

## The Probability that an Element of a Symmetric Group Fixes a Set and Its Application in Graph Theory

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**Abstract:** Let  $G$  be a symmetric group. The commutativity degree is the probability that two elements randomly chosen commute in  $G$ , denoted as  $P(G)$ . This concept is used to determine the abelianness of a group. The probability can be obtained by finding the multiplication table, conjugacy classes and recently using centralizers. The concept of commutativity degree has been generalized by many authors; one of these generalizations is the probability that a group element fixes a set. Thus, the main objective of this paper is to find the probability of an element of a group  $G$  fixes  $\Omega$  where  $\Omega$  is a set consisting of  $(a,b)$ , where  $a$  and  $b$  are commuting elements in  $G$  of size two and the group  $G$  acts on the set of all subset of  $\Omega$  regularly and by conjugation. The results that are obtained from the probability can be applied to graph theory; more precisely the orbit graph and graph related to conjugacy classes. Hence, our second objective is to find the graph related to conjugacy class for the mentioned probability.

**Key words:** Commutativity degree . symmetric group . orbit graph . conjugacy classes . graph related to conjugacy classes . group action

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### INTRODUCTION

Throughout this paper,  $G$  denotes the symmetric group, namely  $S_n$ . The probability that two randomly selected elements commute in  $G$  is called the commutativity degree of  $G$ , denoted by  $P(G)$ . The probability is defined as follows:

$$P(G) = \frac{|\{(x,y) \in G \times G \mid xy = yx\}|}{|G|^2}$$

Clearly, the above probability is equal to one if  $G$  is an abelian group. Gustafson [1] and MacHale [2] proved that the probability that two elements commute is less than or equal to  $5/8$  for finite non-abelian groups. This probability can be computed using conjugacy classes [1, 2]. Numerous researches have been done on the commutativity degree and it has been generalized by lots of researchers. We use one of those generalizations which is the probability that a group element fixes a set that was firstly introduced by

Omer *et al.* [3]. This probability can be obtained by calculating the conjugacy classes under some group action on a set.

In the following, we state some fundamental concepts related to graph theory which are needed in this paper.

A graph  $\Gamma$  consists of two sets, namely vertices  $V(\Gamma)$  and edges  $E(\Gamma)$  together with relation of incidence. The directed graph is a graph whose edges are identified with ordered pair of vertices. Otherwise,  $\Gamma$  is called undirected. In this paper all graphs are considered simple and undirected. Two vertices in  $\Gamma$  are adjacent if they are joined by an edge. A complete graph  $K_n$  is a graph where each order pair of distinct vertices is adjacent [4].

The independent set  $B$  is a non-empty set of  $V(\Gamma)$ , where there is no adjacent between two elements of  $B$  in  $\Gamma$ , while the independent number is the number of vertices in the maximum independent set and is denoted by  $\alpha(\Gamma)$ . However, the maximum number  $m$  for which  $\Gamma$  is  $m$ -vertex colorable is the chromatic number and is denoted by  $\chi(\Gamma)$ . The maximum distance between any

two vertices of  $\Gamma$  is called the diameter and is denoted by  $d(\Gamma)$  [4]. Furthermore, the clique number is the maximum number of complete subgraphs in  $\Gamma$  and is denoted by  $\omega(\Gamma)$ . The dominating set  $X$  of  $V(\Gamma)$  is a set where for each  $v$  outside  $X$ , there exists  $x \in X$  such that  $v$  is adjacent to  $x$ . The dominating number is the minimum size of  $X$  and is denoted by  $\gamma(\Gamma)$  [4].

This paper is structured as follows: The first section is the introduction. In the second section, we state some previous works that are related to the commutativity degree, in particular related to the probability that a group element fixes a set. Our main results are given in the third section, which include the probability mentioned in the second section and some results on the orbit graph and graph related to conjugacy classes.

## PRELIMINARIES

In this section, some previous works that are needed in this paper are included. This section is divided into two parts. The first part provides some works related to the probability that a group element fixes a set or a subgroup element. The second part states some previous researches on graph theory specifically the orbit graph and graph related to conjugacy classes.

**The commutativity degree:** We start with a concept introduced by Sherman [5] in 1975, namely the probability of an automorphism of a finite group fixes an arbitrary element in the group.

**Definition 2.1:** [5] Let  $G$  be a group. Let  $X$  be a non-empty set of  $G$  (i.e.,  $G$  is a group of permutations of  $X$ ). Then the probability of an automorphism of a group fixes a random element from  $X$  is defined as follows:

$$P_G(X) = \frac{\left| \{ (g, x) \mid gx = x \forall g \in G, x \in X \} \right|}{|X||G|}.$$

In 2011, Moghaddam *et al.* [6] explored Sherman's definition and introduced a new probability which is called the probability of an automorphism fixes a subgroup element of a finite group. This probability is stated as follows:

$$P_{A_G}(H, G) = \frac{\left| \{ (\alpha, h) \mid h^\alpha = h \forall h \in H, \alpha \in A_G \} \right|}{|H||G|}$$

where  $A_G$  is the group of automorphisms of a group  $G$ . It is obvious that when  $H = G$ , then

$$P_{A_G}(G, G) = P_{A_G}(G)$$

Omer *et al.* [3] found the probability that an element of a group fixes a set of size two of commuting elements in  $G$ . Their results are listed in the following.

**Definition 2.2:** [3] Let  $G$  be a group. Let  $S$  be a set of all subsets of commuting elements of size two in  $G$ , where  $G$  acts on  $S$  by conjugation. Then the probability of an element of a group fixes a set is given as follows:

$$P_G(S) = \frac{\left| \{ (g, S) \mid gS = S \forall g \in G, S \in S \} \right|}{|S||G|}$$

**Theorem 2.3:** [3] Let  $G$  be a finite group and let  $X$  be a set of elements of  $G$  of size two in the form of  $(a, b)$  where  $a$  and  $b$  commute. Let  $S$  be the set of all subsets of commuting elements of  $G$  of size 2 and  $G$  acts on  $S$  by conjugation. Then the probability that an element of a group fixes a set is given by:

$$P_G(S) = \frac{K}{|S|}$$

where  $K$  is the number of conjugacy classes of  $S$  in  $G$ .

In addition, the probability that a group element fixes a set under some group action was also found for some finite non-abelian 2-groups such as quasi-dihedral groups and semi-dihedral groups and others [7].

**Graph related to conjugacy classes:** In this part, some earlier and recent publication that are related to graph related to conjugacy classes are discussed.

In 1990, Bertram *et al.* [8] introduced a graph which is called a graph related to conjugacy classes. Let  $\Gamma$  be a graph,  $V(\Gamma)$  is the set of vertices in  $\Gamma$ . The vertices of this graph are non-central conjugacy classes, thus  $V(\Gamma) = K(G) - Z(G)$ , where  $K(G)$  is the number of conjugacy classes and  $Z(G)$  is the center of a group. In this graph, two vertices are adjacent if the cardinalities are not coprime.

Moreto *et al.* [9], classified the finite groups such that their conjugacy classes lengths are set-wise relatively prime for any five distinct classes.

Recently, Bianchi *et al.* [10] studied the regularity of the graph related to conjugacy classes and provided some results. Later, Erfanian and Tölue [11] introduced a new graph which is called a conjugate graph. The vertices of this graph are non-central elements of a finite non-abelian group. Two vertices of this graph are adjacent if they are conjugate.

Furthermore, the orbit graph is a result of generalization of conjugate graph. This graph was

firstly introduced by Omer *et al.* [12]. The vertices of an orbit graph are

$$\left| V(\Gamma_G^\Omega) \right| = |\Omega| - |A|$$

where  $\Omega$  can be disjoint union of distinct orbit under action of  $G$  on the set  $\Omega$ , while

$$A = \{v \in \Omega \mid v g = g v, g \in G\}$$

Two vertices of this graph are linked by an edge if and only if there exists  $g \in G$  such that  $g\omega_1 = \omega_2$ , where  $\omega_1, \omega_2 \in \Omega$ .

Ilangovan and Sarmin [13], found some graph properties of graph related to conjugacy classes of two-generator two-groups of class two.

Recently, Moradipour *et al.* [14] used the graph related to conjugacy classes to find some graph properties of some finite metacyclic 2-groups. Moreover, Omer *et al.* [15] found the probability that an element of a metacyclic group fixes a set and associated their results with graph related to conjugacy classes.

## RESULTS AND DISCUSSION

In this section, we provide our main results. The probability that a group element fixes a set can be computed by finding the conjugacy classes of  $\Omega$  under the action. Thus, the probability is the ratio of conjugacy classes of  $\Omega$  to the order of  $\Omega$ . The main point of finding the conjugacy classes is that to associate our results with graph theory, more specifically with the graph related to conjugacy classes. In addition, the orbit graph is a graph whose vertices are non-central orbits under some group actions on a set. Thus, in this section, we firstly find the probability that a symmetric group element fixes a set. Then, the orbit graph and graph related to conjugacy classes are introduced in the second part.

**The probability that a group element fixes a set:** We start finding this probability in the case that the group acts on the set  $\Omega$  by conjugation.

**Theorem 3.1:** Let  $G$  be a symmetric group, namely  $S_n$ . Let  $S$  be a set of elements of  $G$  of size two in the form of  $(a, b)$  where  $a$  and  $b$  commute. Let  $\Omega$  be the set of all subsets of commuting elements of  $G$  of size two and  $G$  acts on  $\Omega$  by conjugation. Then

$$P_G(\Omega) = \frac{K(\Omega)}{|\Omega|}$$

where  $K(\Omega)$  is the number of conjugacy classes under the group action on  $\Omega$ .

**Proof:** If  $G$  acts on  $\Omega$  by conjugation, then there exists  $\psi: G \times \Omega \rightarrow \Omega$  such that  $\psi_g(\omega) = g\omega g^{-1}$ , where  $\omega \in \Omega$ ,  $g \in G$ . The elements of order two in  $G$  are  $(ab)$  and  $(ab)(cd)$ . Thus, the elements of  $\Omega$  are in the form of  $(1, (ab))$ ,  $(1, (ab)(cd))$ ,  $((ab), (cd))$  and  $((ab)(cd), (ac)(bd))$ . If  $G$  acts on  $\Omega$  by conjugation, then  $cl(\omega) = g\omega g^{-1}$ , where  $\omega \in \Omega$ ,  $g \in G$ . Two elements are conjugated if they have the same cycle size and based on [3],

$$P_G(\Omega) = \frac{K(\Omega)}{|\Omega|}$$

as required.

**Theorem 3.2:** Let  $G$  be a symmetric group, namely  $S_n$ . Let  $S$  be a set of elements of  $G$  of size two in the form of  $(a, b)$  where  $a$  and  $b$  commute. Let  $\Omega$  be the set of all subsets of commuting elements of  $G$  of size two and  $G$  acts regularly on  $\Omega$ . Then

$$P_G(\Omega) = \frac{K(\Omega)}{|\Omega|}$$

**Proof:** If  $\Omega$  acts regularly on  $\Omega$ , then for every  $v_1, v_2 \in \Omega$  there exists  $g \in G$  such that  $gv_1 = v_2$ . As a consequence, there are  $K(\Omega)$  of conjugacy classes. Thus,

$$P_G(\Omega) = \frac{K(\Omega)}{|\Omega|}$$

The above theorems are only true when  $n$  is three thus we put some restrictions of the order of  $\Omega$  elements. Therefore, the elements of  $\Omega$  are in the form of  $(a, b)$  where  $a$  and  $b$  commute and both  $a$  and  $b$  have order two. In other words,  $|\Omega| = 1 \text{ cm}(|a|, |b|)$ , where  $|a| = |b| = 2$ . The following theorem illustrates the restrictions that have been done on Theorem 3.1 and Theorem 3.2.

**Theorem 3.3:** Let  $G$  be a symmetric group, namely  $S_n$ . Let  $S$  be a set of elements of  $G$  of size two in the form of  $(a, b)$  where  $a$  and  $b$  commute and  $|a| = |b| = 2$ . Let  $\Omega$  be the set of all subsets of commuting elements of  $G$  of size two and  $G$  acts on  $\Omega$  by conjugation. Then

$$P_G(\Omega) = \frac{2}{|\Omega|}$$

**Proof:** Suppose  $G$  be a symmetric group of order  $n!$ . If  $G$  acts on  $\Omega$  by conjugation, then there exists  $\psi : G \times \Omega \rightarrow \Omega$  such that

$$\psi_g(\omega) = g\omega g^{-1}, \omega \in \Omega, g \in G$$

The elements of order two in  $G$  are  $(ab)$  and  $((ab)(cd))$ , where  $a$  and  $b$  are cycles of length two. Thus, the elements of  $\Omega$  are in the form  $((ab), (cd))$  and  $((ab)(cd), (ac)(bd))$ , since two permutations are conjugate if they have the same cycle structure. It follows that the number of conjugacy classes under the action is two. The proof then follows.

In the following we associate our results with the orbit graph and graph related to conjugacy classes.

**The orbit graph:** In this section, we apply our results in the first part to the orbit graph.

**Theorem 3.4:** Let  $G$  be a symmetric group, namely  $S_n$ . Let  $S$  be a set of elements of  $G$  of size two in the form of  $(a,b)$  where  $a$  and  $b$  commute and  $|a| = |b| = 2$ . Let  $\Omega$  be the set of all subsets of commuting elements of  $G$  of size two and  $G$  acts regularly on  $\Omega$ . Then the orbit graph

$$\Gamma_G^\Omega = \bigcup_{i=1}^{\frac{|\Omega|}{2}} K_{2_i}$$

**Proof:** If  $G$  acts regularly on  $\Omega$ , then the number of vertices in  $\Gamma_G^\Omega$  are equal to

$$|V(\Gamma_G^\Omega)| = |\Omega| - |A|$$

Thus two vertices are adjacent if  $gv_1 = v_2, v_1, v_2 \in \Omega$ . Therefore, there are  $|\Omega|/2$  complete components of  $K_2$  since each vertex in the form  $((ab)(cd))$  is joined to the vertex  $((ab)(cd), (ac)(bd))$ . The proof then follows.

**Proposition 3.1:** Let  $G$  be a symmetric group, namely  $S_n$ . Let  $S$  be a set of elements of  $G$  of size two in the form of  $(a,b)$  where  $a$  and  $b$  commute and  $|a| = |b| = 2$ . Let  $\Omega$  be the set of all subsets of commuting elements of  $G$  of size two. If  $G$  acts regularly on  $\Omega$  and

$$\Gamma_G^\Omega = \bigcup_{i=1}^{\frac{|\Omega|}{2}} K_{2_i}, \text{ then } |E(\Gamma_G^\Omega)| = \frac{|\Omega|}{2}$$

**Proof:** According to [11] the number of edges is

$$|E(\Gamma_G^\Omega)| = \sum_{i=1}^{K(\Gamma_G^\Omega) - |A|} \binom{|x_i^\Omega|}{2}$$

Referring to Theorem 3.4, the number of edges is

$$|E(\Gamma_G^\Omega)| = \sum_{i=1}^{\frac{|\Omega|}{2}} \binom{|2|}{2}$$

thus

$$|E(\Gamma_G^\Omega)| = \frac{|\Omega|}{2}$$

as claimed.

**Proposition 3.2:** Let  $G$  be a symmetric group, namely  $S_n$ . Let  $S$  be a set of elements of  $G$  of size two in the form of  $(a,b)$  where  $a$  and  $b$  commute and  $|a| = |b| = 2$ . Let  $\Omega$  be the set of all subsets of commuting elements of  $G$  of size two. If  $G$  acts regularly on  $\Omega$  and

$$|E(\Gamma_G^\Omega)| = \frac{|\Omega|}{2}$$

thus

$$\chi(\Gamma_G^\Omega) = \omega(\Gamma_G^\Omega) = 2$$

and

$$\alpha(\Gamma_G^\Omega) = \gamma(\Gamma_G^\Omega) = \frac{|\Omega|}{2}$$

**Proof:** The chromatic number and clique number are identical since the maximum size of conjugacy classes is two. Based on [11] the independent number and dominating number are equal to  $|\Omega|/2$ .

**Theorem 3.5:** Let  $G$  be a symmetric group,  $S_n$  of order  $n!$ . Let  $S$  be a set of elements of  $G$  of size two in the form of  $(a,b)$  where  $a$  and  $b$  commute and  $|a| = |b| = 2$ . Let  $\Omega$  be the set of all subsets of commuting elements of  $G$  of size two. If  $G$  acts on  $\Omega$  by conjugation, then the number of vertices in the orbit graph are

$$|V(\Gamma_G^\Omega)| = |\Omega|$$

and the number of edges are

$$|E(\Gamma_G^\Omega)| = \frac{K \frac{|\Omega|}{2}!}{\left( K \frac{|\Omega|}{2} - 2 \right)!}$$

**Proof:** The number of vertices in the orbit graph is

$$|V(\Gamma_G^\Omega)| = |\Omega| - |A|$$

Based on Theorem 3.3

$$|V(\Gamma_G^\Omega)| = |\Omega|$$

Two vertices are connected by an edge if they are conjugated and according to Theorem 3.3 there are two conjugacy classes, thus we have two complete components of  $K_{|\Omega|/2}$ . The number of edges are

$$|E(\Gamma_G^\Omega)| = \sum_{i=1}^{|V(\Gamma_G^\Omega)|} \binom{|x_i^\Omega|}{2}$$

Therefore,

$$|E(\Gamma_G^\Omega)| = \sum_{i=1}^2 \frac{\frac{K_{|\Omega|}}{2}!}{2! \binom{K_{|\Omega|} - 2}{2}}$$

The proof then follows.

**Graph related to conjugacy classes:** In this section, the graph related to conjugacy classes is found for the results obtained in the first part.

**Theorem 3.6:** Let  $G$  be a symmetric group,  $S_n$ . Let  $S$  be a set of elements of  $G$  of size two in the form of  $(a,b)$  where  $a$  and  $b$  commute. Let  $\Omega$  be the set of all subsets of commuting elements of  $G$  of size two and  $G$  acts on  $\Omega$  by conjugation. Then  $\Gamma_G^\Omega$  is an empty graph.

**Proof:** If  $G$  acts on  $\Omega$  by conjugation, then there exists  $\psi: G \times \Omega \rightarrow \Omega$  such that

$$\psi_g(\omega) = g\omega g^{-1}, \omega \in \Omega, g \in G$$

According to Theorem 3.1, there is only one conjugacy class, thus the number of vertices equals to  $\Omega$ . Hence the graph is an empty graph.

**Theorem 3.7:** Let  $G$  be a symmetric group,  $S_n$  of order  $n!$ . Let  $S$  be a set of elements of  $G$  of size two in the form of  $(a,b)$  where  $a$  and  $b$  commute and  $|a| = |b| = 2$ . Let  $\Omega$  be the set of all subsets of commuting elements of  $G$  of size two. If  $G$  acts on  $\Omega$  by conjugation, then the graph related to conjugacy classes  $C_G^\Omega = K_2$ .

**Proof:** Based on Theorem 3.3, there are two conjugacy classes, thus

$$|V(C_G^\Omega)| = K(\Omega) - |A|. \text{ Thus, } |V(C_G^\Omega)| = 2$$

Two vertices are adjacent if they are coprime, thus the vertices in the form  $((ab),(cd))$  are adjacent to the vertices  $((ab)(cd),(ac)(bd))$ . Thus, we have complete components of  $K_2$ , as required.

## CONCLUSION

In this paper, the probability that a group element fixes a set of size two for symmetric groups has been found. The probability was found in the case that a group acts on a set regularly and by conjugation. Some restrictions have been done on the order of elements of  $\Omega$ . The results were then applied to graph theory, particularly to the orbit graph and graph related to conjugacy classes.

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