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Applications of Graphs Related to the Probability that an Element of Finite Metacyclic 2-Group Fixes a Set

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Abstract

In this paper, G denotes a metacyclic 2-group of positive type of nilpotency class at least three and Ω is the set of all subsets of commuting elements of G of size two in the form of (a, b), where a and b commute and $\mathrm{lcm}(|a|, |b|) = 2$. The probability that a group element of G fixes a set is one of the generalizations of the commutativity degree that has been recently introduced. In this paper, the probability that an element of fixes a set for metacyclic 2-groups of positive type of nilpotency class at least three is computed. The results obtained are then applied to graph theory, more precisely to the orbit graph and generalized conjugacy class graph.

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1 Introduction

This section provides some backgrounds related to the commutativity degree and graph theory. We begin with the commutativity degree. The probability that two random elements of G commute is called the commutativity degree, denoted by $P(G)$. It was proven that $P(G) \leq \frac{5}{8}$, [1] and [2]. The commutativity degree has been generalized by several authors, where various results have been obtained. Omer *et al.* [3]. have extended the commutativity degree by defining the probability that a group element fixes a set. In the following some basic concepts related to graph theory are included, which will be used in the later discussion.

A graph Γ is a mathematical structure consisting of two sets, namely vertices and edges which are denoted by $V(\Gamma)$ and $E(\Gamma)$, respectively. A connected graph is a graph in which there is a partition of vertex V into non empty subsets, $V_1, V_2, ..., V_n$ such that two vertices V_1 and V_2 are connected if and only if they belong to the same set V_i. Subgraphs $\Gamma(V_1), \Gamma(V_2), ..., \Gamma(V_n)$ are all components of Γ. The graph Γ is connected if it has precisely one component. However, a graph is a complete graph if each order pair of distinct vertices is adjacent, and it is denoted by K_n, where n is the number of adjacent vertices. The graph is called empty if there is no adjacent between its vertices. In addition, a graph is called null if it has no vertices and in this paper we denote K_0 the null graph. ([4] and [5]).

Furthermore, a non-empty set S of V(Γ) is called an *independent set* of Γ if there is no adjacent between two elements of S in Γ. Meanwhile, the *independent number* is the number of vertices in maximum independent set and it is denoted by $\alpha(\Gamma)$. However, the maximum number c for which Γ is c-vertex colorable is known as *chromatic number* and is denoted by $\chi(\Gamma)$. The *diameter* is the maximum distance between any two vertices of Γ and $d(\Gamma)$ is used as a notation. Furthermore, a *clique* is a complete subgraph in Γ, while the *clique number* is the size of the largest clique in Γ and is denoted by $\omega(\Gamma)$. The *dominating set* $X \subseteq V(\Gamma)$ is a set where for each v outside X, $\exists\ x \in X$ such that v adjacent to x. The minimum size of X is called the *dominating number* denoted by $\gamma(\Gamma)$ ([4] and [5]).

Since the groups under consideration in this paper are metacyclic 2-groups of positive type of nilpotency class at least three, their classification is given in the following theorem.

Theorem 1.1 *[6] Let G be a metacyclic 2-group of positive type of nilpotency class at least three. Then G is isomorphic to one of the following types:*

(1) $G \cong \langle a, b : a^{2^\alpha} = b^{2^\beta} = 1, [b, a] = a^{2^{\alpha-\gamma}} \rangle, 1 + \gamma < \alpha < 2\gamma, \beta \geq \gamma,$

(2) $G \cong \langle a, b : a^{2^\alpha} = 1, b^{2^\beta} = a^{2^{\alpha-\varepsilon}}, [b, a] = a^{2^{\alpha-\gamma}} \rangle, 1 + \gamma < \alpha < 2\gamma, \gamma \leq \beta$ *and* $\alpha \leq \beta + \varepsilon.$

Throughout this paper, we refer to these two classifications as groups of type (1) and (2).

This paper is divided into three sections. The first section focuses on some background about some topics in graph theory, while the second section provides some earlier and recent publications that are related to the commutativity degree and graph theory, more specifically to generalized conjugacy class graph and the orbit graph. In the third section, we presented our results which include the probability that a group element fixes a set, orbit graph and generalized conjugacy class graph.

2 Preliminary Notes

In this section, some works that are related to the probability that an element of a group fixes a set and graph theory are stated. We commence with brief information about the probability of a group element fixes a set, followed by some related work on graph theory, more precisely to graph related to conjugacy classes and the orbit graph. In 2013, the probability that a group element fixes a set, denoted by $P_G(\Omega)$ was firstly introduced by Omer *et al.* [3]. The following theorem is one of Omer *et al.* results that is used in this paper.

Theorem 2.1 *[3] Let G be a finite group. Let S be a set of elements of G of size two in the form of (a, b) where a and b commute and lcm$(|a|, |b|) = 2$. Let Ω be the set of all subsets of commuting elements of G of size two and G acts on Ω. Then the probability that an element of a group fixes a set is given by:*

$$P_G(\Omega) = \frac{K(\Omega)}{|\Omega|},$$

where $K(\Omega)$ is the number of orbits of Ω in G.

The work in [3] has then been extended by finding the probability for some finite non-abelian groups such as the symmetric groups and alternating groups [7]. In this paper, we connect this concept to graph theory by using the orbits that are obtained under group action on a set to the orbit graph and generalized conjugacy class graph.

Some related works on conjugacy class graph are presented in the following.

Bianchi *et al.* [8] studied the regularity of the graph related to conjugacy classes and provided some results. In addition, Moreto *et al.* [9] classified the groups in which conjugacy classes sizes are not coprime for any five distinct classes. Moreover, Moradipour *et al.* [10] used the graph related to conjugacy classes to find some graph properties of some finite metacyclic 2-groups. The graph related to conjugacy classes was also generalized by Omer *et al.* [11], in which the vertices are orbits under the group action on a set. The following is the definition of the generalized conjugacy class graph.

Definition 2.2 *[11] Let G be a finite non-abelian group and Ω is a set of G. If G acts on Ω, then the number of vertices of generalized conjugacy classes graph is $|V(\Gamma_G^{\Omega_c})| = K(\Omega) - |A|$, where A is $\{g\omega = \omega g : \omega \in \Omega\}$. Two vertices ω_1 and ω_2 in $\Gamma_G^{\Omega_c}$ are adjacent if their cardinalities are not coprime.*

The conjugate graph was firstly introduced by Erfanian and Tolue [12], in which two vertices of this graph are connected if they are conjugate. This graph is then generalized by defining a new graph called the orbit graph, denoted by Γ_G^{Ω}, Omer *et al.* [7] in which two vertices ω_1 and ω_2 of Γ_G^{Ω} are adjacent if $\omega_1 = {\omega_2}^g$. The orbit graph has been found for some finite non-abelian groups as in [7] and [13].

3 Main Results

In this section, we compute the probability that an element of a metacyclic 2-group of positive type fixes a set. The results that are obtained from the probability are then applied to graph theory, more precisely to the orbit graph and generalized conjugacy class graph.

3.1 The probability that an element of a group fixes a set

We begin this section with the first result on the presentation of metacyclic 2-groups of positive type of nilpotency class at least three.

Theorem 3.1 *Let G be a group of type (1), $G \cong \langle a, b : a^{2^\alpha} = b^{2^\beta} = 1, [b, a] = a^{2^{\alpha-\gamma}}\rangle, 1 + \gamma < \alpha < 2\gamma, \beta \geq \gamma$. Let S be a set of elements of G of size two in the form of (a, b) where a and b commute and $lcm(|a|, |b|) = 2$. Let*

Ω be the set of all subsets of commuting elements of G of size two. If G acts on Ω by conjugation, then $P_G(\Omega) = \frac{2}{3}$.

proof 3.2 *The elements of G of order two are $a^{2^{\alpha-1}}, b^{2^{\beta-1}}$ and $a^{2^{\alpha-1}}b^{2^{\beta-1}}$. Therefore, the elements of Ω of size two are described as follows. One element is in the form of $(1, a^{2^{\alpha-1}})$, two elements are in the form of $(1, a^{2^{\alpha-1}i}b^{2^{\beta-1}}), 0 \leq i \leq 2^\alpha$ where i is even, two elements are in the form of $(a^{2^{\alpha-1}}, a^{2^{\alpha-1}i}b^{2^{\beta-1}}), 0 \leq i \leq 2^\alpha$ where i is even and one element is in the form of $(a^{2^{\alpha-1}}b^{2^{\beta-1}}, a^{2^{\alpha-1}i}b^{2^{\beta-1}})$, $0 \leq i \leq 2^\alpha$ where i is even. Therefore, $|\Omega| = 6$. Since the action here is by conjugation, thus there are four orbits divided as follow. One orbit is in the form of $\{(1, a^{2^{\alpha-1}i}b^{2^{\beta-1}})\}, 0 \leq i \leq 2^\alpha$ where i is even, one orbit of $\{(a^{2^{\alpha-1}}, a^{2^{\alpha-1}i}b^{2^{\beta-1}})\}$, one orbit of $\{(1, a^{2^{\alpha-1}})\}$ and one in the form of $\{(a^{2^{\alpha-1}}b^{2^{\beta-1}}, a^{2^{\alpha-1}}b^{2^{\beta-1}})\}, 0 \leq i \leq 2^\alpha$. Using Theorem 2.1, $P_G(\Omega) = \frac{4}{6}$, as claimed.*

Theorem 3.3 *Let G be a group of type (2), $G \cong \langle a, b : a^{2^\alpha} = 1, b^{2^\beta} = a^{2^{\alpha-\varepsilon}}, [b, a] = a^{2^{\alpha-\gamma}}\rangle, 1 + \gamma < \alpha < 2\gamma, \gamma \leq \beta$ and $\alpha \leq \beta + \varepsilon$. Let S be a set of elements of G of size two in the form of (a, b) where a and b commute and $lcm(|a|, |b|) = 2$. Let Ω be the set of all subsets of commuting elements of G of size two. If G acts on Ω by conjugation. Then*

$$P_G(\Omega) = \begin{cases} \frac{2}{3}, \text{if } \beta < 1 + \gamma \text{ and } \varepsilon = 3, \\ 1, \text{otherwise.} \end{cases}$$

proof 3.4 *By manual calculations, the elements of G that have order two are $a^{2^{\alpha-1}}$, $a^{7\times2^{\alpha-4}}b^{2^{\beta-1}}$ and $a^{7\times2^{\alpha-4}+2^{\alpha-1}}b^{2^{\beta-1}}$. Thus, the elements of Ω are in the following forms. One element is in the form of $(1, a^{2^{\alpha-1}})$, two elements are in the form of $(1, a^{7\times2^{\alpha-4}+2^{\alpha-1}i}b^{2^{\beta-1}}), 0 \leq i \leq 1$, two elements are in the form of $(a^{2^{\alpha-1}}, a^{7\times2^{\alpha-4}+2^{\alpha-1}i}b^{2^{\beta-1}}), 0 \leq i \leq 1$ and only one element is in the form of $(a^{7\times2^{\alpha-4}}b^{2^{\beta-1}}, a^{7\times2^{\alpha-4}+2^{\alpha-1}i}b^{2^{\beta-1}}), 0 \leq i \leq 1$. Therefore, $|\Omega| = 6$. Since the action here is by conjugation, thus the elements of Ω are actually in the center of G thus when G acts on Ω, the number of orbits is six. In this case and by Theorem 2.1, $P_G(\Omega) = 1$. However, when $\varepsilon = 3$ and $\beta < 1 + \gamma$ the proof is similar to Theorem 3.1.*

In the next section, we apply the obtained results to graph theory, more precisely to the orbit graph and generalized conjugacy class graph.

3.2 The Orbit Graph and Generalized Conjugacy Class Graph

3.2.1 The orbit graph

In this section, the results obtained are applied to the orbit graph. First, we provide a theorem that is considered as a key connect between the probability that a group element fixes a set and the orbit graph.

Theorem 3.5 *Let G be a finite non-abelian group and let Ω be the set of all subsets of commuting elements of G of size two. If G acts on Ω by conjugation and $P_G(\Omega) = 1$, then Γ_G^Ω is a null graph.*

proof 3.6 *Suppose $P_G(\Omega) = 1$. Then by Theorem 2.1, $K(\Omega) = |\Omega|$ thus $cl(\omega) = \omega$ for all $\omega \in \Omega$. Thus all elements a and b are in the center of G. Since $|V(\Gamma_G^\Omega)| = |\Omega| - |A|$ and $A = \{\omega g = g\omega, \omega \in \Omega\}$ leads to $|\Omega| = |A|$, thus the graph is null.*

Next, the orbit graph of metacyclic 2-groups of positive type of nilpotency class at least three is found. We begin with the orbit graph for the first presentation.

Theorem 3.7 *Let G be a group of type (1), $G \cong \langle a, b : a^{2^\alpha} = b^{2^\beta} = 1, [b, a] = a^{2^{\alpha-\gamma}}\rangle, 1 + \gamma < \alpha < 2\gamma, \beta \geq \gamma$. Let S be a set of elements of G of size two in the form of (a, b) where a and b commute and $lcm(|a|, |b|) = 2$. Let Ω be the set of all subsets of commuting elements of G of size two. If G acts on Ω by conjugation, then $\Gamma_G^\Omega = K_2 \cup K_2$.*

proof 3.8 *Using Theorem 3.1, the number of vertices is five. Since two vertices ω_1 and ω_2 are adjacent if $\omega_1 = {\omega_2}^g$, thus there are two complete components of K_2. The proof then follows.*

Theorem 3.9 *Let G be a group of type (2), $G \cong \langle a, b : a^{2^\alpha} = 1, b^{2^\beta} = a^{2^{\alpha-\varepsilon}}, [b, a] = a^{2^{\alpha-\gamma}}\rangle, 1 + \gamma < \alpha < 2\gamma, \gamma \leq \beta$ and $\alpha \leq \beta + \varepsilon$. Let S be a set of elements of G of size two in the form of (a, b) where a and b commute and $lcm(|a|, |b|) = 2$. Let Ω be the set of all subsets of commuting elements of G of size two. If G acts on Ω by conjugation, then*

$$\Gamma_G^\Omega = \begin{cases} K_2 \cup K_2, \textit{if } \beta < 1 + \gamma \textit{ and } \varepsilon = 3, \\ K_0, \qquad\qquad \textit{otherwise.} \end{cases}$$

proof 3.10 *Based on Theorem 3.3 when $\beta < 1 + \gamma$, the number of vertices in Γ_G^Ω is five and and since two vertices are adjacent if they are conjugate hence Γ_G^Ω consists of two complete components of K_2 and one isolated vertex, namely $(b^{2^{\beta-1}}, a^{2^{\alpha-1}} b^{2^{\beta-1}})$. When $\beta > 1 + \gamma$, based on Theorem 3.3 the probability that a group element fixes a set, $P_G(\Omega) = 1$. Using Theorem 3.5, $\Gamma_G^\Omega = K_0$. The proof then follows.*

3.2.2 The generalized conjugacy class graph

In the following we find the generalized conjugacy class graph for both presentations of metacyclic 2-groups of positive type of nilpotency class at three.

Theorem 3.11 *Let G be a group of type (1), $G \cong \langle a, b : a^{2^\alpha} = b^{2^\beta} = 1, [b, a] = a^{2^{\alpha-\gamma}} \rangle, 1 + \gamma < \alpha < 2\gamma, \beta \geq \gamma$. Let S be a set of elements of G of size two in the form of (a, b) where a and b commute and $lcm(|a|, |b|) = 2$. Let Ω be the set of all subsets of commuting elements of G of size two. If G acts on Ω by conjugation, then the generalized conjugacy class graph $\Gamma_G^{\Omega_c} = K_2$.*

proof 3.12 *According to Theorem 3.1 and the definition of generalized conjugacy class graph, the number of vertices in $\Gamma_G^{\Omega_c}$ is three. Based on vertices adjacency, there is one complete component of K_2 and one isolated vertex. The proof is thus complete.*

An immediate consequence of Theorem 3.11 is given in the following corollary.

Corollary 3.13 *Let $G \cong \langle a, b : a^{2^\alpha} = b^{2^\beta} = 1, [b, a] = a^{2^{\alpha-\gamma}} \rangle$, be a group of type (1), where $1 + \gamma < \alpha < 2\gamma, \beta \geq \gamma$. Let S be a set of elements of G of size two in the form of (a, b) where a and b commute and $lcm(|a|, |b|) = 2$. Let Ω be the set of all subsets of commuting elements of G of size two. If G acts on Ω by conjugation and $\Gamma_G^{\Omega_c} = K_2$, then $\chi(\Gamma_G^{\Omega_c}) = \omega(\Gamma_G^{\Omega_c}) = 2$ and $\alpha(\Gamma_G^{\Omega_c}) = \gamma(\Gamma_G^{\Omega_c}) = 2$.*

Theorem 3.14 *Let G be a group of type (2), $G \cong \langle a, b : a^{2^\alpha} = 1, b^{2^\beta} = a^{2^{\alpha-\varepsilon}}, [b, a] = a^{2^{\alpha-\gamma}} \rangle, 1 + \gamma < \alpha < 2\gamma, \gamma \leq \beta$ and $\alpha \leq \beta + \varepsilon$. Let S be a set of elements of G of size two in the form of (a, b) where a and b commute. and $lcm(|a|, |b|) = 2$ Let Ω be the set of all subsets of commuting elements of G of size two. If G acts on Ω by conjugation, then*

$$\Gamma_G^{\Omega_c} = \begin{cases} K_2, \textit{if } \beta < 1 + \gamma, \varepsilon = 3, \\ K_0, \textit{otherwise.} \end{cases}$$

proof 3.15 *According to Theorem 3.3 and if $\varepsilon = 3$, $\beta < 1 + \gamma$, the number of vertices in $\Gamma_G^{\Omega_c}$ is three. Based on vertices adjacency of generalized conjugacy class graph, there are one complete component of K_2 and one isolated vertex. In the case that $\beta \geq 1 + \gamma$, the number of vertices is zero since $K(\Omega)$ and $|A|$ are identical hence $\Gamma_G^{\Omega_c}$ is null.*

4 Conclusion

In this paper, the probability that an element of a group fixes a set is found for the positive type of metacyclic 2-groups of nilpotency class at least three. Besides, the results obtained were then applied to graph theory, specifically to the orbit graph and generalized conjugacy class graph.

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