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ORIGINAL ARTICLE

Ordered semigroups characterized by $(\in, \in \vee q_k)$-fuzzy generalized bi-ideals

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Abstract In this paper, we introduce a considerable machinery that permits us to characterize a number of special (fuzzy) subsets in ordered semigroups. In this regard, we generalize (Davvaz and Khan in Inform Sci 181:1759–1770 2011) and define $(\in, \in \vee q_k)$-fuzzy generalized bi-ideals in ordered semigroups, which is a generalization of the concept of an (α, β)-fuzzy generalized bi-ideal in an ordered semigroup. We also define $(\in, \in \vee q_k)$-fuzzy left (resp. right)-ideals. Using these concept, some characterization theorems of regular, left (resp. right) regular, completely regular and weakly regular ordered semigroups are provided. The upper/lower parts of an $(\in, \in \vee q_k)$-fuzzy generalized bi-ideal and $(\in, \in \vee q_k)$-fuzzy left (resp. right)-ideal are given, and some characterizations are provided.

Keywords Ordered semigroups · Regular, left (resp. right) regular, completely regular and weakly regular ordered semigroups · Fuzzy generalized bi-ideals · Fuzzy left (resp. right) ideals · $(\in, \in \vee q_k)$-fuzzy left (right) ideals · $(\in, \in \vee q_k)$-fuzzy bi-ideals

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1 Introduction

A new type of fuzzy subgroup, that is, the (α, β)-fuzzy subgroup, was introduced in an earlier paper of Bhakat and Das [1, 2] by using the combined notions of "belongingness" and "quasi-coincidence" of a fuzzy point and a fuzzy set. In particular, the concept of an $(\in, \in \vee q)$-fuzzy subgroup is a useful generalization of Rosenfeld's fuzzy subgroup [23]. Davvaz [6] introduced the concept of $(\in, \in \vee q)$-fuzzy sub-near-ring (R-subgroups, ideals) of a near-ring and investigated some of their properties. Jun and Song [8] discussed the general forms of fuzzy interior ideals in semigroups. Davvaz and Khan [3] discussed some characterizations of regular ordered semigroups in terms of (α, β)-fuzzy generalized bi-ideals, where $\alpha, \beta \in \{\in, q, \in \vee q, \in \wedge q\}$ and $\alpha \neq \in \wedge q$. Also, Khan et al. introduced the concept of an (α, β)-fuzzy interior ideals and gave some basic properties of ordered semigroups in terms of this notion (see [7, 12]). Kazanci and Yamak [13] introduced the concept of a generalized fuzzy bi-ideal in semigroups and gave some properties of fuzzy bi-ideals in terms of $(\in, \in \vee q)$-fuzzy bi-ideals. In [25], regular semigroups are characterized by the properties of $(\in, \in \vee \mathrm{q})$-fuzzy ideals. Jun [9, 10] introduced the concept of $(\in, \in \vee q_k)$-fuzzy subalgebras of a BCK/BCI-algebra. In [26], Shabir et al. gave the concept of more generalized forms of (α, β)-fuzzy ideals and defined $(\in, \in \vee q_k)$-fuzzy ideals of semigroups, by generalizing the concept of $x_\lambda q \lambda$, and defined $x_\lambda q_k \lambda$, as $\lambda\ (x) + t + k > 1$, where $k \in [0, 1)$ (also see [11]). For further reading regarding (α, β)-fuzzy subsets and its generalization, we refer the reader to [4, 5, 21, 22].

The topic of these investigations belongs to the theoretical soft computing (fuzzy structure). Indeed, it is well known that semigroups are basic structures in many applied branches like automata and formal languages, coding theory, finite state machines and others. Due to these possibilities of applications, semigroups and related structures are presently extensively investigated in fuzzy settings (see, e.g., monograph [20]).

Our aim in this paper is to introduce a new sort of fuzzy generalized bi-ideals and fuzzy left (resp. right)-ideals in ordered semigroup, called $(\in, \in \vee q_k)$-fuzzy generalized bi-ideals and $(\in, \in \vee q_k)$-fuzzy left (resp. right)-ideals. Then, some results are given in terms of $(\in, \in \vee q_k)$-fuzzy generalized bi-ideals and $(\in, \in \vee q_k)$-fuzzy left (resp. right)-ideals in ordered semigroups and define the lower/upper parts of these generalized fuzzy ideals. We characterize regular, left and right regular, completely regular and weakly regular ordered semigroups in terms of these notions.

2 Basic definitions and preliminaries

By an *ordered semigroup* (or *po-semigroup*), we mean a structure $(S, \cdot, \leq)$ in which the following conditions are satisfied:

(OS1) $(S, \cdot)$ is a semigroup,
(OS2) $(S, \leq)$ is a poset,
(OS3) $a \leq b \longrightarrow ax \leq bx$ and $a \leq b \longrightarrow xa \leq xb$ for all $a, b, x \in S$.

For subsets A, B of an ordered semigroup S, we denote by $AB = \{ab \in S | a \in A, b \in B\}$. If $A \subseteq S$ we denote $(A] = \{t \in S \mid t \leq h \text{ for some } h \in A\}$. If $A = \{a\}$, then we write $(a]$ instead of $(\{a\}]$. If $A, B \subseteq S$, then $A \subseteq (A], (A](B] \subseteq (AB]$, and $((A\ ]\ ] = (A]$.

Let $(S, \cdot, \leq)$ be an ordered semigroup. A non-empty subset A of S is called a *subsemigroup* of S if $A^2 \subseteq A$. A non-empty subset A of S is called a *left* (resp. *right*) ideal of S if

1. $(\forall a \in S)(\forall b \in A)(a \leq b \longrightarrow a \in A)$,
2. $AS \subseteq A$ (resp. $SA \subseteq A$).

A non-empty subset A of an ordered semigroup S is called a *generalized bi-ideal* [24] of S if

1. $(\forall a \in S)(\forall b \in A)(a \leq b \longrightarrow a \in A)$,
2. $ASA \subseteq A$.

A non-empty subset A of an ordered semigroup S is called a *bi-ideal* [14] of S if

1. $(\forall a \in S)(\forall b \in A)(a \leq b \longrightarrow a \in A)$,
2. $A^2 \subseteq A$,
3. $ASA \subseteq A$.

Note that every bi-ideal of S is a generalized bi-ideal of S, but the converse is not true, as given in [24].

An ordered semigroup S is *regular* [14] if for every $a \in S$ there exists, $x \in S$ such that $a \leq axa$, or equivalently, we have (1) $a \in (aSa]\ \ \forall a \in S$ and (2) $A \subseteq (ASA]\ \ \forall A \subseteq S$. An ordered semigroup S is called *left* (resp. *right*) *regular* [19] if for every $a \in S$ there exists $x \in S$ such that $a \leq xa^2$ (resp. $a \leq a^2x$), or equivalently, (1) $a \in (Sa^2]$(resp. $a \in (a^2S]$) $\ \forall a \in S$ and (2) $A \subseteq (SA^2]$(resp. $A \subseteq (A^2S]$) $\ \forall A \subseteq S$. An ordered semigroup S is called *left* (resp. *right*) *simple* [14, 15] if for every left (resp. right) ideal A of S we have $A = S$ and S is called *simple* [14] if it is both left and right simple. An ordered semigroup S is called *completely regular* [14], if it is left regular, right regular and regular. An ordered semigroup S is called *left weakly regular* [24], if for every $a \in S$, there exist $x, y \in S$ such that $a \leq xaya$, or equivalently, (1) $a \in ((Sa)^2]\ \ \forall a \in S$ and (2) $A \subseteq ((SA)^2]\ \ \forall A \subseteq S$. *Right weakly regular* ordered semigroups are defined similarly. An ordered semigroup S is called *weakly regular* if it is both a left and right weakly regular.

Note that if S is a commutative, then the concepts of regular and weakly regular ordered semigroups coincide.

By $B(a)$ (resp. $L(a)$, $R(a)$ and $I(a)$), we mean the generalized bi- (resp. left, right and two-sided) ideal of S generated by $a (a \in S)$[24], and we have $B(a) = (a \cup aSa]$, $L(a) = (a\ \cup Sa]$, $R(a) = (a\ \cup aS]$ and $I(a) = (a\ \cup Sa\ \cup aS\ \cup SaS]$.

Now, we give some fuzzy logic concepts.

A function $\lambda : S \longrightarrow [0, 1]$ is called a *fuzzy subset* of S.

The study of fuzzification of algebraic structures has started in the pioneering paper of Rosenfeld [23]. Rosenfeld introduced the notion of fuzzy groups and successfully extended many results from groups in the theory of fuzzy groups. Kuroki [16, 17] studied fuzzy ideals, fuzzy bi-ideals and semiprime fuzzy ideals in semigroups (also see [18]).

If λ and μ are fuzzy subsets of S, then $\lambda \preceq \mu$ means $\lambda(x) \leq\ \mu(x)$ for all $x \in S$ and the symbols $\wedge$ and $\vee$ will mean the following fuzzy subsets:

$$\lambda \wedge \mu : S \longrightarrow [0, 1] | x \longmapsto (\lambda \wedge \mu)(x) = \lambda(x) \wedge \mu(x) = \min\{\lambda(x), \mu(x)\}$$

$$\lambda \vee \mu : S \longrightarrow [0, 1] | x \longmapsto (\lambda \vee \mu)(x) = \lambda(x) \vee \mu(x) = \max\{\lambda(x), \mu(x)\},$$

for all $x \in S$.

A fuzzy subset λ of S is called a *fuzzy subsemigroup* if $\lambda(xy) \geq \min\{\lambda(x), \lambda(y)\}$ for all $x, y \in S$.

A fuzzy subset λ of S is called a *fuzzy generalized bi-ideal* [24] of S if

1. $x \leq y \longrightarrow \lambda(x) \geq \lambda(y)$,
2. $\lambda(xyz) \geq \min\{\lambda(x), \lambda(z)\}$ for all $x, y, z \in S$.

A fuzzy subset λ of S is called a *fuzzy left (resp. right)-ideal* [14] of S if

1. $x \le y \longrightarrow \lambda(x) \ge \lambda(y)$,
2. $\lambda(xy) \ge \lambda(y)(\lambda(xy) \ge \lambda(x))$ for all $x, y \in S$.

A fuzzy subset of S is called a *fuzzy ideal* if it is both a fuzzy left and right ideal of S.

A fuzzy subsemigroup λ is called a *fuzzy bi-ideal* [14] of S if

1. $x \le y \longrightarrow \lambda(x) \ge \lambda(y)$,
2. $\lambda(xyz) \ge \min\{\lambda(x), \lambda(z)\}$ for all $x, y, z \in S$.

Note that every fuzzy bi-ideal is a generalized fuzzy bi-ideal of S. But the converse is not true, as given in [24].

Let S be an ordered semigroup and λ is a fuzzy subset of S. Then, for all $t \in (0, 1]$, the set $U(\lambda; t) = \{x \in S | \lambda(x) \ge t\}$ is called a *level set* of λ.

Theorem 2.1 [14] *A fuzzy subset λ of an ordered semigroup S is a fuzzy left (resp. right)-ideal of S if and only if $U(\lambda; t)(\neq \emptyset)$ where $t \in (0, 1]$ is a left (resp. right)-ideal of S.*

Theorem 2.2 [3] *A fuzzy subset λ of an ordered semigroup S is a fuzzy generalized bi-ideal of S if and only if $U(\lambda; t)(\neq \emptyset)$ where $t \in (0, 1]$ is a generalized bi-ideal of S.*

Theorem 2.3 [14] *A non-empty subset A of an ordered semigroup S is a left (resp. right)-ideal of S if and only if*

$$\chi_A : S \longrightarrow [0, 1] | x \longmapsto \chi_A(x) = \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{if } x \notin A. \end{cases}$$

is a fuzzy left (resp. right)-ideal of S.

Theorem 2.4 [3] *A non-empty subset A of an ordered semigroup S is a generalized bi-ideal of S if and only if*

$$\chi_A : S \longrightarrow [0, 1] | x \longmapsto \chi_A(x) = \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{if } x \notin A. \end{cases}$$

is a fuzzy generalized bi-ideal of S.

If $a \in S$ and A is a non-empty subset of S, then

$A_a = \{(y, z) \in S \times S | a \le yz\}$.

If λ and μ are two fuzzy subsets of S, then the product $\lambda \circ \mu$ of λ and μ is defined by:

$$\lambda \circ \mu : S \longrightarrow [0, 1] | a \longmapsto (\lambda \circ \mu)(a)$$
$$= \begin{cases} \bigvee_{(y,z) \in A_a} (\lambda(y) \wedge \mu(z)) & \text{if } A_a \neq \emptyset, \\ 0 & \text{if } A_a = \emptyset. \end{cases}$$

Let λ be a fuzzy subset of S, then the set of the form:

$$\lambda(y) := \begin{cases} t \in (0, 1] & \text{if } y = x \\ 0 & \text{if } y \neq x, \end{cases}$$

is called a *fuzzy point* with support x and value t and is denoted by $[x; t]$. A fuzzy point $[x; t]$ is said to *belong to* (resp. *quasi-coincident* with) a fuzzy set λ, written as $[x; t] \in \lambda$ (resp. $[x; t]q\lambda$) if $\lambda(x) \ge t$ (resp. $\lambda(x) + t > 1$). If $[x; t] \in \lambda$ or $[x; t]q\lambda$, then we write $[x; t] \in \vee q\lambda$. The symbol $\overline{\in \vee q}$ means $\in \vee q$ does not hold.

Generalizing the concept of $[x; t]q\lambda$, in semigroups, Jun [9, 10] defined $[x; t]q_k\lambda$, as $\lambda(x) + t + k > 1$, where $k \in [0, 1)$.

3 $(\in, \in \vee q_k)$-fuzzy generalized bi-ideals

In what follows, let S denote an ordered semigroup unless otherwise specified. In this section, we define a more generalized form of (α, β)-fuzzy generalized bi-ideals of an ordered semigroups S and introduce $(\in, \in \vee q_k)$-fuzzy generalized bi-ideals S where $\alpha \in \{\in, q, \in \wedge q, \in \vee q\}$ and k is an arbitrary element of [0,1) unless otherwise specified.

Definition 3.1 A fuzzy subset λ of S is called an $(\in, \in \vee q_k)$-fuzzy generalized bi-ideal of S if it satisfies the conditions:

1. $(\forall x, y \in S)\, (\forall t \in (0, 1])(x \le y, [y; t] \in \lambda \longrightarrow [x; t] \in \vee q_k \lambda)$,
2. $(\forall x, y, z \in S)\, (\forall t, r \in (0, 1])([x; s] \in \lambda, [y; t] \in \lambda \longrightarrow [xyz; s \wedge t] \in \vee q_k \lambda)$.

Definition 3.2 A fuzzy subset λ which is both an $(\in, \in \vee q_k)$-fuzzy generalized bi-ideal and an $(\in, \in \vee q_k)$-fuzzy subsemigroup of S is called an $(\in, \in \vee q_k)$-*fuzzy bi-ideal* of S.

Example 3.3 Consider the ordered semigroup $S = \{a, b, c, d\}$

$\cdot$	a	b	c	d
a	a	a	a	a
b	a	a	a	a
c	a	a	b	a
d	a	a	b	b

$\le := \{(a, a), (b, b), (c, c), (d, d), (a, b)\}$.

Define a fuzzy subset λ of S as follows:

$$\lambda : S \longrightarrow [0, 1] | x \longmapsto \lambda(x) = \begin{cases} 0.8 & \text{if } x = a \\ 0.7 & \text{if } x = d \\ 0.4 & \text{if } x = c \\ 0.3 & \text{if } x = b \end{cases}$$

Then, it is easy to calculate that λ is an $(\in, \in \vee q_k)$-fuzzy bi-ideal of S for $k = 0.4$.

Theorem 3.4 *Let A be a generalized bi-ideal of S and λ a fuzzy subset in S defined by*

$$\lambda(x) = \begin{cases} \geq \frac{1-k}{2} & \text{if } x \in A \\ 0 & \text{otherwise.} \end{cases}$$

Then

1. λ is a $(q, \in \vee q_k)$-fuzzy generalized bi-ideal of S.
2. λ is an $(\in, \in \vee q_k)$-fuzzy generalized bi-ideal of S.

Proof

1. Let $x, y \in S, x \leq y$ and $t \in (0, 1]$ be such that $[y; t]\mathrm{q}\lambda$. Then $y \in A, \lambda(y) + t > 1$. Since A is a generalized bi-ideal of S and $x \leq y \in A$, we have $x \in A$. Thus $\lambda(x) \geq \frac{1-k}{2}$. If $t \leq \frac{1-k}{2}$, then λ $(x) \geq t$ and so $[x; t] \in \lambda$. If $t > \frac{1-k}{2}$, then $\lambda(x) + t + k > \frac{1-k}{2} + \frac{1-k}{2} + k = 1$ and so $[x; t]\mathrm{q}_k\lambda$. Therefore, $[x; t] \in \vee q_k\lambda$. Let $x, y, z \in S$ and $t, r \in (0, 1]$ be such that $[x; t]\mathrm{q}\lambda$ and $[z; r]\mathrm{q}\lambda$. Then $x, z \in A, \lambda(x) + t > 1$ and λ $(y) + t > 1$. Since A is a generalized bi-ideal of S, we have $xyz \in A$. Thus $\lambda(xyz) \geq \frac{1-k}{2}$. If $t \wedge r > \frac{1-k}{2}$, then $\lambda(x) + t \wedge r + k > \frac{1-k}{2} + \frac{1-k}{2} + k = 1$ and so $[x;\ t \wedge r]\mathrm{q}_k\lambda$. If $t \wedge r \leq \frac{1-k}{2}$, then λ $(x) \geq t \wedge r$ and so $[x; t \wedge r] \in \lambda$. Therefore, $[x; t \wedge r] \in \vee q_k\lambda$.
2. Let $x, y \in S, x \leq y$ and $t \in (0, 1]$ be such that $[y; t] \in \lambda$. Then λ $(y) \geq t$ and $y \in A$. Since $x \leq y \in A$, we have $x \in A$. Thus $\lambda(x) \geq \frac{1-k}{2}$. If $t \leq \frac{1-k}{2}$, then λ $(x) \geq t$ and so $[x; t] \in \lambda$. If $t > \frac{1-k}{2}$, then $\lambda(x) + t + k > \frac{1-k}{2} + \frac{1-k}{2} + k = 1$ and so $[x;\ t]\mathrm{q}_k\lambda$. Therefore, $[x; t] \in \vee q_k\lambda$. Let $x, y, z \in S$ and $t, r \in (0, 1]$ be such that $[x; t in\lambda$ and $[z; r] \in \lambda$. Then $x, z \in A$ and $xyz \in A$. Thus $\lambda(xyz) \geq \frac{1-k}{2}$. If $t \wedge r > \frac{1-k}{2}$, then $\lambda(x) + t \wedge r + k > \frac{1-k}{2} + \frac{1-k}{2} + k = 1$ and so $[x;\ t \wedge r]\mathrm{q}_k\lambda$. If $t \wedge r \leq \frac{1-k}{2}$, then λ $(x) \geq t \wedge r$ and so $[x; t \wedge r] \in \lambda$.Therefore, $[x; t \wedge r] \in \vee q_k\lambda$. Consequently, λ is an $(\in, \in \vee q_k)$-fuzzy generalized bi-ideal of S. □

If we take $k = 0$ in Theorem 3.4, then we get the following corollary:

Corollary 3.5 [3] *Let A be a generalized bi-ideal of S and λ a fuzzy subset in S defined by*

$$\lambda(x) = \begin{cases} \geq \frac{1-k}{2} & \text{if } x \in A \\ 0 & \text{otherwise.} \end{cases}$$

Then

1. λ is a $(q, \in \vee q)$-fuzzy generalized bi-ideal of S.
2. λ is an $(\in, \in \vee q)$-fuzzy generalized bi-ideal of S.

Theorem 3.6 *Let λ be a fuzzy subset of S. Then, λ is an $(\in, \in \vee q_k)$-fuzzy generalized bi-ideal of S if and only if*

1. $(\forall x, y \in S)(x \leq y \longrightarrow \lambda(x) \geq \lambda(y) \wedge \frac{1-k}{2})$,
2. $(\forall x, y, z \in S)(\lambda(xyz) \geq \lambda(x) \wedge \lambda(z) \wedge \frac{1-k}{2})$.

Proof Let λ be an $(\in, \in \vee q_k)$-fuzzy generalized bi-ideal of S. On the contrary, assume that there exist $x, y \in S$, and $x \leq y$ such that $\lambda(x) < \lambda(y) \wedge \frac{1-k}{2}$. Choose $t \in (0, 1]$ such that $\lambda(x) < t \leq \lambda(y) \wedge \frac{1-k}{2}$. Then $[y; t] \in \lambda$ but λ $(x) < t$ and $\lambda(x) + t + k < \frac{1-k}{2} + \frac{1-k}{2} + k = 1$, so $[x; t]\overline{\in \vee q_k}\lambda$, which is a contradiction. Hence $\lambda(x) \geq \lambda(y) \wedge \frac{1-k}{2}$ for all $x, y \in S$ with $x \leq y$. If there exist $x, y, z \in S$ such that $\lambda(xyz) < \lambda(x) \wedge \lambda(z) \wedge \frac{1-k}{2}$. Choose $t \in (0, 1]$ such that $\lambda(xyz) < t \leq \lambda(x) \wedge \lambda(z) \wedge \frac{1-k}{2}$. Then $[z; t] \in \lambda, [z; t] \in \lambda$ but λ $(xyz) < t$ and $\lambda(xyz) + t + k < \frac{1-k}{2} + \frac{1-k}{2} + k = 1$, so $[xyz;\ t]\mathrm{q}_k\lambda$. Thus, $[xyz; t]\overline{\in \vee q_k}\lambda$, which is a contradiction. Therefore, $\lambda(xyz) \geq \lambda(x) \wedge \lambda(z) \wedge \frac{1-k}{2}$ for all $x, y, z \in S$.

Conversely, let $[y; t] \in \lambda$ for $t \in (0, 1]$. Then $\lambda(y) \geq t$. Now, $\lambda(x) \geq \lambda(y) \wedge \frac{1-k}{2} \geq t \wedge \frac{1-k}{2}$. If $t > \frac{1-k}{2}$, then $\lambda(x) \geq \frac{1-k}{2}$ and $\lambda(x) + t + k > \frac{1-k}{2} + \frac{1-k}{2} + k = 1$, it follows that $[x; t]\mathrm{q}_k\lambda$.If $t \leq \frac{1-k}{2}$, then λ $(x) \geq t$ and so $[x; t] \in \lambda$. Thus, $[x; t] \in \vee q_k\lambda$. Let $[x; t] \in \lambda$ and $[z; r] \in \lambda$, then λ $(x) \geq t$ and λ $(z) \geq r$. Thus $\lambda(xyz) \geq \lambda(x) \wedge \lambda(z) \wedge \frac{1-k}{2} \geq t \wedge r \wedge \frac{1-k}{2}$. If $t \wedge r > \frac{1-k}{2}$, then $\lambda(xyz) \geq \frac{1-k}{2}$ and $\lambda(xyz) + t \wedge r + k > \frac{1-k}{2} + \frac{1-k}{2} + k = 1$ and so $[xyz;\ t \wedge\ r]\mathrm{q}_k\lambda$. If $t \wedge r \leq \frac{1-k}{2}$, then λ $(xyz) \geq t \wedge r$ and hence, $[xyz; t \wedge r] \in \lambda$. Thus $[xyz; t \wedge r] \in \vee q_k\lambda$ and consequently, λ is an $(\in, \in \vee q_k)$-fuzzy generalized bi-ideal of S. □

If we take $k = 0$ in Theorem 3.6, we have the following corollary.

Corollary 3.7 [3] *Let λ be a fuzzy subset of S. Then λ is an $(\in, \in \vee q)$-fuzzy generalized bi-ideal of S if and only if*

1. $(\forall x, y \in S)(x \leq y \longrightarrow \lambda(x) \geq \lambda(y) \wedge 0.5)$,
2. $(\forall x, y, z \in S)(\lambda(xyz) \geq \lambda(x) \wedge \lambda(z) \wedge 0.5)$.

Theorem 3.8 *A fuzzy subset λ of S is an $(\in, \in \vee q_k)$-fuzzy generalized bi-ideal of S if and only if $U(\lambda; t)(\neq \emptyset)$ is a generalized bi-ideal of S for all $t \in (0, \frac{1-k}{2}]$.*

Proof Assume that λ is an $(\in, \in \vee q_k)$-fuzzy generalized bi-ideal of S. Let $x, y \in S$ be such that $x \leq y \in U(\lambda; t)$ where $t \in (0, \frac{1-k}{2}]$. Then $\lambda(y) \geq t$ and by Theorem 3.6, $\lambda(x) \geq \lambda(y) \wedge \frac{1-k}{2} \geq t \wedge \frac{1-k}{2} = t$. It follows that $x \in U(\lambda; t)$. Let $x, z \in U(\lambda; t)$ and $y \in S$. Then λ $(x) \geq t$ and λ $(z) \geq t$ and by Theorem 3.6,

$$\lambda(xyz) \geq \lambda(x) \wedge \lambda(z) \wedge \frac{1-k}{2} \geq t \wedge \frac{1-k}{2} = t.$$

Thus λ $(xyz) \geq t$ and so $xyz \in U(\lambda; t)$. Therefore, $U(\lambda; t)$ is a generalized bi-ideal.

Conversely, assume that $U(\lambda; t)(\neq \emptyset)$ is a generalized bi-ideal of S for all $t \in (0, \frac{1-k}{2}]$. If there exist $x, y \in S$ with $x \leq y$ such that $\lambda(x) < \lambda(y) \wedge \frac{1-k}{2}$. Then $\lambda(x) < t \leq \lambda(y) \wedge$

$\frac{1-k}{2}$ for some $t \in (0, \frac{1-k}{2}]$. Then, $y \in U(\lambda;t)$ but $x \notin U(\lambda;t)$, a contradiction. Thus $\lambda(x) \geq \lambda(y) \wedge \frac{1-k}{2}$ for all $x, y \in S$ with $x \leq y$. If there exist $x, y, z \in S$ such that $\lambda(xyz) < \lambda(x) \wedge \lambda(z) \wedge \frac{1-k}{2}$.Then there exists $t \in (0, \frac{1-k}{2}]$ such that

$$\lambda(xyz) < t \leq \lambda(x) \wedge \lambda(z) \wedge \frac{1-k}{2}.$$

It shows that $\lambda(x) \geq t$ and $\lambda(z) \geq t$ but $\lambda(xyz) < t$ i.e., $x, y \in U(\lambda;t)$ but $xay \notin U(\lambda;t)$, a contradiction. Therefore, $\lambda(xyz) \geq \lambda(x) \wedge \lambda(y) \wedge \frac{1-k}{2}$ for all $x, y, z \in S$. Hence by Theorem 3.6, λ is an $(\in, \in \vee q_k)$-fuzzy generalized bi-ideal of S. □

Example 3.9 Consider the ordered semigroup $S = \{a, b, c, d\}$

$\cdot$	a	b	c	d
a	a	a	a	a
b	a	a	a	a
c	a	a	b	a
d	a	a	b	b

$$\leq \; := \{(a,a), (b,b), (c,c), (d,d), (a,b)\}.$$

Then $\{a\}$, $\{a, b\}$, $\{a, c\}$, $\{a, d\}$, $\{a, b, c\}$, $\{a, c, d\}$ and $\{a, b, c, d\}$ are generalized bi-ideals of S. But $\{a, c\}$, $\{a, d\}$ and $\{a, c, d\}$ are not bi-ideals of S. Define a fuzzy subset λ of S as follows:

$$\lambda : S \longrightarrow [0,1] | x \longmapsto \lambda(x) = \begin{cases} 0.8 & \text{if } x = a \\ 0.7 & \text{if } x = d \\ 0.4 & \text{if } x = c \\ 0.3 & \text{if } x = b \end{cases}$$

Then

$$U(\lambda;t) = \begin{cases} S & \text{if } 0 < t \leq 0.3 \\ \{a,c,d\} & \text{if } 0.3 < t \leq 0.4 \\ \{a,d\} & \text{if } 0.4 < t \leq 0.7 \\ \emptyset & \text{if } 0.8 < t \leq 1 \end{cases}$$

Then, by Theorem 3.8, λ is an $(\in, \in \vee q_k)$-fuzzy generalized bi-ideal of S for $k \in (0, \frac{1-k}{2}]$ with $k = 0.4$.

Note that every $(\in, \in \vee q_k)$-fuzzy bi-ideal of S is an $(\in, \in \vee q_k)$-fuzzy generalized bi-ideal of S. However, the converse is not true, in general, as shown in the following example:

Example 3.10 Consider the ordered semigroup as shown in Example 3.9, and define a fuzzy subset λ as follows:

$$\lambda : S \longrightarrow [0,1] | x \longmapsto \lambda(x) = \begin{cases} 0.8 & \text{if } x = a \\ 0 & \text{if } x = b \\ 0.7 & \text{if } x = c \\ 0 & \text{if } x = d \end{cases}$$

Then, we have

$$U(\lambda;t) = \begin{cases} \{a,c\} & \text{if } 0 < t \leq 0.7 \\ \{a\} & \text{if } 0.7 < t \leq 0.8 \\ \emptyset & \text{if } 0.8 < t \leq 1 \end{cases}$$

Then by Theorem 3.8, λ is an $(\in, \in \vee q_k)$-fuzzy generalized bi-ideal of S for every $k \in [0, 1)$ but λ is not an $(\in, \in \vee q_k)$-fuzzy bi-ideal of S, because $U(\lambda;\ 0.6) = \{a,c\}$ is a generalized bi-ideal but not a bi-ideal of S.

Proposition 3.11 *Every $(\in, \in \vee q_k)$-fuzzy generalized bi-ideal of a regular ordered semigroup S is an $(\in, \in \vee q_k)$-fuzzy bi-ideal of S.*

Proof Let $a, b \in S$ and λ an $(\in, \in \vee q_k)$-fuzzy generalized bi-ideal of S. Since S is regular, there exists $x \in S$ such that $b \leq bxb$. Then

$$\lambda(ab) \geq \lambda(a(bxb) = \lambda(a(bx)b) \geq \left\{\lambda(a) \wedge \lambda(b) \wedge \frac{1-k}{2}\right\}.$$

This means that λ is an $(\in, \in \vee q_k)$-fuzzy subsemigroup of S. Thus, λ is an $(\in, \in \vee q_k)$-fuzzy bi-ideal of S. □

Proposition 3.12 *Every $(\in, \in \vee q_k)$-fuzzy generalized bi-ideal of a left weakly regular ordered semigroup S is an $(\in, \in \vee q_k)$-fuzzy bi-ideal of S.*

Proof Let $a, b \in S$ and λ an $(\in, \in \vee q_k)$-fuzzy generalized bi-ideal of S. Since S is left weakly regular, there exist $x, y \in S$ such that $b \leq xbyb$. Then

$$\lambda(ab) \geq \lambda(a(xbyb) = \lambda(a(xby)b) \geq \left\{\lambda(a) \wedge \lambda(b) \wedge \frac{1-k}{2}\right\}.$$

This means that λ is an $(\in, \in \vee q_k)$-fuzzy subsemigroup of S. Thus λ is an $(\in, \in \vee q_k)$-fuzzy bi-ideal of S. □

Remark 3.13 From Proposition 3.11 and 3.12, it follows that in regular and left weakly regular ordered semigroups, the concepts of $(\in, \in \vee q_k)$-fuzzy generalized bi-ideals and $(\in, \in \vee q_k)$-fuzzy bi-ideals coincide.

Proposition 3.14 *If λ is a nonzero $(\in, \in \vee q_k)$-fuzzy generalized bi-ideal of S. Then the set $\lambda_0 = \{x \in S | \lambda(x) > 0\}$ is a generalized bi-ideal of S.*

Proof Let λ be an $(\in, \in \vee q_k)$-fuzzy generalized bi-ideal of S. Let $x, y \in S, x \leq y$ and $y \in \lambda_0$. Then, $\lambda(y) > 0$. Since λ is an $(\in, \in \vee q_k)$-fuzzy generalized bi-ideal of S, we have

$$\lambda(x) \geq \lambda(y) \wedge \frac{1-k}{2} > 0, \text{ because } \lambda(y) > 0.$$

Thus $\lambda(x) > 0$ and so $x \in \lambda_0$. Let $x, z \in \lambda_0$ and $y \in S$. Then, $\lambda(x) > 0$ and $\lambda(z) > 0$. Now,

$$\lambda(xyz) \geq \lambda(x) \wedge \lambda(z) \wedge \frac{1-k}{2} > 0, \text{ because } \lambda(x) > 0 \text{ and } \lambda(z) > 0.$$

Thus $xyz \in \lambda_0$ and consequently, λ_0 is a generalized bi-ideal of S. □

Lemma 3.15 *A non-empty subset A of S is a generalized bi-ideal if and only if the characteristic function χ_A of A is an $(\in, \in \vee q_k)$-fuzzy generalized bi-ideal of S.*

Proof The proof is straightforward. □

Definition 3.16 A fuzzy subset λ of S is called a *$(\in, \in \vee q_k)$-fuzzy left (resp. right) ideal* of S if it satisfies the conditions:

1. $(\forall x, y \in S, \forall t \in (0,1]$ such that $x \leq y)$ $([y; t] \in \lambda \longrightarrow [x; t] \in \vee q_k \lambda)$,
2. $(\forall x, y \in S)(\forall t \in (0,1])([y; t] \in \lambda \longrightarrow$ $[(xy); t] \in \vee q_k \lambda)(\text{resp.}[(yx); t] \in \vee q_k \lambda)$.

A fuzzy subset λ is called $(\in, \in \vee q_k)$-*fuzzy ideal*if it is both an $(\in, \in \vee q_k)$-fuzzy left and right ideal of S.

Theorem 3.17 *Let A be a left (resp. right)-ideal of S and λ a fuzzy subset in S defined by*

$$\lambda(x) = \begin{cases} \geq \frac{1-k}{2} & \text{if } x \in A \\ 0 & \text{otherwise.} \end{cases}$$

Then

1. λ is a $(q, \in \vee q_k)$-fuzzy left (resp. right)-ideal of S.
2. λ is an $(\in, \in \vee q_k)$-fuzzy left (resp. right)-ideal of S.

Proof The proof follows from Theorem 3.4. □

Theorem 3.18 *Let λ be a fuzzy subset of S. Then λ is an $(\in, \in \vee q_k)$-fuzzy left (resp. right)-ideal of S if and only if*

1. $(\forall x, y \in S)(x \leq y \longrightarrow \lambda(x) \geq \lambda(y) \wedge \frac{1-k}{2})$,
2. $(\forall x, y \in S)(\lambda(xy) \geq \lambda(y) \wedge \frac{1-k}{2}$ (resp.$\lambda(xy) \geq \lambda(x)$ $\wedge \frac{1-k}{2}))$.

Proof The proof follows from Theorem 3.6. □

Theorem 3.19 *A fuzzy subset λ of S is an $(\in, \in \vee q_k)$-fuzzy left (resp. right)-ideal of S if and only if $U(\lambda; t)$ $(\neq \emptyset)$ is a left (resp. right)-ideal of S for all $t \in (0, \frac{1-k}{2}]$.*

Proof The proof follows from Theorem 3.8. □

4 Upper and lower parts of $(\in, \in \vee q_k)$-fuzzy generalized bi-ideals

In this section, we define the upper/lower parts of an $(\in, \in \vee q_k)$-fuzzy generalized bi-ideal and characterize regular and left weakly regular ordered semigroups in terms of $(\in, \in \vee q_k)$-fuzzy generalized bi-ideals and $(\in, \in \vee q_k)$-fuzzy left (resp. right)-ideals.

Definition 4.1 Let λ and μ be a fuzzy subsets of S. Then, the fuzzy subsets $\overline{\lambda}^k$, $(\lambda \wedge^k \mu)^-$, $(\lambda \vee^k \mu)^-$, and $(\lambda \circ^k \mu)^-$ of S as follows:

$$\overline{\lambda}^k : S \longrightarrow [0,1] | x \longmapsto \lambda^k(x) = \lambda(x) \wedge \frac{1-k}{2},$$

$$(\lambda \wedge^k \mu)^- : S \longrightarrow [0,1] | x \longmapsto (\lambda \wedge^k \mu)(x) = (\lambda \wedge \mu)(x) \wedge \frac{1-k}{2},$$

$$(\lambda \vee^k \mu)^- : S \longrightarrow [0,1] | x \longmapsto (\lambda \vee^k \mu)(x) = (\lambda \vee \mu)(x) \wedge \frac{1-k}{2},$$

$$(\lambda \circ^k \mu)^- : S \longrightarrow [0,1] | x \longmapsto (\lambda \circ^k \mu)(x) = (\lambda \circ \mu)(x) \wedge \frac{1-k}{2},$$

for all $x \in S$.

Lemma 4.2 *Let λ and μ be fuzzy subsets of S. Then the following hold:*

1. $\left(\lambda \wedge^k \mu\right)^- = \left(\overline{\lambda}^k \wedge \overline{\mu}^k\right)$,
2. $\left(\lambda \vee^k \mu\right)^- = \left(\overline{\lambda}^k \vee \overline{\mu}^k\right)$,
3. $\left(\lambda \circ^k \mu\right)^- = \left(\overline{\lambda}^k \circ \overline{\mu}^k\right)$.

Proof The proof follows from [26]. □

Let A be a non-empty subset of S, then the upper and lower parts of the characteristic function χ_A^k are defined as follows:

$$\overline{\chi}_A^k : S \longrightarrow [0,1] | x \longmapsto \overline{\chi}_A^k(x) = \begin{cases} \frac{1-k}{2} & \text{if } x \in A \\ 0 & \text{otherwise,} \end{cases}$$

Lemma 4.3 *Let A and B be non-empty subset of S. Then the following hold*:

1. $(\chi_A \wedge^k \chi_B)^- = \overline{\chi}_{A \cap B}^k$,
2. $(\chi_A \vee^k \chi_B)^- = \overline{\chi}_{A \cup B}^k$,
3. $(\chi_A \circ^k \chi_B)^- = \overline{\chi}_{(AB]}^k$.

Proof The proofs of (1) and (2) are obvious.

(3) Let $x \in (AB]$. Then $\chi_{(AB]}(x) = 1$ and hence $\overline{\chi}_{(AB]}^k(x) = 1 \wedge \frac{1-k}{2} = \frac{1-k}{2}$. Since $x \in (AB]$, we have $x \leq ab$ for some $a \in A$ and $b \in B$. Then $(a, b) \in A_x$ and $A_x \neq \emptyset$. Thus

$$\begin{aligned}
(\chi_A \circ^k \chi_B)^-(x) &= (\chi_A \circ \chi_B)(x) \wedge \frac{1-k}{2} \\
&= \left[\bigvee_{(y,z) \in A_x} (\chi_A(y) \wedge \chi_B(z))\right] \wedge \frac{1-k}{2} \\
&\geq (\chi_A(a) \wedge \chi_B(b)) \wedge \frac{1-k}{2}.
\end{aligned}$$

Since $a \in A$ and $b \in B$, we have $\chi_A(a) = 1$ and $\chi_B(b) = 1$ and so

$$(\chi_A \circ^k \chi_B)^-(x) \geq (\chi_A(a) \wedge \chi_B(b)) \wedge \frac{1-k}{2}$$
$$= (1 \wedge 1) \wedge \frac{1-k}{2} = \frac{1-k}{2}.$$

Thus, $(\chi_A \circ^k \chi_B)^-(x) = \frac{1-k}{2} = \overline{\chi}^k_{(AB]}(x)$. Let $x \notin (AB]$, then $\chi_{(AB]}(x) = 0$ and hence, $\overline{\chi}^k_{(AB]}(x) = 0 \wedge \frac{1-k}{2} = 0$. Let $(y, z) \in A_x$. Then

$$(\chi_A \circ^k \chi_B)^-(x) = (\chi_A \circ \chi_B)(x) \wedge \frac{1-k}{2}$$
$$= \left[\bigvee_{(y,z)\in A_x} (\chi_A(y) \wedge \chi_B(z))\right] \wedge \frac{1-k}{2}$$

Since $(y, z) \in A_x$, then $x \leq yz$. If $y \in A$ and $z \in B$, then $yz \in AB$ and so $x \in (AB]$. This is a contradiction. If $y \notin A$ and $z \in B$, then

$$\left[\bigvee_{(y,z)\in A_x} (\chi_A(y) \wedge \chi_B(z))\right] \wedge \frac{1-k}{2}$$
$$= \left[\bigvee_{(y,z)\in A_x} (0 \wedge 1)\right] \wedge \frac{1-k}{2} = 0.$$

Hence, $\overline{\chi}^k_{(AB]}(x) = 0 = (\chi_A \circ^k \chi_B)^-(x)$. Similarly, for $y \in A$ and $z \notin B$, we have $\overline{\chi}^k_{(AB]}(x) = 0 = (\chi_A \circ^k \chi_B)^-(x)$. □

Lemma 4.4 *The lower part $\overline{\chi}^k_A$ of the characteristic function χ_A of A is an $(\in, \in \vee q_k)$-fuzzy generalized bi-ideal of S if and only if A is a generalized bi-ideal of S.*

Proof Let A be a generalized bi-ideal of S. Then, by Theorems 2.3 and 3.15, $\overline{\chi}^k_A$ is an $(\in, \in \vee q_k)$-fuzzy generalized bi-ideal of S. Conversely, assume that $\overline{\chi}^k_A$ is an $(\in, \in \vee q_k)$-fuzzy generalized bi-ideal of S. Let $x, y \in S$, $x \leq y$. If $y \in A$, then $\overline{\chi}^k_A(y) = \frac{1-k}{2}$. Since $\overline{\chi}^k_A$ is an $(\in, \in \vee q_k)$-fuzzy generalized bi-ideal of S, and $x \leq y$, we have, $\overline{\chi}^k_A(x) \geq \overline{\chi}^k_A(y) \wedge \frac{1-k}{2} = \frac{1-k}{2}$. It follows that $\overline{\chi}^k_A(x) = \frac{1-k}{2}$ and so $x \in A$. Let $x, z \in A$ and $y \in S$. Then, $\overline{\chi}^k_A(x) = \frac{1-k}{2}$ and $\overline{\chi}^k_A(z) = \frac{1-k}{2}$. Now,

$$\overline{\chi}^k_A(xyz) \geq \overline{\chi}^k_A(x) \wedge \overline{\chi}^k_A(z) \wedge \frac{1-k}{2} = \frac{1-k}{2}.$$

Hence $\overline{\chi}^k_A(xyz) = \frac{1-k}{2}$ and so $xyz \in A$. Therefore, A is a generalized bi-ideal of S. □

Lemma 4.5 *The lower part $\overline{\chi}^k_A$ of the characteristic function χ_A of A is an $(\in, \in \vee q_k)$-fuzzy left (resp. right)-ideal of S if and only if A is a left (resp. right)-ideal of S.*

Proof The proof follows from Lemma 4.4. □

In the following Proposition, we show that if λ is an $(\in, \in \vee q_k)$-fuzzy generalized bi-ideal of S, then $\overline{\lambda}^k$ is a fuzzy generalized bi-ideal of S.

Proposition 4.6 *If λ is an $(\in, \in \vee q_k)$-fuzzy generalized bi-ideal of S, then $\overline{\lambda}^k$ is a fuzzy generalized bi-ideal of S.*

Proof Let $x, y \in S$, $x \leq y$. Since λ is an $(\in, \in \vee q_k)$-fuzzy generalized bi-ideal of S and $x \leq y$, we have $\lambda(x) \geq \lambda(y) \wedge \frac{1-k}{2}$. It follows that $\lambda(x) \wedge \frac{1-k}{2} \geq \lambda(y) \wedge \frac{1-k}{2}$, and hence $\overline{\lambda}^k(x) \geq \overline{\lambda}^k(x)$. For $x, y, z \in S$, we have $\lambda(xyz) \geq \lambda(x) \wedge \lambda(z) \wedge \frac{1-k}{2}$. Then $\lambda(xyz) \wedge \frac{1-k}{2} \geq \lambda(x) \wedge \lambda(z) \wedge \frac{1-k}{2} = \left(\lambda(x) \wedge \frac{1-k}{2}\right) \wedge \left(\lambda(z) \wedge \frac{1-k}{2}\right)$, and so $\overline{\lambda}^k(xyz) \geq \overline{\lambda}^k(x) \wedge \overline{\lambda}^k(z)$. Consequently, $\overline{\lambda}^k$ is a fuzzy generalized bi-ideal of S. □

In [24], regular and left weakly regular ordered semigroups are characterized by the properties of their fuzzy left (resp. right) and fuzzy generalized bi-ideals. In the following, we characterize regular, left weakly regular, left and right simple and completely regular ordered semigroups in terms of $(\in, \in \vee q_k)$-fuzzy left (resp. right) and $(\in, \in \vee q_k)$-fuzzy generalized bi-ideals.

Lemma 4.7 [14] *An ordered semigroup S is left (resp. right) simple if and only if $(Sa] = S$ (resp. $(aS] = S$) for every $a \in S$.*

Proposition 4.8 *If S is regular, left and right simple ordered semigroup, then for every $(\in, \in \vee q_k)$-fuzzy generalized bi-ideal λ of S we have $\overline{\lambda}^k(a) = \overline{\lambda}^k(b)$, for every $a, b \in S$.*

Proof Assume that S is regular, left and right simple and λ an $(\in, \in \vee q_k)$-fuzzy generalized bi-ideal of S. We consider, $E_S = \{e \in S | e \leq e^2\}$, then $E_S \neq \emptyset$, because S is regular, hence for every $a \in S$, there exists $x \in S$ such that $a \leq axa$, then $ax \leq (axa)x = (ax)(ax) = (ax)^2$ and so $ax \in E_S$. Let $b, e \in E_S$. Since S is left and right simple, by Lemma 4.7, it follows that $S = (Sb]$ and $S = (bS]$. Since $e \in S$, we have $e \in (Sb]$ and $e \in (bS]$, then $e \leq xb$ and $e \leq by$ for some $x, y \in S$, and we have

$$e^2 = (by)(xb) \leq b(yx)b.$$

Since λ is an $(\in, \in \vee q_k)$-fuzzy generalized bi-ideal of S, we have

$$\lambda(e^2) \geq \lambda(b(yx)b) \wedge \frac{1-k}{2}$$
$$\geq \left(\lambda(b) \wedge \lambda(b) \wedge \frac{1-k}{2}\right) \wedge \frac{1-k}{2}$$
$$= \left(\lambda(b) \wedge \frac{1-k}{2}\right)$$

Thus $\lambda(e^2) \wedge \frac{1-k}{2} \geq \lambda(b) \wedge \frac{1-k}{2}$ and we have,

$\overline{\lambda}^k(e^2) \geq \overline{\lambda}^k(b)$ $(*)$.

Since $e \in E_S$, we have $e \leq e^2$ and so $\lambda(e) \geq \lambda(e^2) \wedge \frac{1-k}{2}$, it follows that $\lambda(e) \wedge \frac{1-k}{2} \geq \lambda(e^2) \wedge \frac{1-k}{2}$, and so $\overline{\lambda}^k(e) \geq \overline{\lambda}^k(e^2)$. Thus, by $(*)$, we have $\overline{\lambda}^k(e) \geq \overline{\lambda}^k(b)$. On the other hand, since $e \in S$, by Lemma 4.7, we have $(Se] = S = (eS]$. Since $b \in S$, we have $b \in (Se]$ and $b \in (eS]$, then $b \leq se$ and $b \leq te$ for some $s, t \in S$. Thus, by the same arguments as above, we get $\overline{\lambda}^k(b) \geq \overline{\lambda}^k(e)$. It follows that $\overline{\lambda}^k(b) = \overline{\lambda}^k(e)$ and hence $\overline{\lambda}^k$ is constant on E_S.

Now, let $a \in S$, then there exists $x \in S$ such that $a \leq axa$. It follows that $ax \leq (axa)x = (ax)(ax) = (ax)^2$ and $xa \leq x(axa) = xa)(xa) = (xa)^2$. Thus, $ax, xa \in E_S$, it follows by the previous arguments, $\overline{\lambda}^k(ax) = \overline{\lambda}^k(b) = \overline{\lambda}^k(xa)$. Since $(ax)a(xa) = (axa)xa \geq axa \geq a$, we have,

$$\begin{aligned}\lambda(a) &\geq \lambda((ax)a(xa)) \wedge \frac{1-k}{2}\\ &\geq \left(\lambda(ax) \wedge \lambda(xa) \wedge \frac{1-k}{2}\right) \wedge \frac{1-k}{2}\\ &= \left(\lambda(ax) \wedge \frac{1-k}{2}\right) \wedge \left(\lambda(xa) \wedge \frac{1-k}{2}\right)\end{aligned}$$

Thus $\lambda(a) \wedge \frac{1-k}{2} \geq \left(\lambda(ax) \wedge \frac{1-k}{2}\right) \wedge \left(\lambda(xa) \wedge \frac{1-k}{2}\right)$ and we have $\overline{\lambda}^k(a) \geq \overline{\lambda}^k(ax) \wedge \overline{\lambda}^k(xa) = \overline{\lambda}^k(b)$. Since $b \in (Sa]$ and $b \in (aS]$, we have $b \leq pa$ and $b \leq aq$ for some $p, q \in S$. Then $b^2 \leq (aq)(pa) = a(qp)a$ and we have

$$\begin{aligned}\lambda(b^2) &\geq \lambda(a(qp)a) \wedge \frac{1-k}{2}\\ &\geq \left(\lambda(a) \wedge \lambda(a) \wedge \frac{1-k}{2}\right) \wedge \frac{1-k}{2}\\ &= \left(\lambda(a) \wedge \frac{1-k}{2}\right),\end{aligned}$$

hence, $\lambda(b^2) \wedge \frac{1-k}{2} \geq \lambda(a) \wedge \frac{1-k}{2}$ and we have, $\overline{\lambda}^k(b^2) \geq \overline{\lambda}^k(a)$. Since $b \in E_S$, we have, $b^2 \geq b$, then $\lambda(b) \geq \lambda(b^2) \wedge \frac{1-k}{2}$ and so $\lambda(b) \wedge \frac{1-k}{2} \geq \lambda(b^2) \wedge \frac{1-k}{2}$, it follows that $\overline{\lambda}^k(b) \geq \overline{\lambda}^k(b^2)$ and so $\overline{\lambda}^k(b) \geq \overline{\lambda}^k(a)$. Thus, $\overline{\lambda}^k(b) = \overline{\lambda}^k(a)$ and so, $\overline{\lambda}^k$ is a constant function on S. □

Lemma 4.9 [14] *An ordered semigroup S is completely regular if and only if for every $A \subseteq S$, we have, $A \subseteq (A^2SA^2]$.*

Theorem 4.10 *An ordered semigroup S is completely regular if and only if for every $(\in, \in \vee q_k)$-fuzzy generalized bi-ideal of S, we have*

$$\overline{\lambda}^k(a) = \overline{\lambda}^k(a^2) \quad \text{for every } a \in S.$$

Proof $\longrightarrow$ Let $a \in S$. Since S is completely regular, by Lemma 4.9, $a \in (a^2Sa^2]$. Then, there exists $x \in S$, such that $a \leq a^2xa^2$. Since λ is an $(\in, \in \vee q_k)$-fuzzy generalized bi-ideal of S, we have

$$\begin{aligned}\lambda(a) &\geq \lambda(a^2xa^2) \wedge \frac{1-k}{2}\\ &\geq \left(\lambda(a^2) \wedge \lambda(a^2) \wedge \frac{1-k}{2}\right) \wedge \frac{1-k}{2}\\ &= \left(\lambda(a^2) \wedge \frac{1-k}{2}\right)\\ &\geq \left(\lambda(a) \wedge \lambda(a) \wedge \frac{1-k}{2}\right) \wedge \frac{1-k}{2}\\ &= \left(\lambda(a) \wedge \frac{1-k}{2}\right).\end{aligned}$$

Thus, $\lambda(a) \wedge \frac{1-k}{2} \geq \lambda(a^2) \wedge \frac{1-k}{2} \geq \lambda(a) \wedge \frac{1-k}{2}$, and it follows that $\overline{\lambda}^k(a) \geq \overline{\lambda}^k(a^2) \geq \overline{\lambda}^k(a)$. Thus $\overline{\lambda}^k(a) = \overline{\lambda}^k(a^2)$ for every $a \in S$.

$\Longleftarrow$ Let $a \in S$. We consider the generalized bi-ideal $B(a^2) = (a^2 \cup a^2Sa^2]$ of S, generated by $a^2 (a \in S)$. Then, by Lemma 4.4,

$$\overline{\chi}^k_{B(a^2)}(a) = \begin{cases} \frac{1-k}{2} & \text{if } a \in B(a^2)\\ 0 & \text{otherwise}\end{cases}$$

is an $(\in, \in \vee q_k)$-fuzzy generalized bi-ideal of S. By hypothesis, we have

$$\overline{\chi}^k_{B(a^2)}(a^2) = \overline{\chi}^k_{B(a^2)}(a).$$

Since $a^2 \in B(a^2)$, we have, $\overline{\chi}^k_{B(a^2)}(a^2) = \frac{1-k}{2}$ and hence, $\overline{\chi}^k_{B(a^2)}(a) = \frac{1-k}{2}$. Thus, $a \in B(a^2)$ and hence, $a \leq a^2$ or $a \leq a^2xa^2$. If $a \leq a^2$, then $a \leq a^2 = aa \leq a^2a^2 = aaa^2 \leq a^2aa^2 \in a^2Sa^2$ and $a \in (a^2Sa^2]$. If $a \leq a^2xa^2$, then $a \in a^2Sa^2$ and $a \in (a^2Sa^2]$. Thus, S is completely regular. □

Lemma 4.11 [24] *Let S be an ordered semigroup, then the following conditions are equivalent*:

1. S is regular,
2. $B \cap L \subseteq (BL]$ for every generalized bi-ideal B and left ideal L of S,
3. $B(a) \cap L(a) \subseteq (B(a)L(a)]$ for every $a \in S$.

Theorem 4.12 *An ordered semigroup S is regular if and only if for every $(\in, \in \vee q_k)$-fuzzy generalized bi-ideal λ and $(\in, \in \vee q_k)$-fuzzy left ideal μ of S, we have*

$$\left(\lambda \wedge^k \mu\right)^- \preceq \left(\lambda \circ^k \mu\right)^-.$$

Proof Suppose that λ is an $(\in, \in \vee q_k)$-fuzzy generalized bi-ideal and μ an $(\in, \in \vee q_k)$-fuzzy left ideal of a regular ordered semigroup S. Let $a \in S$. Since S is regular, there exits $x \in S$ such that $a \leq axa \leq (axa)(xa)$. Then $(axa, xa) \in A_a$ and $A_a \neq \emptyset$. Thus,

$$\begin{aligned}\left(\lambda \circ^k \mu\right)^-(a) &= (\lambda \circ \mu)(a) \wedge \frac{1-k}{2}\\ &= \left[\bigvee_{(y,z)\in A_a} (\lambda(y) \wedge \mu(z))\right] \wedge \frac{1-k}{2}\\ &\geq (\lambda(axa) \wedge \mu(xa)) \wedge \frac{1-k}{2}.\end{aligned}$$

Since λ is an $(\in, \in \vee q_k)$-fuzzy generalized bi-ideal and μ an $(\in, \in \vee q_k)$-fuzzy left ideal of S, we have $\lambda(axa) \geq \lambda(a) \wedge \lambda(a) \wedge \frac{1-k}{2} = \lambda(a) \wedge \frac{1-k}{2}$ and $\mu(xa) \geq \mu(a) \wedge \frac{1-k}{2}$. Therefore,

$$\begin{aligned}\left[\lambda(axa) \wedge \mu(xa) \wedge \frac{1-k}{2}\right] &\geq \left(\lambda(a) \wedge \frac{1-k}{2}\right) \wedge \left(\mu(a) \wedge \frac{1-k}{2}\right)\\ &= \lambda^-(a) \wedge^k \mu^-(a)\end{aligned}$$

Thus $\left(\lambda \circ^k \mu\right)^-(a) \geq \left(\lambda \wedge^k \mu\right)^-(a)$.

Conversely, assume that $\left(\lambda \wedge^k \mu\right)^- \preceq \left(\lambda \circ^k \mu\right)^-$ for every $(\in, \in \vee q_k)$-fuzzy generalized bi-ideal λ and every $(\in, \in \vee q_k)$-fuzzy left ideal μ of S. To prove that S is regular, by Lemma 4.11, it is enough to prove that

$B \cap L \subseteq (BL]$ for generalized bi-ideal B and left ideal L of S.

Let $x \in B \cap L$. Then $x \in B$ and $x \in L$. Since B is a generalized bi-ideal and L a left ideal of S, by Lemma 4.4 and 4.5, $\overline{\chi}_B^k$ is an $(\in, \in \vee q_k)$-fuzzy generalized bi-ideal and $\overline{\chi}_L^k$ an $(\in, \in \vee q_k)$-fuzzy left ideal of S. By hypothesis, $(\chi_B \circ^k \chi_L)^-(x) \geq (\chi_B \wedge^k \chi_L)^-(x) = (\chi_B^k \wedge \chi_L^k)(x) \wedge \frac{1-k}{2}$. Since $x \in B$ and $x \in L$, we have $\overline{\chi}_B^k(x) = \frac{1-k}{2}$ and $\overline{\chi}_L^k(x) = \frac{1-k}{2}$. Thus $(\chi_B^k \wedge \chi_L^k)(x) \wedge \frac{1-k}{2} = \chi_B^k(x) \wedge \chi_L^k(x) \wedge \frac{1-k}{2} = \frac{1-k}{2}$. It follows that $(\chi_B \circ^k \chi_L)^-(x) = \frac{1-k}{2}$. By using Lemma 4.3 (3), we have $(\chi_B \circ^k \chi_L)^- = \overline{\chi}_{(BL]}^k$. Therefore, $\overline{\chi}_{(BL]}^k(x) = \frac{1-k}{2}$ and so $x \in (BL]$. Consequently, S is regular. □

Lemma 4.13 [24] *Let S be an ordered semigroup, then the following conditions are equivalent*:

1. S is regular,
2. $B \cap I = (BIB]$ for every generalized bi-ideal B and ideal I of S,
3. $B(a) \cap I(a) = (B(a)I(a)B(a)]$ for every $a \in S$.

Theorem 4.14 *An ordered semigroup S is regular if and only if for every $(\in, \in \vee q_k)$-fuzzy generalized bi-ideal λ and every $(\in, \in \vee q_k)$-fuzzy ideal μ of S, we have*

$\left(\lambda \wedge^k \mu\right)^- \preceq \left(\lambda \circ^k \mu \circ^k \lambda\right)^-$.

Proof Suppose that λ is an $(\in, \in \vee q_k)$-fuzzy generalized bi-ideal and μ an $(\in, \in \vee q_k)$-fuzzy left ideal of a regular ordered semigroup S. Let $a \in S$. Since S is regular, there exits $x \in S$ such that $a \leq axa \leq (axa)(xa) = a(xaxa)$. Then $(a, xaxa) \in A_a$ and $A_a \neq \emptyset$. Thus,

$$\begin{aligned}\left(\lambda \circ^k \mu \circ^k \lambda\right)^-(a) &= (\lambda \circ^k \mu \circ \lambda)(a) \wedge \frac{1-k}{2}\\ &= \left[\bigvee_{(y,z)\in A_a} (\lambda \circ^k \mu)(y) \wedge \lambda(z)\right] \wedge \frac{1-k}{2}\\ &= \bigvee_{(y,z)\in A_a} \left[\bigvee_{(p,q)\in A_a} \left((\lambda(p) \wedge \mu(q)) \wedge \frac{1-k}{2}\right) \wedge \lambda(z)\right] \wedge \frac{1-k}{2}\\ &= \bigvee_{(y,z)\in A_a} \bigvee_{(p,q)\in A_a} ((\lambda(p) \wedge \mu(q)) \wedge \lambda(z)) \wedge \frac{1-k}{2}\\ &= \bigvee_{(y,z)\in A_a} \bigvee_{(p,q)\in A_a} (\lambda(p) \wedge \mu(q) \wedge \lambda(z)) \wedge \frac{1-k}{2}\\ &\geq (\lambda(a) \wedge \mu(xax) \wedge \lambda(a)) \wedge \frac{1-k}{2}.\end{aligned}$$

Since μ an $(\in, \in \vee q_k)$-fuzzy ideal of S, we have $\mu(xax) \geq \mu(ax) \wedge \frac{1-k}{2} \geq \left(\mu(a) \wedge \frac{1-k}{2}\right) \wedge \frac{1-k}{2}$. Therefore,

$$\begin{aligned}&\left[\lambda(a) \wedge \mu(xax) \wedge \lambda(a) \wedge \frac{1-k}{2}\right]\\ &\geq \left(\lambda(a) \wedge \frac{1-k}{2}\right) \wedge \left(\mu(a) \wedge \frac{1-k}{2}\right)\\ &= \lambda^-(a) \wedge^k \mu^-(a)\end{aligned}$$

Thus $\left(\lambda \circ^k \mu \circ^k \lambda\right)^-(a) \geq \left(\lambda \wedge^k \mu\right)^-(a)$.

Conversely, assume that $\left(\lambda \wedge^k \mu\right)^- \preceq \left(\lambda \circ^k \mu \circ^k \lambda\right)^-$ for every $(\in, \in \vee q_k)$-fuzzy generalized bi-ideal λ and every $(\in, \in \vee q_k)$-fuzzy ideal μ of S. To prove that S is regular, by Lemma 4.13, it is enough to prove that

$B \cap I \subseteq (BIB]$ for generalized bi-ideal B and ideal I of S.

Let $x \in B \cap I$. Then $x \in B$ and $x \in I$. Since B is a generalized bi-ideal and I an ideal of S, by Lemma 4.4 and 4.5, $\overline{\chi}_B^k$ is an $(\in, \in \vee q_k)$-fuzzy generalized bi-ideal and $\overline{\chi}_I^k$ an $(\in, \in \vee q_k)$-fuzzy ideal of S. By hypothesis, $(\chi_B \circ^k \chi_I \circ^k \chi_B)^-(x) \geq (\chi_B \wedge^k \chi_I)^-(x) = (\chi_B^k \wedge \chi_I^k)(x) \wedge \frac{1-k}{2}$. Since $x \in B$ and $x \in I$, we have $\overline{\chi}_B^k(x) = \frac{1-k}{2}$ and $\overline{\chi}_I^k(x) = \frac{1-k}{2}$. Thus $(\chi_B^k \wedge \chi_I^k)(x) \wedge \frac{1-k}{2} = \chi_B^k(x) \wedge \chi_I^k(x) \wedge \frac{1-k}{2} = \frac{1-k}{2}$. It follows that $(\chi_B \circ^k \chi_I \circ^k \chi_B)^-(x) = \frac{1-k}{2}$. By using Lemma 4.3 (3), we have $(\chi_B \circ^k \chi_I \circ^k \chi_B)^- = \overline{\chi}_{(BIB]}^k$. Therefore, $\overline{\chi}_{(BIB]}^k(x) = \frac{1-k}{2}$ and so $x \in (BIB]$. Consequently, S is regular. □

Lemma 4.15 [24] *Let S be an ordered semigroup, then the following conditions are equivalent*:

1. S is regular,
2. $R \cap B \cap L \subseteq (RBL]$ for every right ideal R, generalized bi-ideal B and left ideal L of S,
3. $R(a) \cap B(a) \cap L(a) \subseteq (R(a)B(a)L(a)]$ for every $a \in S$.

Theorem 4.16 *An ordered semigroup S is regular if and only if for every $(\in, \in \vee q_k)$-fuzzy generalized bi-ideal μ, every $(\in, \in \vee q_k)$-fuzzy right ideal λ and every $(\in, \in \vee q_k)$-fuzzy left ideal ρ of S, we have,*

$(\lambda \wedge \mu \wedge \rho)^{-} \preceq (\lambda \circ \mu \circ \rho)^{-}$.

Proof Let S be a regular ordered semigroup, μ an $(\in, \in \vee q_k)$-fuzzy generalized bi-ideal, λ an $(\in, \in \vee q_k)$-fuzzy right ideal and ρ an $(\in, \in \vee q_k)$-fuzzy left ideal of S. Let $a \in S$. Since S is regular, there exits $x \in S$ such that $a \leq axa = axa \leq (axa)(xa) \leq (axa)x(axa) = (ax)(axa)(xa)$. Then $((ax)(axa), xa) \in A_a$ and $A_a \neq \emptyset$. Thus,

$$\begin{aligned}
\left(\lambda \circ^k \mu \circ^k \rho\right)^{-}(a) &= (\lambda \circ^k \mu \circ \rho)(a) \wedge \frac{1-k}{2} \\
&= \left[\bigvee_{(y,z)\in A_a} (\lambda \circ^k \mu)(y) \wedge \rho(z)\right] \wedge \frac{1-k}{2} \\
&= \bigvee_{(y,z)\in A_a}\left[\bigvee_{(p,q)\in A_a}\left((\lambda(p) \wedge \mu(q)) \wedge \frac{1-k}{2}\right) \wedge \rho(z)\right] \wedge \frac{1-k}{2} \\
&= \bigvee_{(y,z)\in A_a}\bigvee_{(p,q)\in A_a} ((\lambda(p) \wedge \mu(q)) \wedge \rho(z)) \wedge \frac{1-k}{2} \\
&= \bigvee_{(y,z)\in A_a}\bigvee_{(p,q)\in A_a} (\lambda(p) \wedge \mu(q) \wedge \rho(z)) \wedge \frac{1-k}{2} \\
&\geq (\lambda(ax) \wedge \mu(axa) \wedge \rho(xa)) \wedge \frac{1-k}{2}.
\end{aligned}$$

Since λ an $(\in, \in \vee q_k)$-fuzzy right ideal μ an $(\in, \in \vee q_k)$-fuzzy generalized ideal and ρ an $(\in, \in \vee q_k)$-fuzzy left ideal of S, we have $\lambda(ax) \geq \lambda(a) \wedge \frac{1-k}{2}$, $\mu(axa) \geq \mu(a) \wedge \mu(a) \wedge \frac{1-k}{2}$ and $\rho(xa) \geq \rho(a) \wedge \frac{1-k}{2}$. Therefore,

$$\begin{aligned}
&\left[\lambda(ax) \wedge \mu(axa) \wedge \rho(xa) \wedge \frac{1-k}{2}\right] \\
&\geq \left(\lambda(a) \wedge \frac{1-k}{2}\right) \wedge \left(\mu(a) \wedge \mu(a) \wedge \frac{1-k}{2}\right) \wedge \left(\rho(a) \wedge \frac{1-k}{2}\right) \\
&= \left(\lambda(a) \wedge \frac{1-k}{2}\right) \wedge \left(\mu(a) \wedge \frac{1-k}{2}\right) \wedge \left(\rho(a) \wedge \frac{1-k}{2}\right) \\
&= \lambda^{-}(a) \wedge^k \mu^{-}(a) \wedge^k \rho^{-}(a)
\end{aligned}$$

Thus $\left(\lambda \circ^k \mu \circ^k \rho\right)^{-}(a) \geq \left(\lambda \wedge^k \mu \wedge^k \rho\right)^{-}(a)$.

Conversely, assume that $\left(\lambda \wedge^k \mu \wedge^k \rho\right)^{-} \preceq \left(\lambda \circ^k \mu \circ^k \rho\right)^{-}$ for every $(\in, \in \vee q_k)$-fuzzy right ideal λ , every $(\in, \in \vee q_k)$-fuzzy generalized ideal μ and every $(\in, \in \vee q_k)$-fuzzy left ideal ρ of S. To prove that S is regular, by Lemma 4.15, it is enough to prove that

$R \cap B \cap L \subseteq (RBL]$ for right ideal R, generalized bi-ideal B and left ideal L of S.

Let $x \in R \cap B \cap L$. Then $x \in R$, $x \in B$ and $x \in L$. Since R is a right ideal, B a generalized bi-ideal and L a left ideal of S, by Lemma 4.4 and 4.5, $\overline{\chi}_R^k$ is an $(\in, \in \vee q_k)$-fuzzy right ideal, $\overline{\chi}_B^k$ an $(\in, \in \vee q_k)$-fuzzy generalized bi-ideal and $\overline{\chi}_L^k$ an $(\in, \in \vee q_k)$-fuzzy left ideal of S. By hypothesis, $(\chi_R \circ^k \chi_B \circ^k \chi_L)^{-}(x) \geq (\chi_R \wedge^k \chi_B \wedge^k \chi_L)^{-}(x) = (\chi_B^k \wedge \chi_I^k \wedge \chi_L^k)(x) \wedge \frac{1-k}{2}$. Since $x \in R$, $x \in B$ and $x \in L$, we have $\overline{\chi}_R^k(x) = \frac{1-k}{2}$, $\overline{\chi}_B^k(x) = \frac{1-k}{2}$ and $\overline{\chi}_L^k(x) = \frac{1-k}{2}$. Thus $(\chi_R^k \wedge \chi_B^k \wedge \chi_L^k)(x) \wedge \frac{1-k}{2} = \chi_R^k(x) \wedge \chi_B^k \wedge \chi_L^k(x) \wedge \frac{1-k}{2} = \frac{1-k}{2}$. It follows that $(\chi_R \circ^k \chi_B \circ^k \chi_L)^{-}(x) = \frac{1-k}{2}$. By using Lemma 4.3 (3), we have $(\chi_R \circ^k \chi_B \circ^k \chi_L)^{-} = \overline{\chi}_{(RBL]}^k$. Therefore, $\overline{\chi}_{(RBL]}^k(x) = \frac{1-k}{2}$ and so $x \in (RBL]$. Consequently, S is regular. □

Lemma 4.17 [24] *Let $(S, \cdot, \leq)$ be an ordered semigroup, then the following conditions are equivalent:*

1. *S is left weakly regular,*
2. *$I \cap L \subseteq (IL]$ for every ideal I and left ideal L of S,*
3. *$I(a) \cap L(a) \subseteq (I(a)L(a)]$ for every $a \in S$.*

Theorem 4.18 *An ordered semigroup $(S, \cdot, \leq)$ is left weakly regular if and only if for every $(\in, \in \vee q_k)$-fuzzy ideal λ and every $(\in, \in \vee q_k)$-fuzzy left ideal μ of S, we have*

$(\lambda \wedge^k \mu)^{-} \preceq (\lambda \circ^k \mu)^{-}$.

Proof Suppose that λ is an $(\in, \in \vee q_k)$-fuzzy ideal and μ an $(\in, \in \vee q_k)$-fuzzy left ideal of a left weakly regular ordered semigroup S. Let $a \in S$. Since S is left weakly regular, there exits $x, y \in S$ such that $a \leq xaya = (xa)(ya)$. Then $(xa, ya) \in A_a$ and $A_a \neq \emptyset$. Thus,

$$\begin{aligned}
\left(\lambda \circ^k \mu\right)^{-}(a) &= (\lambda \circ \mu)(a) \wedge \frac{1-k}{2} \\
&= \left[\bigvee_{(y,z)\in A_a} (\lambda(y) \wedge \mu(z))\right] \wedge \frac{1-k}{2} \\
&\geq (\lambda(xa) \wedge \mu(ya)) \wedge \frac{1-k}{2}.
\end{aligned}$$

Since λ is an $(\in, \in \vee q_k)$-fuzzy ideal and μ an $(\in, \in \vee q_k)$-fuzzy left ideal of S, we have $\lambda(xa) \geq \lambda(a) \wedge \frac{1-k}{2}$ and $\mu(ya) \geq \mu(a) \wedge \frac{1-k}{2}$. Therefore,

$$\begin{aligned}
\left[\lambda(xa) \wedge \mu(ya) \wedge \frac{1-k}{2}\right] &\geq \left(\lambda(a) \wedge \frac{1-k}{2}\right) \\
&\quad \wedge \left(\mu(a) \wedge \frac{1-k}{2}\right) \\
&= \lambda^{-}(a) \wedge^k \mu^{-}(a)
\end{aligned}$$

Thus $\left(\lambda \circ^k \mu\right)^{-}(a) \geq \left(\lambda \wedge^k \mu\right)^{-}(a)$.

Conversely, assume that $\left(\lambda \wedge^k \mu\right)^{-} \preceq \left(\lambda \circ^k \mu\right)^{-}$ for every $(\in, \in \vee q_k)$-fuzzy ideal λ and every $(\in, \in \vee q_k)$-fuzzy left ideal μ of S. To prove that S is left weakly regular, by Lemma 4.17, it is enough to prove that

$I \cap L \subseteq (IL]$ for ideal I and left ideal L of S.

Let $x \in I \cap L$. Then $x \in I$ and $x \in L$. Since I is an ideal and L a left ideal of S, by Lemma 4.4 and 4.5, $\overline{\chi}_I^k$ is an $(\in, \in \vee q_k)$-fuzzy ideal and $\overline{\chi}_L^k$ an $(\in, \in \vee q_k)$-fuzzy left ideal of S. By hypothesis, $(\chi_I \circ^k \chi_L)^-(x) \geq (\chi_I \wedge^k \chi_L)^-(x) = (\chi_I^k \wedge \chi_L^k)(x) \wedge \frac{1-k}{2}$. Since $x \in I$ and $x \in L$, we have $\overline{\chi}_I^k(x) = \frac{1-k}{2}$ and $\overline{\chi}_L^k(x) = \frac{1-k}{2}$. Thus $(\chi_I^k \wedge \chi_L^k)(x) \wedge \frac{1-k}{2} = \chi_I^k(x) \wedge \chi_L^k(x) \wedge \frac{1-k}{2} = \frac{1-k}{2}$. It follows that $(\chi_I \circ^k \chi_L)^-(x) = \frac{1-k}{2}$. By using Lemma 4.3 (3), we have $(\chi_I \circ^k \chi_L)^- = \overline{\chi}_{(IL]}^k$. Therefore, $\overline{\chi}_{(IL]}^k(x) = \frac{1-k}{2}$ and so $x \in (IL]$. Consequently, S is left weakly regular. □

Lemma 4.19 [24] *Let S be an ordered semigroup, then the following conditions are equivalent:*

1. *S is left weakly regular,*
2. *$I \cap B \subseteq (IB]$ for every generalized bi-ideal B and every ideal I of S,*
3. *$I(a) \cap B(a) \subseteq (I(a)B(a)]$ for every $a \in S$.*

Theorem 4.20 *An ordered semigroup $(S, \cdot, \leq)$ is left weakly regular if and only if for every $(\in, \in \vee q_k)$-fuzzy ideal λ and every $(\in, \in \vee q_k)$-fuzzy generalized bi-ideal μ of S, we have*

$$(\lambda \wedge^k \mu)^- \preceq (\lambda \circ^k \mu)^-.$$

Proof Suppose that λ is an $(\in, \in \vee q_k)$-fuzzy ideal and μ an $(\in, \in \vee q_k)$-fuzzy generalized bi-ideal of a left weakly regular ordered semigroup S. Let $a \in S$. Since S is left weakly regular, there exits $x, y \in S$ such that $a \leq xaya \leq x(xaya)ya = (x^2ay)(aya)$. Then $(x^2ay, aya) \in A_a$ and $A_a \neq \emptyset$. Thus,

$$\begin{aligned}(\lambda \circ^k \mu)^-(a) &= (\lambda \circ \mu)(a) \wedge \frac{1-k}{2} \\ &= \left[\bigvee_{(y,z) \in A_a} (\lambda(y) \wedge \mu(z))\right] \wedge \frac{1-k}{2} \\ &\geq (\lambda(xa) \wedge \mu(ya)) \wedge \frac{1-k}{2}.\end{aligned}$$

Since λ is an $(\in, \in \vee q_k)$-fuzzy ideal and μ an $(\in, \in \vee q_k)$-fuzzy generalized bi-ideal of S, we have $\lambda(x^2ay) \geq \lambda(ay) \wedge \frac{1-k}{2} \geq \left(\lambda(a) \wedge \frac{1-k}{2}\right) \wedge \frac{1-k}{2} = \lambda(a) \wedge \frac{1-k}{2}$ and $\mu(aya) \geq \mu(a) \wedge \mu(a) \wedge \frac{1-k}{2} = \mu(a) \wedge \frac{1-k}{2}$. Therefore

$$\begin{aligned}\left[\lambda(x^2ay) \wedge \mu(aya) \wedge \frac{1-k}{2}\right] &\geq \left(\lambda(a) \wedge \frac{1-k}{2}\right) \\ &\quad \wedge \left(\mu(a) \wedge \frac{1-k}{2}\right) \\ &= \lambda^-(a) \wedge^k \mu^-(a)\end{aligned}$$

Thus $(\lambda \circ^k \mu)^-(a) \geq (\lambda \wedge^k \mu)^-(a)$.

Conversely, assume that $(\lambda \wedge^k \mu)^- \preceq (\lambda \circ^k \mu)^-$ for every $(\in, \in \vee q_k)$-fuzzy ideal λ and every $(\in, \in \vee q_k)$-fuzzy generalized bi-ideal μ of S. To prove that S is left weakly regular, by Lemma 4.19, it is enough to prove that

$I \cap B \subseteq (IL]$ for ideal I and generalized bi-ideal B of S.

Let $x \in I \cap B$. Then $x \in I$ and $x \in B$. Since I is an ideal and B a generalized bi-ideal of S, by Lemma 4.4 and 4.5, $\overline{\chi}_I^k$ is an $(\in, \in \vee q_k)$-fuzzy ideal and $\overline{\chi}_B^k$ an $(\in, \in \vee q_k)$-fuzzy generalized bi-ideal of S. By hypothesis, $(\chi_I \circ^k \chi_B)^-(x) \geq (\chi_I \wedge^k \chi_B)^-(x) = (\chi_I^k \wedge \chi_B^k)(x) \wedge \frac{1-k}{2}$. Since $x \in I$ and $x \in B$, we have $\overline{\chi}_I^k(x) = \frac{1-k}{2}$ and $\overline{\chi}_B^k(x) = \frac{1-k}{2}$. Thus $(\chi_I^k \wedge \chi_B^k)(x) \wedge \frac{1-k}{2} = \chi_I^k(x) \wedge \chi_B^k(x) \wedge \frac{1-k}{2} = \frac{1-k}{2}$. It follows that $(\chi_I \circ^k \chi_B)^-(x) = \frac{1-k}{2}$. By using Lemma 4.3 (3), we have $(\chi_I \circ^k \chi_B)^- = \overline{\chi}_{(IB]}^k$. Therefore, $\overline{\chi}_{(IB]}^k(x) = \frac{1-k}{2}$ and so $x \in (IB]$. Consequently, S is left weakly regular. □

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