, \frac{1-k}{2}\right]$. It follows that $a \in U\left(\tilde{F}; \tilde{t}_{b}\right) \subseteq \left[\tilde{F}\right]_{\tilde{t}_{b}}^{\tilde{k}}$ but $b \notin Q^{\tilde{k}}\left(\tilde{F}; \tilde{t}\right)$. Also, we have $\tilde{F}\left(b\right) + \tilde{t}_{b} < 2\tilde{t}_{b} \leq [1,1] - \tilde{k}$, and so $b_{\tilde{t}_{b}} \overline{q_{k}} \tilde{F}$, i.e.,

 $b \notin Q^{\tilde{k}}(\tilde{F};\tilde{t})$. Therefore, $b \notin [\tilde{F}]_{\tilde{t}}^{\tilde{k}}$, a contradiction. Hence, $\tilde{F}(y) \ge r \min \left\{ \tilde{F}(x), \left\lceil \frac{1-k^-}{2}, \frac{1-k^+}{2} \right\rceil \right\}$ for all $x, y \in S$ with $x \leq y$. Suppose that there exist $a, b \in S$ such that $\tilde{F}(ab) < \tilde{t} \le r \min\left\{\tilde{F}(a), \tilde{F}(b), \left[\frac{1-k^{-}}{2}, \frac{1-k^{+}}{2}\right]\right\}.$ some $\tilde{t} \in D\left(0, \frac{1-k}{2}\right)$. It follows that for $a \in U(\tilde{F}; \tilde{t}) \subseteq [\tilde{F}]_{\tilde{t}}^{\tilde{k}}$ and $b \in U(\tilde{F}; \tilde{t}) \subseteq [\tilde{F}]_{\tilde{t}}^{\tilde{k}}$, so from that $ab \in \left[\tilde{F}\right]_{\tilde{t}}^{\tilde{k}}$. Thus, $\tilde{F}(ab) \geq \tilde{t}$ or (b2) $\tilde{F}(ab) + \tilde{t} > [1,1] - \tilde{k}$, a contradiction. Therefore, $\tilde{F}(xy) \ge r \min\left\{\tilde{F}(x), \tilde{F}(y), \left[\frac{1-k^{-}}{2}, \frac{1-k^{+}}{2}\right]\right\}$ for all $x, y \in S$. Assume that there exist $a, b \in S$ such that $\tilde{F}(a) \le r \min\left\{\tilde{F}(ab), \left[\frac{1-k^-}{2}, \frac{1-k^+}{2}\right]\right\}$. Then, $\tilde{F}(a) < \tilde{t}_a \le r \min\left\{\tilde{F}(ab), \left[\frac{1-k^-}{2}, \frac{1-k^+}{2}\right]\right\}$

for some $\tilde{t}_a \in D\left(0, \frac{1-k}{2}\right)$. It follows that $ab \in U\left(\tilde{F}; \tilde{t}_a\right) \subseteq \left[\tilde{F}\right]_{\tilde{t}_a}^{\tilde{k}}$ so from (b3) that $a \in \left[\tilde{F}\right]_{\tilde{t}_a}^{\tilde{k}}$. Thus, $\tilde{F}(a) \ge \tilde{t}_a$ or $\tilde{F}(a) + \tilde{t}_a > [1,1] - \tilde{k}$, a contradiction. Therefore, $\tilde{F}(x) \ge r \min\left\{\tilde{F}(xy), \left[\frac{1-k^-}{2}, \frac{1-k^+}{2}\right]\right\}$ for all $x, y \in S$. Using

Theorem 3.5, we conclude that \tilde{F} is an interval valued $(\in, \in \lor q_{\tilde{k}})$ -fuzzy left filter of *S*. Similarly, we obtain

the right case. 3.19 Corollary: For any interval valued fuzzy subset \tilde{F} of S, the following are equivalent:

(1) \tilde{F} is an interval valued $(\in, \in \lor q)$ -fuzzy left (resp. right) filter of *S*,

(2)
$$(\forall \tilde{t} \in D(0,1]) \begin{pmatrix} \left[\tilde{F}\right]_{\tilde{t}}^{\tilde{k}} \neq \emptyset \\ \Rightarrow \left[\tilde{F}\right]_{\tilde{t}}^{\tilde{k}} \text{ is a left (resp. right) filter of } S \end{pmatrix}$$
.

An interval valued fuzzy subset \tilde{F} of *S* is said to be *proper* if $\text{Im}(\tilde{F})$ has at least two elements. Two interval valued fuzzy subsets are said to be *equivalent* if they have same family of level subsets. Otherwise, they are said to be non-equivalent.

3.20 *Theorem:* Let \tilde{F} be an interval valued $(\in, \in \lor q_{\tilde{k}})$ -fuzzy left (resp. right) filter of *S* such that

$$#\left\{\tilde{F}(x) \mid \tilde{F}(x) < \left[\frac{1-k^{-}}{2}, \frac{1-k^{+}}{2}\right]\right\} \geq 2.$$

Then, there exist two proper non-equivalent interval valued $(\in, \in \lor q_{\tilde{k}})$ -fuzzy left (resp. right) filters of *S* such that \tilde{F} can be expressed as the union of them. *Proof:*

Let
$$\left\{ \tilde{F}(x) \mid \tilde{F}(x) < \left[\frac{1-k^{-}}{2}, \frac{1-k^{+}}{2}\right] \right\} = \{\tilde{t}_{1}, \tilde{t}_{2}, ..., \tilde{t}_{n}\},$$

where $\tilde{t}_1 > \tilde{t}_2 > ... > \tilde{t}_n$ and $n \ge 2$. Then, the chain of $(\in, \in \lor q_{\tilde{k}})$ -level left (resp. right) filters of \tilde{F} is

$$[\tilde{F}]^{\tilde{k}}_{\left[\frac{1-k^{-}}{2},\frac{1-k^{+}}{2}\right]} \subseteq [\tilde{F}]^{\tilde{k}}_{\tilde{t}} \subseteq [\tilde{F}]^{\tilde{k}}_{\tilde{t}_{2}} \subseteq \dots [\tilde{F}]^{\tilde{k}}_{\tilde{t}_{n}} = S.$$

Let $\tilde{\Xi}$ and $\tilde{\Sigma}$ be interval valued fuzzy subsets of *S* defined by

$$\tilde{\Xi}(x) = \begin{cases} \tilde{t}_1 & \text{if } x \in [\tilde{\Xi}]_{\tilde{t}_1}^k, \\ \tilde{t}_2 & \text{if } x \in [\tilde{\Xi}]_{\tilde{t}_2}^{\tilde{k}} \setminus [\tilde{\Xi}]_{\tilde{t}_1}^{\tilde{k}}, \\ \dots & \\ \tilde{t}_n & \text{if } x \in [\tilde{\Xi}]_{\tilde{t}_n}^{\tilde{k}} \setminus [\tilde{\Xi}]_{\tilde{t}_{n-1}}^{\tilde{k}} \end{cases}$$

and

and

$$\hat{\Sigma}(x) = \begin{cases}
\tilde{\Sigma}(x) & \text{if } x \in [\tilde{\Sigma}]_{\left[\frac{1-k^{-}}{2}, \frac{1-k^{+}}{2}\right]}^{k}, \\
\tilde{k} & \text{if } x \in [\tilde{\Sigma}]_{\tilde{t}_{2}}^{\tilde{k}} \setminus [\tilde{\Sigma}]_{\left[\frac{1-k^{-}}{2}, \frac{1-k^{+}}{2}\right]}^{\tilde{k}}, \\
\tilde{t}_{3} & \text{if } x \in [\tilde{\Sigma}]_{\tilde{t}_{2}}^{\tilde{k}} \setminus [\tilde{\Sigma}]_{\tilde{t}_{1}}^{\tilde{k}}, \\
& \dots \\
\tilde{t}_{n} & \text{if } x \in [\tilde{\Sigma}]_{\tilde{t}_{n}}^{\tilde{k}} \setminus [\tilde{\Sigma}]_{\tilde{t}_{n-1}}^{\tilde{k}},
\end{cases}$$

respectively, where $\tilde{t}_3 < \tilde{k} < \tilde{t}_2$. Then, $\tilde{\Xi}$ and $\tilde{\Sigma}$ are interval valued $(\in, \in \lor q_{\tilde{k}})$ -fuzzy left (resp. right) filters of *S*, and $\tilde{\Xi}, \tilde{\Sigma} \subseteq \tilde{F}$. The chains of $(\in \lor q_{\tilde{k}})$ -level left (resp. right) filters of $\tilde{\Xi}$ and $\tilde{\Sigma}$ are, respectively, given by

 $\begin{bmatrix} \tilde{\Xi} \end{bmatrix}_{\tilde{t}_1}^{\tilde{k}} \subseteq \begin{bmatrix} \tilde{\Xi} \end{bmatrix}_{\tilde{t}_2}^{\tilde{k}} \subseteq \cdots \subseteq \begin{bmatrix} \tilde{\Xi} \end{bmatrix}_{\tilde{t}_n}^{\tilde{k}}$

$$\left[\tilde{\Sigma} \right]_{\frac{1-k}{2}}^{\tilde{k}} \subseteq \left[\tilde{\Sigma} \right]_{\tilde{t}_1}^{\tilde{k}} \subseteq \ldots \subseteq \left[\tilde{\Sigma} \right]_{\tilde{t}_n}^{\tilde{k}}$$

Therefore, $\tilde{\Xi}$ and $\tilde{\Sigma}$ are non-equivalent and clearly $\tilde{F} = \tilde{\Xi} \bigcup \tilde{\Sigma}$. This completes the proof.

4. Implication-based interval valued fuzzy left (resp. right) filters

Fuzzy logic is an extension of set theoretic multivalued logic in which the truth values are linguistic variables or terms of the linguistic variable truth. Some operators, for example $\lor,\land,\neg,\rightarrow$ in fuzzy logic are also defined by using truth tables and the extension principle can be applied to derive definitions of the operators. In fuzzy logic, the truth value of fuzzy proposition Φ is denoted by $[\Phi]$. For a universe *U* of discourse, we display the fuzzy logical and corresponding set-theoretical notations used in this paper

$$x \in [F] = F(x), \tag{4.1}$$

$$[\Phi \land \Psi] = \min\{[\Phi], [\Psi]\}, \qquad (4.2)$$

$$[\Phi \rightarrow \Psi] = \min\{1, 1 - [\Phi] + [\Psi]\}, \qquad (4.3)$$

$$[\forall \Phi(x)] = \inf_{x \in U} [\Phi(x)], \qquad (4.4)$$

$$\Phi$$
 if and only if $[\Phi] = 1$ for all valuation. (4.5)

The truth valuation rules given in (4.3) are those in the \pounds ukasiewicz system of continuous-valued logic. Of course, various implication operators have been defined. We show only a selection of them in the following.

(a) Gaines-Rescher implication operator (I_{GR}) :

$$I_{GR}(a,b) = \begin{cases} 1 & \text{if } a \le b, \\ 0 & \text{otherwise.} \end{cases}$$

(b) Gödel implication operator (I_G) :

$$I_G(a,b) = \begin{cases} 1 & \text{if } a \le b, \\ b & \text{otherwise} \end{cases}$$

(c) The contraposition of Gödel implication operator (I_{cG}) :

$$I_{cG}(a,b) = \begin{cases} 1 & \text{if } a \le b, \\ 1-a & \text{otherwise} \end{cases}$$

Ying [28] introduced the concept of fuzzifying topology. We can expand his/her idea to ordered semigroups, and we define an interval valued fuzzifying left (resp. right) filters as follows.

4.1 Definition: An interval valued fuzzy subset \tilde{F} of S is called an *interval valued fuzzifying left (resp. right)* filter of S if it satisfies the following conditions:

(d1)
$$(\forall x, y \in S) \left(x \le y \Rightarrow \left(\left[x \in \tilde{F} \right] \rightarrow \left[y \in \tilde{F} \right] \right) \right),$$

(d2)
$$(\forall x, y \in S) \left(r \min \left\{ \left\lfloor x \in \tilde{F} \right\rfloor, \left\lfloor y \in \tilde{F} \right\rfloor \right\} \rightarrow \left\lfloor xy \in \tilde{F} \right\rfloor \right\},$$

(d3)
$$(\forall x, y \in S) \left(\left[xy \in \tilde{F} \right] \rightarrow \left[x \in \tilde{F} \right] \left(\operatorname{resp.} \left[y \in \tilde{F} \right] \right) \right).$$

Obviously, the conditions (d1), (d2) and (d3) are equivalent to (b4), (b5) and (b6), respectively. Therefore,

an interval valued fuzzifying left (resp. right) filter is an ordinary interval valued fuzzy left (resp. right) filter. In [17], the concept of \tilde{t} -tautology is introduced, i.e., for all valuations

$$= \tilde{\Phi} \text{ if and only if } [\tilde{\Phi}] \ge \tilde{t}, \qquad (4.6)$$

4.2 Definition: An interval valued fuzzy subset \tilde{F} of S and $\tilde{t} \in D(0,1]$ is called a \tilde{t} -implication-based interval valued fuzzy left (resp. right) filter of S if it satisfies:

(d4)
$$(\forall x, y \in S)$$
 $(x \le y \Rightarrow \left(\models_{\widetilde{t}} \left[x \in \widetilde{F} \right] \rightarrow \left[y \in \widetilde{F} \right] \right),$
(d5) $(\forall x, y \in S)$
 $\left(\models_{\widetilde{t}} \min\left\{ \left[x \in \widetilde{F} \right], \left[y \in \widetilde{F} \right] \right\} \rightarrow \left[xy \in \widetilde{F} \right] \right),$
(d6) $(\forall x, y \in S)$
 $\left(\models_{\widetilde{t}} \left[xy \in \widetilde{F} \right] \rightarrow \left[x \in \widetilde{F} \right] \left(\operatorname{resp} \left[y \in \widetilde{F} \right] \right) \right).$

Let *I* be an implication operator. Clearly, \overline{F} is a \tilde{t} -implication-based interval valued fuzzy left (resp. right) filter of *S* if and only if it satisfies:

(d7)
$$(\forall x, y \in S) \left(x \le y \Rightarrow I\left(\tilde{F}(x), \tilde{F}(y)\right) \ge \tilde{t}\right),$$

(d8) $(\forall x, y \in S) \left(I\left(r\min\left\{\tilde{F}(x), \tilde{F}(y)\right\}, \tilde{F}(xy)\right) \ge \tilde{t}\right),$
(d9) $(\forall x, y \in S) \left(I\left(\tilde{F}(xy), \tilde{F}(x)\right) \ge \tilde{t}\right) \left(\operatorname{resp}\left(\tilde{F}(xy), \tilde{F}(y)\right) \ge \tilde{t}\right).$

4.3 Theorem: For any interval valued fuzzy subset \overline{F} of S, we have,

(1) If $I = I_{GR}$, then \tilde{F} is a 0.5 -implication-based interval fuzzy left (resp. right) filter of *S* if and only if \tilde{F} is an interval valued fuzzy left (resp. right) filter of *S*.

(2) If $I = I_G$, then \tilde{F} is a $\frac{1-k}{2}$ -implication-based interval valued fuzzy left (resp. right) filter of *S* if and only if \tilde{F} is an interval valued $(\in, \in \lor q_{\tilde{k}})$ -fuzzy left (resp. right) filter of *S*.

(3) If $I = I_{cG}$, then \tilde{F} is a $\frac{1-k}{2}$ -implication-based interval valued fuzzy left (resp. right) filter of *S* if and only if \tilde{F} satisfies the following conditions:

(3.1)
$$x \le y \Rightarrow r \max \left\{ \tilde{F}(y), \left[\frac{1-k^{-}}{2}, \frac{1-k^{+}}{2} \right] \right\} \ge r \min \left\{ \tilde{F}(x), [1,1] \right\},$$

$$(3.2) \quad r \max\left\{\tilde{F}(xy), \left[\frac{1-k^{-}}{2}, \frac{1-k^{+}}{2}\right]\right\} \geq r \min\left\{\tilde{F}(x), \tilde{F}(y), [1,1]\right\},$$

$$(3.3) \quad r \max\left\{\tilde{F}(x), \left[\frac{1-k^{-}}{2}, \frac{1-k^{+}}{2}\right]\right\} \geq r \min\left\{\tilde{F}(xy), [1,1]\right\}$$

$$\left(\operatorname{resp.max}\left\{\tilde{F}(y), \left[\frac{1-k^{-}}{2}, \frac{1-k^{+}}{2}\right]\right\} \geq r \min\left\{\tilde{F}(xy), [1,1]\right\},$$

for all $x, y \in S$.

Proof: (1) Straightforward.

(2) Assume that
$$\tilde{F}$$
 is a $\frac{1-k}{2}$ - implication-

based interval valued fuzzy left filter of S. The

(i)
$$(\forall r, v \in S)$$

(i)
$$(\forall x, y \in S)$$

 $\left(x \leq y \Rightarrow I_{G}\left(\tilde{F}(x), \tilde{F}(y)\right) \geq \left[\frac{1-k^{-}}{2}, \frac{1-k^{+}}{2}\right]\right),$
(ii) $(\forall x, y \in S)$
 $\left(I_{G}\left(r\min\left\{\tilde{F}(x), \tilde{F}(y)\right\}, \tilde{F}(xy)\right) \geq \left[\frac{1-k^{-}}{2}, \frac{1-k^{+}}{2}\right]\right),$
(iii) $(\forall x, y \in S) \left(I_{G}\left(\tilde{F}(xy), \tilde{F}(x)\right) \geq \left[\frac{1-k^{-}}{2}, \frac{1-k^{+}}{2}\right]\right).$
Let $x, y \in S$ be such that $x \leq y$. Using (i) we have

$$\widetilde{F}(y) \ge \widetilde{F}(x) \text{ or } \widetilde{F}(x) > \widetilde{F}(y) \ge \left[\frac{1-k^-}{2}, \frac{1-k^+}{2}\right].$$
Hence, $\widetilde{F}(y) \ge r \min\left\{\widetilde{F}(x), \left[\frac{1-k^-}{2}, \frac{1-k^+}{2}\right]\right\}.$ From

(ii), we get
$$\tilde{F}(xy) \ge r \min\left\{\tilde{F}(x), \tilde{F}(y)\right\}$$
 or

$$r\min\left\{\tilde{F}(x), \tilde{F}(y)\right\} > \tilde{F}(xy) \ge \left\lfloor \frac{1-k^-}{2}, \frac{1-k^+}{2} \right\rfloor.$$
 Thus,
$$\tilde{F}(xy) \ge r\min\left\{\tilde{F}(x), \tilde{F}(y), \left\lfloor \frac{1-k^-}{2}, \frac{1-k^+}{2} \right\rfloor\right\}.$$
 Case (iii)

implies that $\tilde{F}(x) \ge \tilde{F}(xy)$ or $\tilde{F}(xy) \ge \tilde{F}(x) \ge \frac{1-k^{-}}{2}, \frac{1-k^{+}}{2}$, and so $\tilde{F}(x) \ge r \min$ $\left\{\tilde{F}(xy), \left[\frac{1-k^{-}}{2}, \frac{1-k^{+}}{2}\right]\right\}$. Using Theorem 3.5, we conclude that \tilde{F} is an interval valued $(\in, \in \lor q_{\tilde{i}})$ -fuzzy left filter of S.

Conversely, suppose that \tilde{F} is an interval valued $(\in, \in \lor q_{\tilde{k}})$ -fuzzy left filter of *S*. Let $x, y \in S$ such that $x \le y$. Using Theorem 3.5 (1), we have

$$I_{G}\left(\tilde{F}(x), \tilde{F}(y)\right)$$

$$=\begin{cases} [1,1] \ge \left[\frac{1-k^{-}}{2}, \frac{1-k^{+}}{2}\right] \\ \text{if } r \min\left\{\tilde{F}(x), \left[\frac{1-k^{-}}{2}, \frac{1-k^{+}}{2}\right]\right\} = \tilde{F}(x), \\ \tilde{F}(y) \ge \left[\frac{1-k^{-}}{2}, \frac{1-k^{+}}{2}\right] \\ \text{if } r \min\left\{\tilde{F}(x), \left[\frac{1-k^{-}}{2}, \frac{1-k^{+}}{2}\right]\right\} \\ = \left[\frac{1-k^{-}}{2}, \frac{1-k^{+}}{2}\right]. \end{cases}$$

From Theorem 3.5 (2), if $r\min\left\{\tilde{F}(x),\tilde{F}(x),\left[\frac{1-k^{-}}{2},\frac{1-k^{+}}{2}\right]\right\}=r\min\{\tilde{F}(x),\tilde{F}(y)\},$ then

$$\tilde{F}(xy) \ge r \min\left\{\tilde{F}(x), \tilde{F}(y)\right\}$$

And so

If

$$I_{G}\left(r\min\left\{\tilde{F}\left(x\right),\tilde{F}\left(y\right)\right\},\tilde{F}\left(xy\right)\right)=[1,1]$$

$$\geq \left[\frac{1-k^{-}}{2},\frac{1-k^{+}}{2}\right].$$
If
$$r\min\left\{\tilde{F}\left(x\right),\tilde{F}\left(y\right),\left[\frac{1-k^{-}}{2},\frac{1-k^{+}}{2}\right]\right\}$$

$$=\left[\frac{1-k^{-}}{2},\frac{1-k^{+}}{2}\right],$$
then
$$\tilde{F}\left(xy\right)\geq \left[\frac{1-k^{-}}{2},\frac{1-k^{+}}{2}\right] \text{ and thus}$$

$$I_{G}\left(r\min\left\{\tilde{F}\left(x\right),\tilde{F}\left(y\right)\right\},\tilde{F}\left(xy\right)\right)\geq \left[\frac{1-k^{-}}{2},\frac{1-k^{+}}{2}\right].$$
From Theorem 3.5 (3), if
$$r\min\left\{\tilde{F}\left(xy\right),\left[\frac{1-k^{-}}{2},\frac{1-k^{+}}{2}\right]\right\}=\tilde{F}\left(xy\right),$$

then

$$I_{G}\left(\tilde{F}(xy),\tilde{F}(x)\right) = [1,1] \ge \left[\frac{1-k^{-}}{2},\frac{1-k^{+}}{2}\right]$$

Otherwise,

$$I_{G}\left(\tilde{F}(xy),\tilde{F}(x)\right) \ge \left[\frac{1-k^{-}}{2},\frac{1-k^{+}}{2}\right]$$

Consequently, \tilde{F} is a $\frac{1-k}{2}$ -implication-based fuzzy I left filter of *S*. Similarly, we obtain the right case.

(3) Suppose that F satisfies (3.1), (3.2) and (3.3). Let $x, y \in S$ be such that $x \leq y$. In (3.1), if $r\min\{F(x),[1,1]\}=[1,1]$ then $r \max\left\{\tilde{F}(y), \left[\frac{1-k^{-}}{2}, \frac{1-k^{+}}{2}\right]\right\} = [1,1]$ and hence

 $\tilde{F}(y) = [1,1] \ge \tilde{F}(x)$. Therefore,

$$I_{cG}\left(\tilde{F}(x),\tilde{F}(y)\right) = [1,1] \ge \left[\frac{1-k^{-}}{2},\frac{1-k^{+}}{2}\right].$$

If F(x) < [1,1], then

$$r \max\left\{\tilde{F}(y), \left[\frac{1-k^{-}}{2}, \frac{1-k^{+}}{2}\right]\right\} \ge \tilde{F}(x).$$
(4.7)
If $\tilde{F}(y) > \left[\frac{1-k^{-}}{2}, \frac{1-k^{+}}{2}\right]$ in (4.7), then $\tilde{F}(y) \ge \tilde{F}(x)$

and thus

$$I_{cG}\left(\tilde{F}(x),\tilde{F}(y)\right) = [1,1] \ge \left[\frac{1-k^{-}}{2},\frac{1-k^{+}}{2}\right].$$

If $\tilde{F}(y) \le \left[\frac{1-k^{-}}{2},\frac{1-k^{+}}{2}\right]$ in (4.7), then
 $\tilde{F}(x) \le \left[\frac{1-k^{-}}{2},\frac{1-k^{+}}{2}\right].$ Hence,
 $I_{cG}\left(\tilde{F}(x),\tilde{F}(y)\right)$
 $= \begin{cases} 1,1] \ge \left[\frac{1-k^{-}}{2},\frac{1-k^{+}}{2}\right] & \text{if } \mu(y) \ge \mu(x) \\ 1,1] - \tilde{F}(x) \ge \left[\frac{1-k^{-}}{2},\frac{1-k^{+}}{2}\right] & \text{otherwise.} \end{cases}$

In (3.2), if $r \min \left\{ \tilde{F}(x), \tilde{F}(y), [1,1] \right\} = [1,1]$, then $r \max\left\{\tilde{F}(xy), \left[\frac{1-k^{-}}{2}, \frac{1-k^{+}}{2}\right]\right\} = [1,1]$ and so $\tilde{F}(xy) = [1,1] \ge r \min \left\{ \tilde{F}(x), \tilde{F}(y) \right\}.$

Therefore,

$$\begin{split} I_{cG}\bigg(r\min\bigg\{\tilde{F}(x),\tilde{F}(y)\bigg\},\tilde{F}(xy)\bigg) &= [1,1]\\ &\ge \bigg[\frac{1-k^-}{2},\frac{1-k^+}{2}\bigg].\\ \text{If } r\min\bigg\{\tilde{F}(x),\tilde{F}(y),[1,1]\bigg\} &= r\min\bigg\{\tilde{F}(x),\tilde{F}(y)\bigg\}, \text{ then }\\ r\max\bigg\{\tilde{F}(xy),\bigg[\frac{1-k^-}{2},\frac{1-k^+}{2}\bigg]\bigg\} &\ge r\min\bigg\{\tilde{F}(x),\tilde{F}(y)\bigg\}.\\ \text{Thus, if }\\ r\max\bigg\{\tilde{F}(xy),\bigg[\frac{1-k^-}{2},\frac{1-k^+}{2}\bigg]\bigg\} &= \bigg[\frac{1-k^-}{2},\frac{1-k^+}{2}\bigg] \text{ in } (4.8),\\ \text{then }\\ \tilde{F}(xy) &\le \bigg[\frac{1-k^-}{2},\frac{1-k^+}{2}\bigg] \end{split}$$

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and

$$\min\left\{\tilde{F}(x),\tilde{F}(y)\right\} \leq \left[\frac{1-k^{-}}{2},\frac{1-k^{+}}{2}\right]$$

Therefore,

r

$$I_{cG}\left(r\min\left\{\tilde{F}(x),\tilde{F}(y)\right\},\tilde{F}(xy)\right) = [1,1]$$
$$\geq \left[\frac{1-k^{-}}{2},\frac{1-k^{+}}{2}\right]$$

whenever
$$F(xy) \ge r \min\{F(x), F \sim (y)\}$$
, and
 $I_{cG}\left(r \min\left\{\tilde{F}(x), \tilde{F}(y)\right\}, \tilde{F}(xy)\right)$
 $= [1,1] - r \min\left\{\tilde{F}(x), \tilde{F}(y)\right\} \ge \left[\frac{1-k^{-}}{2}, \frac{1-k^{+}}{2}\right]$

whenever
$$\tilde{F}(xy) < r \min{\{\tilde{F}(x), \tilde{F}(y)\}}$$
. Now, if
 $r \max\left\{\tilde{F}(xy), \left[\frac{1-k^-}{2}, \frac{1-k^+}{2}\right]\right\} = \tilde{F}(xy)$
in (4.8), then $\tilde{F}(xy) \ge r \min\left\{\tilde{F}(x), \tilde{F}(y)\right\}$ and so
 $I_{cG}\left(r \min\left\{\tilde{F}(x), \tilde{F}(y)\right\}, \tilde{F}(xy)\right) = [1,1]$
 $\ge \left[\frac{1-k^-}{2}, \frac{1-k^+}{2}\right].$

In (3.3), if F(xy) = [1,1], then $r \max\left\{\tilde{F}(x), \left[\frac{1-k^{-}}{2}, \frac{1-k^{+}}{2}\right]\right\} = [1,1]$ and hence

$$I_{cG}\left(\tilde{F}(xy),\tilde{F}(x)\right) = [1,1] \ge \left[\frac{1-k^-}{2},\frac{1-k^+}{2}\right].$$

If F(xy) < [1,1], then

$$r \max\left\{\tilde{F}(x), \left[\frac{1-k^{-}}{2}, \frac{1-k^{+}}{2}\right]\right\} \ge \tilde{F}(xy). \quad (4.9)$$

If

If

$$r \max\left\{\tilde{F}(x), \left[\frac{1-k^{-}}{2}, \frac{1-k^{+}}{2}\right]\right\} = \tilde{F}(x)$$

in (4.9), then $\tilde{F}(x) \ge \tilde{F}(xy)$. Hence,

$$I_{cG}\left(\tilde{F}(xy), \tilde{F}(x)\right) = [1,1] \ge \left[\frac{1-k^{-}}{2}, \frac{1-k^{+}}{2}\right].$$

$$r \max\left\{\tilde{F}(x), \left[\frac{1-k^{-}}{2}, \frac{1-k^{+}}{2}\right]\right\} = \left[\frac{1-k^{-}}{2}, \frac{1-k^{+}}{2}\right] \quad \text{in}$$

(4.9), then
$$\tilde{F}(xy) \leq \left[\frac{1-k^-}{2}, \frac{1-k^+}{2}\right]$$
 which implies that
 $I_{cG}\left(\tilde{F}(xy), \tilde{F}(x)\right)$
 $\left[1,1\right] \geq \left[\frac{1-k^-}{2}, \frac{1-k^+}{2}\right]$ if $\tilde{F}(x) \geq \tilde{F}(xy)$,

$$= \begin{cases} 1, 1 \\ 1, 1 \end{bmatrix} - \tilde{F}(xy) \ge \begin{bmatrix} \frac{1-k^{-}}{2}, \frac{1-k^{+}}{2} \end{bmatrix} \text{ otherwise.}$$

Consequently, \tilde{F} is a $\frac{1-\kappa}{2}$ -implication-based interval valued fuzzy left filter of *S*.

Conversely assume that \tilde{F} is a $\frac{1-k}{2}$ -implication-based interval valued fuzzy left filter of *S*. Then, (i) $(\forall x, y \in S)$ $\left(x \leq y \Rightarrow I_{cG}\left(\tilde{F}(x), \tilde{F}(y)\right) \geq \left[\frac{1-k^{-}}{2}, \frac{1-k^{+}}{2}\right]\right)$

(ii)
$$(\forall x, y \in S)$$

 $\left(I_{cG}(r\min\{\tilde{F}(x), \tilde{F}(y)\}, \tilde{F}(xy)) \ge \left[\frac{1-k^{-}}{2}, \frac{1-k^{+}}{2}\right]\right),$
(iii) $(\forall x, y \in S) \left(I_{G}\left(\tilde{F}(xy), \tilde{F}(x)\right) \ge \left[\frac{1-k^{-}}{2}, \frac{1-k^{+}}{2}\right]\right).$

Let $x, y \in S$ be such that $x \leq y$. (iv) implies that $I_{cG}\left(\tilde{F}(x), \tilde{F}(y)\right) = [1,1]$ or $[1,1] - \tilde{F}(x) \geq \left[\frac{1-k^{-}}{2}, \frac{1-k^{+}}{2}\right]$ so that $\tilde{F}(x) \leq \tilde{F}(y)$ or $\tilde{F}(x) \leq \left[\frac{1-k^{-}}{2}, \frac{1-k^{+}}{2}\right]$. Therefore,

$$r \max\left\{\tilde{F}(y), \left[\frac{1-k^{-}}{2}, \frac{1-k^{+}}{2}\right]\right\} \ge \tilde{F}(x) = r \min\{\tilde{F}(x), [1,1]\}.$$

From (v), we have
$$P \qquad I_{cG}\left(r \min\left\{\tilde{F}(x), \tilde{F}(y)\right\}, \mu(xy)\right) = [1,1],$$

i.e., $r \min\{\tilde{F}(x), \tilde{F}(y)\} \le \tilde{F}(xy),$
or $[1,1] - r \min\left\{\tilde{F}(x), \tilde{F}(y)\right\} \ge \left[\frac{1-k^{-}}{2}, \frac{1-k^{+}}{2}\right].$
Hence,
$$r \max\left\{\tilde{F}(y), \left[\frac{1-k^{-}}{2}, \frac{1-k^{+}}{2}\right]\right\} \ge$$

$$r \max\left\{F(y), \left\lfloor \frac{1}{2}, \frac{1}{2}\right\rfloor\right\} \ge \min\left\{\tilde{F}(x), \tilde{F}(y)\right\} = \min\{\tilde{F}(x), \tilde{F}(y), [1,1]\}$$

for all $x, y \in S$. Finally, (vi) implies that

$$\begin{split} I_{cG}\Big(\tilde{F}(x),\tilde{F}(y)\Big) &= [1,1] \text{ or} \\ [1,1] - \tilde{F}(xy) \geq \left[\frac{1-k^-}{2},\frac{1-k^+}{2}\right] \text{ so that } \tilde{F}(xy) \leq \tilde{F}(x) \\ \text{ or } \tilde{F}(xy) \leq \left[\frac{1-k^-}{2},\frac{1-k^+}{2}\right]. \text{ Therefore,} \\ r \max\left\{\tilde{F}(x),\left[\frac{1-k^-}{2},\frac{1-k^+}{2}\right]\right\} \geq \tilde{F}(xy) \\ &= \min\{\tilde{F}(xy),[1,1]\}. \end{split}$$

The right case is similar to the left case. This completes the proof.

4.4 Corollary: If $I = I_G$, then any interval valued fuzzy subset \tilde{F} of S is a 0.5 -implication-based interval valued fuzzy left (resp. right) filter of S if and only if \tilde{F} is an interval valued $(\in, \in \lor q)$ -fuzzy left (resp. right) filter of S.

4.5 *Corollary:* If $I = I_{cG}$, then any interval valued fuzzy subset \tilde{F} of S is a 0.5-implication based interval valued fuzzy left (resp. right) filter of S if and only if \tilde{F} satisfies the following condition: (3.1)

$$x \leq y \Rightarrow r \max\left\{\tilde{F}(y), [0.5, 0.5]\right\} \geq r \min\left\{\tilde{F}(x), [1,1]\right\},$$

$$(3.2)$$

$$r \max\left\{\tilde{F}(xy), [0.5, 0.5]\right\} \geq r \min\left\{\tilde{F}(x), \tilde{F}(y), [1,1]\right\},$$

$$(3.3) \quad r \max\left\{\tilde{F}(x), [0.5, 0.5]\right\} \geq r \min\left\{\tilde{F}(xy), [1,1]\right\}$$

$$\left(\operatorname{resp.rmax}_{\widetilde{F}(y),[0.5,0.5]}\right) \ge \operatorname{rmin}_{\widetilde{F}(xy),[1,1]}\right)$$
for all $x, y \in S$.

5. Conclusions

In 1975, Zadeh [29] introduced the notion of interval fuzzy sets as an extension of fuzzy sets in which the value of the membership degrees are interval of numbers instead of numbers. On the other hand, semigroups are important in many areas of mathematics, for example, language theory, automata coding and theory, combinatorics and mathematical analysis. So, a theory of interval-valued fuzzy sets on ordered semigroups can be developed. Since the concept of fuzzy filters of ordered semigroups play an important role in the study of ordered semigroup structure, we studied a more general form of interval valued $(\in, \in \lor q)$ -fuzzy left (right) filters in ordered semigroups and we established characterizations of an interval valued ($\in, \in \lor q$)-fuzzy

left (right) filter. Moreover the obtained results can be applied to semirings, near-rings, hemirings, rings and etc.

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