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Some Innovative Types of Fuzzy Bi-Ideals in Ordered Semigroups

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An ordered semigroup (algebraic structure) is a semigroup together with a partial order that is compatible with the semigroup operation. In many applied disciplines like computer science, coding theory, sequential machines and formal languages, the use of fuzzified algebraic structures especially ordered semigroups play a remarkable role. A theory of fuzzy sets in terms of fuzzy points on ordered semigroups can be developed. In this paper, we introduce the concepts of (α, β)-fuzzy bi-ideals and $(\overline{\beta}, \overline{\alpha})$-fuzzy bi-ideals of ordered semigroups, where $\alpha, \beta \in \{\in_\gamma, q_\delta, \in_\gamma \wedge q_\delta, \in_\gamma \vee q_\delta\}$, $\overline{\alpha}, \overline{\beta} \in \{\overline{\in_\gamma}, \overline{q_\delta}, \overline{\in_\gamma} \wedge \overline{q_\delta}, \overline{\in_\gamma} \vee \overline{q_\delta}\}$, $\alpha \neq \in_\gamma \wedge q_\delta$ and $\beta \neq \overline{\in_\gamma} \wedge \overline{q_\delta}$, and some related properties are investigated. The important milestone of this paper is, to link ordinary bi-ideals and fuzzy bi-ideals of types $(\in_\gamma, \in_\gamma \vee q_\delta)$ and $(\overline{\in_\gamma}, \overline{\in_\gamma} \vee \overline{q_\delta})$ using level subset $U(\mu; r)$. Special attention is paid to $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy bi-ideals and $(\overline{\in_\gamma}, \overline{\in_\gamma} \vee \overline{q_\delta})$-fuzzy bi-ideals.

Keywords: Ordered Semigroups, Fuzzy Bi-Ideals, (α, β)-Fuzzy Bi-Ideals, $(\overline{\beta}, \overline{\alpha})$-Fuzzy Bi-Ideals.

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1. INTRODUCTION

The major advancement in the fascinating world of fuzzy set started with the work of renowned scientist Zadeh (1965) which open a new era of research around the world. A fuzzy set can be defined as a set without a crisp and clearly sharp boundries which contains the elements with only a partial degree of membership. The use of fuzzy sets (extension of ordinary sets) in real world problem involving uncertainties are considered to be the most powerful tool compare to ordinary sets. The fuzzification of algebraic structures in a variety of applied subjects such as control engineering, fuzzy automata and error-correcting codes is of great interest for the researchers. The importance of such algebraic structures can be seen from the latest research which has been carried out in the last few years. In Pu and Liu (1980), Zadeh's idea of fuzzy sets was applied to generalize some of the basic concepts of general topology.

Mordeson et al. (2003) presented an up to date account of fuzzy sub-semigroups and fuzzy ideals of a semigroup. The book concentrates on theoretical aspects, but also includes applications in the areas of fuzzy coding theory, fuzzy finite state machines, and fuzzy languages. Basic results on fuzzy subsets, semigroups, codes, finite state machines, and languages are reviewed and introduced, as well as certain fuzzy ideals of a semigroup and advanced characterizations and properties of fuzzy semigroups. Kuroki (1981) introduced the notion of fuzzy bi-ideals in semigroups. Kehayopulu and Tsingelis (2002) applied the fuzzy concept in ordered semigroups and studied some properties of fuzzy left (right) ideals and fuzzy filters in ordered semigroups (also refer to Lee and Park, 2003; Shabir and Khan, 2008). Fuzzy implicative and Boolean filters of R_0-algebra were initiated by Liu and Lee (2005) (also see Jun and Liu 2006).

The idea of a quasi-coincidence of a fuzzy point with a fuzzy set, which is mentioned in Bhakat and Das (1996); Bhakat and Das (1996), played a vital role to generate some different types of fuzzy subgroups. It is worth pointing out that Bhakat and Das (see Bhakat and Das 1996) gave the concepts of (α, β)-fuzzy subgroups by using the "*belongs to*" relation $(\in)$ and "*quasi-coincident with*" relation (q) between a fuzzy point and a fuzzy subgroup, and introduced the concept of an $(\in, \in \vee q)$-fuzzy subgroup. In particular, $(\in, \in \vee q)$-fuzzy subgroup is an important and useful generalization of the Rosenfeld's fuzzy subgroup (Rosenfeld, 1971). It is now natural to investigate similar type of generalizations of the existing fuzzy subsystems of other algebraic structures. With this objective in view, Ma et al. (2008), introduced the interval valued

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$(\in, \in \vee q)$-fuzzy ideals of pseudo-MV algebras and gave some important results of pseudo-MV algebras (also see Ma et al. 2009; Ma et al. 2009; Yuan et al. 2003). In Davvaz and Mozafar (2009), Davvaz and Mozafar studied $(\in, \in \vee q)$-fuzzy subalgebras and (ideals) of a Lie algebras and provided some basic results of this algebra. Jun and Song (see Jun and Song 2006) discussed general forms of fuzzy interior ideals in semigroups. Kazanci and Yamak introduced the concept of a generalized fuzzy bi-ideal in semigroups (Kazanci and Yamak, 2008) and gave some properties of fuzzy bi-ideals in terms of $(\in, \in \vee q)$-fuzzy bi-ideals. Jun et al. (see Jun et al. 2009) gave the concept of a generalized fuzzy bi-ideal in ordered semigroups and characterized regular ordered semigroups in terms of this notion. Davvaz et al. used the idea of generalized fuzzy sets in hyperstructures and introduced different generalized fuzzy subsystems e.g., (see Davvaz et al. 2009; Davvaz and Mozafar, 2009; Davvaz and Corsini, 2008; Davvaz et al. 2008; Davvaz, 2008; Davvaz, 2006; Kazanci and Davvaz, 2009; Zhan et al. 2009; Zhan and Davvaz, 2010). In Ma et al. (2009), introduced the concept of a generalized fuzzy filter of R_0-algebra and provided some

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properties in terms of this notion. Many other researchers used the idea of generalized fuzzy sets and gave several characterizations results in different branches of algebra see references. The concept of an (α, β)-fuzzy interior ideals in ordered semigroups was first introduced by Khan and Shabir (2009), where some basic properties of (α, β)-fuzzy interior ideals were discussed. Also, for further study refer to Bai, (2010); Murali, (2004).

Recently, Yin and Zhan (2010), introduced more general forms of $(\in, \in\vee q)$-fuzzy (implicative, positive implicative and fantastic) filters and $(\overline{\in}, \overline{\in}\vee\overline{q})$-fuzzy (implicative, positive implicative and fantastic) filters of BL-algebras and defined $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy (implicative, positive implicative and fantastic) filters and $(\overline{\in_\gamma}, \overline{\in_\gamma}\vee\overline{q_\delta})$-fuzzy (implicative, positive implicative and fantastic) filters of BL-algebras and gave some interesting results in terms of these notions.

In this paper, we deal with a generalization of the paper Jun et al. (2009), we discuss more general forms of $(\in, \in\vee q)$-fuzzy bi-ideals and $(\overline{\in}, \overline{\in}\vee\overline{q})$-fuzzy bi-ideals of ordered semigroups. We introduce the concepts of (α, β)-fuzzy bi-ideals and $(\overline{\beta}, \overline{\alpha})$-fuzzy bi-ideals of ordered semigroups, and some related properties are investigated. Furthermore, ordinary bi-ideals and fuzzy bi-ideals of types $(\in_\gamma, \in_\gamma \vee q_\delta)$ and $(\overline{\in_\gamma}, \overline{\in_\gamma}\vee\overline{q_\delta})$ are linked using level subset. Special attention is given to $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy bi-ideals and $(\overline{\in_\gamma}, \overline{\in_\gamma}\vee\overline{q_\delta})$-fuzzy bi-ideals and some interesting results are obtained.

2. PRELIMINARIES

In this section, we give a review of some fundamental concepts that are necessary for this paper.

By an *ordered semigroup* (or *po-semigroup*) we mean a *structure* $(S, \cdot, \leq)$ in which the following are satisfied:

$(OS1)$ $(S, \cdot)$ is a semigroup,

$(OS2)$ $(S, \leq)$ is a poset,

$(OS3)$ $a \leq b \Longrightarrow ax \leq bx$ and $xa \leq xb$ for all $a, b, x \in S$.

For $A \subseteq S$, we denote $(A] := \{t \in S \mid t \leq h$ for some $h \in A\}$. If $A = \{a\}$, then we write $(a]$ instead of $(\{a\}]$. For $A, B \subseteq S$, we denote,

$$AB := \{ab \mid a \in A, b \in B\}$$

Let $(S, \cdot, \leq)$ be an ordered semigroup. A nonempty subset A of S is called a *subsemigroup* of S if $A^2 \subseteq A$.

A non-empty subset A of an ordered semigroup S is called a *bi-ideal* of S if it satisfies

(i) $(\forall a, b \in S)(\forall b \in A)(a \leq b \Longrightarrow a \in A)$,

(ii) $A^2 \subseteq A$,

(iii) $ASA \subseteq A$.

Now, we recall some fuzzy logic concepts.

A *fuzzy subset* μ from a universe X is a function from X into a unit closed interval $[0, 1]$ of real numbers, i.e., $\mu : X \longrightarrow [0, 1]$.

A fuzzy subset μ of an ordered semigroup $(S, \cdot, \leq)$ is called a *fuzzy bi-ideal* of S if it satisfies

(i) $(\forall x, y \in S)(\mu(xy) \geq \min\{\mu(x), \mu(y)\})$,

(ii) $(\forall x, a, y \in S)(\mu(xay) \geq \min\{\mu(x), \mu(y)\})$,

(iii) $(\forall x, y \in S)(x \leq y \Longrightarrow \mu(x) \geq \mu(y))$.

For a fuzzy subset μ of S and $t \in (0, 1]$, the *crisp set* $U(\mu; t) := \{x \in S \mid \mu(x) \geq t\}$ is called the *level subset* of μ.

Throughout this paper, S will denote an ordered semigroup, unless otherwise stated.

2.1. Theorem (cf Jun et al. 2009)

A fuzzy subset μ of S is a fuzzy bi-ideal of S if and only if each non-empty level subset $U(\mu; t)$, for all $t \in (0, 1]$ is a bi-ideal of S.

Let $A \subseteq S$ and $a \in S$, we denote by $A_a = \{(y, z) \mid a \leq yz\}$.

Let μ and λ be fuzzy subsets of S, define the product $\mu \circ \lambda$ of μ and λ as follows:

$$\begin{aligned} \mu \circ \lambda : S \longrightarrow [0, 1] \mid a \longmapsto (\mu \circ \lambda)(a) \\ = \begin{cases} (y, z) \in A_a \min\{\mu(y), \lambda(z)\} & \text{if } A_a \neq \emptyset \\ 0 & \text{if } A_a = \emptyset \end{cases} \end{aligned}$$

The concept of an ordered semigroup is needed in order to present a concept of regular fuzzy expressions applicable to the various models of fuzzy automata.

Let X be a non-empty finite set and X^* be the free semigroup generated by X with identity Λ. Let $(S, \cdot, \leq)$ be an ordered semigroup. A function f from X^* into S is called an S-*language* over X. An S-*automation* over X is a 4-tuple $A = (R, p, h, g)$, where R is a finite non-empty set, p is a function from $R \times X \times R$ into S, and h and g are functions from R into S. Let $A = (R, p, h, g)$ be an S-automation over X.

(1) Let p^* be the function from $R \times X^* \times R$ into S defined respectively as follows:

$$p^*(r, \Lambda, r') = \begin{cases} 1 & \text{if } r = r' \\ 0 & \text{if } r \neq r' \end{cases}$$

$$p^*(r, ax, r') = \bigvee_{r'' \in R} p(r, a, r''*(r'', x, r')$$

for all $r, r' \in R$, $a \in X$ and $x \in X^*$.

(2) Let f^A be the function from X^* into S defined by for all $x \in X^*$,

$$f^A = \bigvee_{r \in R} \bigvee_{r' \in R} h(r) \cdot p^*(r, x, r') \cdot g(r')$$

In this definition, X is the set of input symbols, R is the set of states, $p(r, a, r')$ is the grade of membership that the next state is r' given that the present state is r and input a is applied, $h(r)$ is the grade of membership that r is the initial state and $g(r)$ is the grade of membership that r is an accepting state. Moreover, f^A is the S-language over X accepted by A.

The above definition includes many of the various existing models of fuzzy automata. These models may be obtained by appropriate choice of S, e.g., max-min automation: $S_M = (S, \cdot, \leq)$, where S is a subset of the real number system, usually $[0, 1]$ and $a \cdot b = a \wedge b$.

A fuzzy subset μ in a set S of the form

$$\mu : S \longrightarrow [0, 1], \quad y \longmapsto \begin{cases} t \in (0, 1] & \text{if } y = x \\ 0 & \text{if } y \neq x \end{cases}$$

is said to be a *fuzzy point with support* x and *value* t and is denoted by $[x; t]$. For a fuzzy point $[x; t]$ and a fuzzy subset μ of a set S, Pu and Liu (1980) gave meaning to the symbol $[x; t]\alpha\mu$, where $\alpha \in \{\in, q, \in\vee q, \in\wedge q\}$.

To say that $[x; t] \in \mu$ (resp. $[x; t]q\mu$) means that $\mu(x) \geq t$ (resp. $\mu(x) + t > 1$), and in this case, $[x; t]$ is said to *belong to* (resp. *be quasi-coincident with*) a fuzzy subset μ. To say that $[x; t] \in \vee q\mu$ (resp. $[x; t] \in \wedge q\mu$) means that $[x; t] \in \mu$ or $[x; t]q\mu$ (resp. $[x; t] \in \mu$ and $[x; t]q\mu$). To say that $[x; t]\overline{\alpha}\mu$ means that $[x; t]\alpha\mu$ does not hold for $\alpha \in \{\in, q, \in\vee q\}$.

2.2. Theorem (cf. Jun et al. 2009)

Let μ be a fuzzy subset of S. Then $U(\mu; t)$ is a bi-ideal of S for all $t \in (0.5, 1]$ if and only if μ satisfies the following conditions:
(i) $(\forall x, y \in S)(\max\{\mu(xy), 0.5\} \geq \min\{\mu(x), \mu(y)\})$,
(ii) $(\forall x, a, y \in S)(\max\{\mu(xay), 0.5\} \geq \min\{\mu(x), \mu(y)\})$
(iii) $(\forall x, y \in S)(\max\{\mu(x), 0.5\} \geq \mu(y)$ with $x \leq y)$.

2.3. Definition (Jun et al. 2009)

A fuzzy subset μ of S is called an $(\in, \in\vee q)$-*fuzzy bi-ideal* of S if it satisfies the following conditions:
(i) $(\forall x, y \in S)(\forall t, r \in (0, 1])([x; t] \in \mu, [y; r] \in \mu \Longrightarrow [xy; \min\{t, r\}] \in \vee q\mu)$,
(ii) $(\forall x, a, y \in S)(\forall t, r \in (0, 1])([x; t] \in \mu, [y; r] \in \mu \Longrightarrow [xay; \min\{t, r\}] \in \vee q\mu)$,
(iii) $(\forall x, y \in S)(\forall t \in (0, 1])(x \leq y, [y; t] \in \mu \Longrightarrow [x; t] \in \vee q\mu)$.

2.4. Theorem (cf. Jun et al. 2009)

A fuzzy subset μ of S is an $(\in, \in\vee q)$-fuzzy bi-ideal of S if and only if it satisfies the following conditions:
(i) $(\forall x, y \in S)(\mu(xy) \geq \min\{\mu(x), \mu(y), 0.5\})$,
(ii) $(\forall x, a, y \in S)(\mu(xay) \geq \min\{\mu(x), \mu(y), 0.5\})$,
(iii) $(\forall x, y \in S)(x \leq y,\ \mu(x) \geq \min\{\mu(y), 0.5\})$.

2.5. Definition

A fuzzy subset μ of S is called an $(\overline{\in}, \overline{\in}\vee\overline{q})$-*fuzzy bi-ideal* of S if it satisfies the following conditions:

(B1) $(\forall x, y \in S)(\forall r \in (0, 1])([x; r]\overline{\in}\mu \Longrightarrow [y; r]\overline{\in}\vee\overline{q}\mu$ with $x \leq y)$,

(B2) $(\forall x, y \in S)(\forall r, t \in (0, 1])([xy; \min\{r, t\}]\overline{\in}\mu \Longrightarrow [x; r]\overline{\in}\vee\overline{q}\mu$ or $[y; t]\overline{\in}\vee\overline{q}\mu)$,

(B3) $(\forall x, a, y \in S)(\forall r, t \in (0, 1])([xay; \min\{r, t\}]\overline{\in}\mu \Longrightarrow [x; r]\overline{\in}\vee\overline{q}\mu$ or $[y; t]\overline{\in}\vee\overline{q}\mu)$.

2.6. Example

Consider $S = \{a, b, c, d, e\}$ with the following multiplication table and order relation:

$\cdot$	a	b	c	d	e
a	a	d	a	d	d
b	a	b	a	d	d
c	a	d	c	d	e
d	a	d	a	d	d
e	a	d	c	d	e

$$\leq\ = \{(a, a), (a, c), (a, d), (a, e), (b, b), (b, d), (b, e), (c, c), (c, e), (d, d), (d, e), (e, e)\}$$

Define a fuzzy subset $\mu : S \longrightarrow [0, 1]$ as follows:
$\mu(a) = 0.9$, $\mu(b) = \mu(c) = \mu(d) = \mu(e) = 0.5$.

Then by a routine calculation, μ is an $(\overline{\in}, \overline{\in}\vee\overline{q})$-fuzzy bi-ideal of S.

2.7. Theorem

A fuzzy subset μ of S is an $(\overline{\in}, \overline{\in}\vee\overline{q})$-fuzzy bi-ideal of S if and only if

(B4) $(\forall x, y \in S)(\max\{\mu(x), 0.5\} \geq \mu(y)$ with $x \leq y)$,

(B5) $(\forall x, y \in S)(\max\{\mu(xy), 0.5\} \geq \min\{\mu(x), \mu(y)\})$,

(B6) $(\forall x, a, y \in S)(\max\{\mu(xay), 0.5\} \geq \min\{\mu(x), \mu(y)\})$.

Proof. (B1)$\Longrightarrow$(B4). If there exist $x, y \in S$ with $x \leq y$ such that $\max\{\mu(x), 0.5\} < r = \mu(y)$, then $0.5 < t \leq 1$, $[x; r]\overline{\in}\mu$ but $[y; t] \in \mu$. By (B1), we have $[y; r]\overline{q}\mu$. Then $\mu(y) \geq t$ and $r + \mu(y) \leq 1$, which implies that $t \leq 0.5$, a contradiction. Hence (B4) is valid.

(B4)$\Longrightarrow$(B1). Let $x, y \in S$ with $x \leq y$ and $r \in (0, 1]$ be such that $[x; r]\overline{\in}\mu$. Then $\mu(x) < r$. If $\mu(x) \geq \mu(y)$, then $\mu(y) \leq \mu(x) < r$. It follows that $[y; r]\overline{\in}\mu$. If $\mu(x) < \mu(y)$, then (B4), we have $0.5 \geq \mu(y)$. Let $[y; r]\overline{\in}\mu$, then $\mu(y) < r$ and $\mu(y) \leq 0.5$. It follows that $[y; r]\overline{q}\mu$, thus $[y; r]\overline{\in}\vee\overline{q}\mu$.

(B2)$\Longrightarrow$(B5). If there exist $x, y \in S$ such that $\max\{\mu(xy), 0.5\} < t = \min\{\mu(x), \mu(y)\}$, then $0.5 < t \leq 1$, $[xy; t]\overline{\in}\mu$ but $[x; t] \in \mu$ and $[y; t] \in \mu$. By (B2), we have $[x; t]\overline{q}\mu$ or $[y; t]\overline{q}\mu$. Then $(\mu(x) \geq t$ and $t + \mu(x) \leq 1)$ or $(\mu(y) \geq t$ and $t + \mu(y) \leq 1)$, which implies that $t \leq 0.5$, a contradiction.

(B5)$\Longrightarrow$(B2). Let $x, y \in S$ and $r, t \in (0, 1]$ be such that $[xy; \min\{r, t\}]\overline{\in}\mu$, then $\mu(xy) < \min\{r, t\}$.
(a) If $\mu(xy) \geq \min\{\mu(x), \mu(y)\}$, then $\min\{\mu(x), \mu(y)\} < \min\{r, t\}$, and consequently, $\mu(x) < r$ or $\mu(y) < t$. It follows that $[x; r]\overline{\in}\mu$ or $[y; t]\overline{\in}\mu$. Thus $[x; r]\overline{\in}\vee\overline{q}\mu$ or $[y; t]\overline{\in}\vee\overline{q}\mu$.

(b) If $\mu(xy) < \min\{\mu(x), \mu(y)\}$, then by (B5), we have $0.5 \geq \min\{\mu(x), \mu(y)\}$. Let $[x; r]\overline{\in}\mu$ or $[y; t]\overline{\in}\mu$, then $\mu(x) < r$ and $\mu(x) \leq 0.5$ or $\mu(y) < t$ and $\mu(y) \leq 0.5$. It follows $[x; r]\overline{q}\mu$ or $[y; t]\overline{q}\mu$, and $[x; r]\overline{\in \vee q}\mu$ or $[y; t]\overline{\in \vee q}\mu$.

(B3)$\Longrightarrow$(B6). If there exist $x, a, y \in S$ such that $\max\{\mu(xay), 0.5\} < t = \min\{\mu(x), \mu(y)\}$, then $0.5 < t \leq 1$, $[xay; t]\overline{\in}\mu$ but $[x; t] \in \mu$ and $[y; t] \in \mu$. By (B3), we have $[x; t]\overline{q}\mu$ or $[y; t]\overline{q}\mu$. Then $(\mu(x) \geq t$ and $t + \mu(x) \leq 1)$ or $(\mu(y) \geq t$ and $t + \mu(y) \leq 1)$, which implies that $t \leq 0.5$, a contradiction.

(B6)$\Longrightarrow$(B3). Let $x, a, y \in S$ and $r, t \in (0, 1]$ be such that $[xay; \min\{r, t\}]\overline{\in}\mu$, then $\mu(xay) < \min\{r, t\}$.
(a) If $\mu(xay) \geq \min\{\mu(x), \mu(y)\}$, then $\min\{\mu(x), \mu(y)\} < \min\{r, t\}$, and consequently, $\mu(x) < r$ or $\mu(y) < t$. It follows that $[x; r]\overline{\in}\mu$ or $[y; t]\overline{\in}\mu$. Thus $[x; r]\overline{\in \vee q}\mu$ or $[y; t]\overline{\in \vee q}\mu$.
(b) If $\mu(xay) < \min\{\mu(x), \mu(y)\}$, then by (B6), we have $0.5 \geq \min\{\mu(x), \mu(y)\}$. Let $[x; r]\overline{\in}\mu$ or $[y; t]\overline{\in}\mu$, then $\mu(x) < r$ and $\mu(x) \leq 0.5$ or $\mu(y) < t$ and $\mu(y) \leq 0.5$. It follows $[x; r]\overline{q}\mu$ or $[y; t]\overline{q}\mu$, and $[x; r]\overline{\in \vee q}\mu$ or $[y; t]\overline{\in \vee q}\mu$. □

2.8. Definition (Jun et al. 2009)
Given $\gamma, \delta \in (0, 1]$ and $\gamma < \delta$, a fuzzy subset μ of S is a *fuzzy bi-ideal with thresholds* (γ, δ) of S if it satisfies the following conditions:
(1) $(\forall x, y \in S)(\forall \gamma, \delta \in (0, 1])(\max\{\mu(xy), \gamma\} \geq \min\{\mu(x), \mu(y), \delta\})$,
(2) $(\forall x, a, y \in S)(\forall \gamma, \delta \in (0, 1])(\max\{\mu(xay), \gamma\} \geq \min\{\mu(x), \mu(y), \delta\})$,
(3) $(\forall x, y \in S)(\forall \gamma, \delta \in (0, 1])(\max\{\mu(x), \gamma\} \geq \min\{\mu(y), \delta\})$.

3. (α, β)-FUZZY BI-IDEALS

In this section, we introduce some new relationships between fuzzy points and fuzzy subsets, and investigate (α, β)-fuzzy bi-ideals of ordered semigroups.

In what follows let $\gamma, \delta \in [0, 1]$ be such that $\gamma < \delta$. For a fuzzy point $[x; r]$ and a fuzzy subset μ of X, we say that
(1) $[x; r] \in_\gamma \mu$ if $\mu(x) \geq r > \gamma$.
(2) $[x; r]q_\delta\mu$ if $\mu(x) + r > 2\delta$.
(3) $[x; r] \in_\gamma \vee q_\delta\mu$ if $[x; r] \in_\gamma \mu$ or $[x; r]q_\delta\mu$.
(4) $[x; r] \in_\gamma \wedge q_\delta\mu$ if $[x; r] \in_\gamma \mu$ and $[x; r]q_\delta\mu$.
(5) $[x; r]\overline{\alpha}\mu$ if $[x; r]\alpha\mu$ does not hold for $\alpha \in \{\in_\gamma, q_\delta, \in_\gamma \vee q_\delta, \in_\gamma \wedge q_\delta\}$.

Note that, the case when $\alpha = \in_\gamma \wedge q_\delta$ is omitted. Because if μ is a fuzzy subset of S such that $\mu(x) \leq \delta$ for all $x \in S$. Let $x \in S$ and $r \in (0, 1]$ be such that $[x; r] \in_\gamma \wedge q_\delta\mu$. Then $\mu(x) \geq r > \gamma$ and $\mu(x) + r > 2\delta$. It follows that $2\mu(x) = \mu(x) + \mu(x) \geq \mu(x) + r > 2\delta$ so that $\mu(x) > \delta$. This means that $\{[x; r] \mid [x; r] \in_\gamma \wedge q_\delta\} = \emptyset$.

Now, we introduce the concept of (α, β)-fuzzy bi-ideals (resp. (α, β)-fuzzy subsemigroups) of ordered semigroups as following:

3.1. Definition
A fuzzy subset μ of S is called an (α, β)-*fuzzy subsemigroup* of S if it satisfies the condition:
$(\forall x, y \in S)(\forall r, t \in (\gamma, 1])([x; r]\alpha\mu$ and $[y; t]\alpha\mu \Longrightarrow [xy; \min\{r, t\}]\beta\mu)$.

3.2. Theorem
Let $2\delta = 1 + \gamma$ and μ be an $(\alpha, \in_\gamma \vee q_\delta)$-fuzzy subsemigroup of S. Then the set $\mu_{\bar{\gamma}}$ is a subsemigroup of S, where $\mu_{\bar{\gamma}} = \{x \in S \mid \mu(x) > \gamma\}$.

Proof. Assume that μ is an $(\alpha, \in_\gamma \vee q_\delta)$-fuzzy subsemigroup of S. Let $x, y \in \mu_{\bar{\gamma}}$. Then $\mu(x) > \gamma$ and $\mu(y) > \gamma$. We consider the following two cases:
Case 1: $\alpha \in \{\in_\gamma, \in_\gamma \vee q_\delta\}$. Then $[x; \mu(x)]\alpha\mu$ and $[y; \mu(y)]\alpha\mu$. It follows that $[xy; \min\{\mu(x), \mu(y)\}] \in_\gamma \vee q_\delta\mu$, i.e., $[xy; \min\{\mu(x), \mu(y)\}] \in_\gamma \mu$ or $[xy; \min\{\mu(x), \mu(y)\}]q_\delta\mu$. Hence we have

$$\mu(xy) \geq \min\{\mu(x), \mu(y)\} > \gamma \quad \text{or}$$
$$\mu(xy) + \min\{\mu(x), \mu(y)\} > 2\delta$$

and so $\mu(xy) \geq \min\{\mu(x), \mu(y)\} > \gamma$ or

$$\mu(xy) > 2\delta - \min\{\mu(x), \mu(y)\} \geq 2\delta - 1 = \gamma$$

Hence $\mu(xy) > \gamma$ and so $xy \in \mu_{\bar{\gamma}}$.
Case 2: $\alpha = q_\delta$. Then $[x; 1]\alpha\mu$ and $[y; 1]\alpha\mu$, since $2\delta = 1 + \gamma$. Analogous to the proof of case 1, we have $xy \in \mu_{\bar{\gamma}}$. Consequently, $\mu_{\bar{\gamma}}$ is a subsemigroup of S. □

3.3. Theorem
Let $2\delta = 1 + \gamma$ and A be a non-empty subset of S. Then A is a subsemigroup of S if and only if the fuzzy subset μ of S defined as follows:

$$\mu(x) = \begin{cases} \geq \delta & \text{for all } x \in A \\ \gamma & \text{otherwise} \end{cases}$$

is an $(\alpha, \in_\gamma \vee q_\delta)$-fuzzy subsemigroup of S.

Proof. Assume that A is a subsemigroup of S. Let $x, y \in S$ and $r, t \in (\gamma, 1]$ be such that $[x; r]\alpha\mu$ and $[y; t]\alpha\mu$. We consider the following four cases:
Case 1: $[x; r] \in_\gamma \mu$ and $[y; t] \in_\gamma \mu$. Then $\mu(x) \geq r > \gamma$ and $\mu(y) \geq t > \gamma$. Thus, $\mu(x) \geq \delta$ and $\mu(y) \geq \delta$, i.e., $x, y \in A$.
Case 2: $[x; r]q_\delta\mu$ and $[y; t]q_\delta\mu$ Then $\mu(x) + r > 2\delta$ and $\mu(y) + t > 2\delta$, and so $\mu(x) > 2\delta - r \geq 2\delta - 1 = \gamma$ and $\mu(y) > 2\delta - t \geq 2\delta - 1 = \gamma$. It follows that $\mu(x) \geq \delta$ and $\mu(y) \geq \delta$, i.e., $x, y \in A$.

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Case 3: $[x; r] \in_\gamma \mu$ and $[y; t]q_\delta\mu$. Then $\mu(x) \geq r > \gamma$ and $\mu(y) + t > 2\delta$. Analogous the proof of case 1 and 2, we have $\mu(x) \geq \delta$ and $\mu(y) \geq \delta$, i.e., $x, y \in A$.
Case 4: $[x; r]q_\delta\mu$ and $[y; t] \in_\gamma \mu$. Then $\mu(x) + r > 2\delta$ and $\mu(x) \geq r > \gamma$. Analogous the proof of case 1 and 2, we have $\mu(x) \geq \delta$ and $\mu(y) \geq \delta$, i.e., $x, y \in A$.

Thus, in any case, $x, y \in A$. Hence $xy \in A$, which implies that $\mu(xy) \geq \delta$. If $\min\{r, t\} \leq \delta$, then $\mu(xy) \geq \delta \geq \min\{r, t\} > \gamma$, i.e., $[xy; \min\{r, t\}] \in_\gamma \mu$. If $\min\{r, t\} > \delta$, then $\mu(xy) + \min\{r, t\} > \delta + \delta = 2\delta$, i.e., $[xy; \min\{r, t\}]q_\delta\mu$. Therefore $[xy; \min\{r, t\}] \in_\gamma \vee q_\delta\mu$.

Conversely, assume that μ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy subsemigroup of S. It is easy to see that $A = \mu_{\bar{\gamma}}$. Hence, it follows from Theorem 3.2, that A is a subsemigroup of S. □

3.4. Proposition
Every $(\in_\gamma \vee q_\delta, \in_\gamma \vee q_\delta)$-fuzzy subsemigroup of S is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy subsemigroup of S.

Proof. It is straightforward since $[x; r] \in_\gamma \mu$ implies $[x; r] \in_\gamma \vee q_\delta\mu$ for all $x \in S$ and $r \in (\gamma, 1]$. □

3.5. Proposition
Every $(\in_\gamma, \in_\gamma)$-fuzzy subsemigroup of S is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy subsemigroup of S.

Proof. The proof is straightforward and is omitted here. □

The following example shows that the converses of Propositions 3.4 and 3.5 may not be true.

3.6. Example
Consider the ordered semigroup as given in Example 2.6 and define a fuzzy subset μ as follows:
$\mu(a) = 0.8$, $\mu(b) = 0.7$, $\mu(d) = 0.4$, $\mu(c) = 0.6$, $\mu(e) = 0.3$.
Then
(1) it is routine to verify that μ is an $(\in_{0.3}, \in_{0.3} \vee q_{0.4})$-fuzzy bi-ideal of S.
(2) μ is not an $(\in_{0.3}, \in_{0.3})$-fuzzy bi-ideal of S, since $[a; 0.78] \in_{0.3} \mu$ and $[b; 0.68] \in_{0.3} \mu$ but $[ab; \min\{0.78, 0.68\}] = [d; 0.68]\overline{\in_{0.3}}\mu$.
(3) μ is not an $(\in_{0.3} \vee q_{0.6}, \in_{0.3} \vee q_{0.6})$-fuzzy bi-ideal of S, since $[a; 0.68] \in_{0.3} \vee q_{0.6}\mu$ and $[b; 0.48] \in_{0.3} \vee q_{0.6}\mu$ but $[ab; \min\{0.68, 0.48\}] = [d; 0.48]\ \overline{\in_{0.3} \vee q_{0.6}}\mu$.

3.7. Definition
A fuzzy subset μ of S is called an (α, β)-*fuzzy bi-ideal* of S if it satisfies the conditions:
(B7) $(\forall x, y \in S)(\forall r \in (\gamma, 1])([y; r]\alpha\mu \Longrightarrow [x; r]\beta\mu$ with $x \leq y)$.
(B8) $(\forall x, y \in S)(\forall r, t \in (\gamma, 1])([x; r]\alpha\mu$ and $[y; t]\alpha\mu \Longrightarrow [xy; \min\{r, t\}]\beta\mu)$.
(B9) $(\forall x, a, y \in S)(\forall r, t \in (\gamma, 1])([x; r]\alpha\mu$ and $[y; t]\alpha\mu \Longrightarrow [xay; \min\{r, t\}]\beta\mu)$.

3.8. Example
Consider the ordered semigroup as given in Example 2.6 and define a fuzzy subset μ of S as follows:
$\mu(a) = 0.9$, $\mu(b) = \mu(c) = \mu(d) = \mu(e) = 0.5$.
Then μ is an $(\in_{0.3}, \in_{0.3} \vee q_{0.6})$-fuzzy bi-ideal of S.

3.9. Theorem
Let $2\delta = 1 + \gamma$ and μ be an $(\alpha, \in_\gamma \vee q_\delta)$-fuzzy bi-ideal of S and $\alpha \neq \in_\gamma \wedge q_\delta$. Then the set $\mu_{\bar{\gamma}}$ is a bi-ideal of S.

Proof. Assume that μ is an $(\alpha, \in_\gamma \vee q_\delta)$-fuzzy bi-ideal of S. Let $x, z \in \mu_{\bar{\gamma}}$. Then $\mu(x) > \gamma$ and $\mu(z) > \gamma$. We consider the following two cases:
Case 1: $\alpha \in \{\in_\gamma, \in_\gamma \vee q_\delta\}$. Then $[x; \mu(x)]\alpha\mu$ and $[z; \mu(z)]\alpha\mu$. By $(B9)$,

$$[xyz; \min\{\mu(x), \mu(z)\}] \in_\gamma \vee q_\delta\mu$$

i.e., $[xyz; \min\{\mu(x), \mu(z)\}] \in_\gamma \mu$ or $[xyz; \min\{\mu(x), \mu(z)\}]q_\delta\mu$. It follows that

$$\mu(xyz) \geq \min\{\mu(x), \mu(z)\} > \gamma \quad \text{or}$$
$$\mu(xyz) + \min\{\mu(x), \mu(z)\} > 2\delta$$

and so $\mu(xyz) \geq \min\{\mu(x), \mu(z)\} > \gamma$ or

$$\mu(xyz) > 2\delta - \min\{\mu(x), \mu(z)\} \geq 2\delta - 1 = \gamma$$

Hence $\mu(xyz) > \gamma$ and so $xyz \in \mu_{\bar{\gamma}}$.
Case 2: $\alpha = q_\delta$. Then $[x; 1]\alpha\mu$ and $[z; 1]\alpha\mu$, since $2\delta = 1 + \gamma$. Analogous to the proof of case 1, we have $xyz \in \mu_{\bar{\gamma}}$. Similarly, for $x, y \in S$ and $x \leq y$ if $y \in \mu_{\bar{\gamma}}$ then $x \in \mu_{\bar{\gamma}}$. The remaining proof follows from Theorem 3.2. Consequently, $\mu_{\bar{\gamma}}$ is a bi-ideal of S. □

3.10. Theorem
Let μ be a fuzzy subset of S. If μ is a $(q_\delta, \in_\gamma \vee q_\delta)$-fuzzy subsemigroup, then the following condition holds:

$$(\forall x, y \in S)(\max\{\mu(xy), \gamma\} \geq \min\{\mu(x), \mu(y), \delta\})$$

Proof. Let μ be a $(q_\delta, \in_\gamma \vee q_\delta)$-fuzzy subsemigroup of S. Assume that there exist $x, y \in S$ such that $\max\{\mu(xy), \gamma\} < \min\{\mu(x), \mu(y), \delta\}$. Then for all $\gamma < r \leq 1$ such that

$$2\delta - \max\{\mu(xy, \gamma\} > r \geq 2\delta - \min\{\mu(x), \mu(y), \delta\}$$

we have

$$2\delta - \mu(xy) \geq 2\delta - \max\{\mu(xy), \gamma\} > r$$
$$> \max\{2\delta - \mu(x), 2\delta - \mu(y), \delta\}$$

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and so

$$\mu(x)+r>2\delta,\quad \mu(y)+r>2\delta,\quad \mu(xy)+r<2\delta$$

and $\mu(xy)<\delta<r$. Hence $[x;r]q_\delta\mu, [y;r]q_\delta\mu$ but $[xy;r]\overline{\in_\gamma\vee q_\delta}\mu$, a contradiction. Therefore

$$\max\{\mu(xy),\gamma\}\geq\min\{\mu(x),\mu(y),\delta\}$$

for all $x,y\in S$. □

3.11. Theorem

Let μ be a fuzzy subset of S. If μ is a $(q_\delta,\in_\gamma\vee q_\delta)$-fuzzy bi-ideal, then the following conditions hold:
(1) $(\forall x,y\in S)(\max\{\mu(x),\gamma\}\geq\min\{\mu(y),\delta\}$ with $x\leq y)$,
(2) $(\forall x,y\in S)(\max\{\mu(xy),\gamma\}\geq\min\{\mu(x),\mu(y),\delta\})$,
(3) $(\forall x,a,y\in S)(\max\{\mu(xay),\gamma\}\geq\min\{\mu(x),\mu(y),\delta\})$.

Proof. The proof of (2) follows from Theorem 3.10. Assume that there exist $x,a,y\in S$ such that $\max\{\mu(xay),\gamma\}<\min\{\mu(x),\mu(y),\delta\}$. Then for all $\gamma<r\leq 1$ such that

$$2\delta-\max\{\mu(xay,\gamma\}>r\geq 2\delta-\min\{\mu(x),\mu(y),\delta\}$$

we have

$$2\delta-\mu(xay)\geq 2\delta-\max\{\mu(xay),\gamma\}>r$$
$$>\max\{2\delta-\mu(x),2\delta-\mu(y),\delta\}$$

and so

$$\mu(x)+r>2\delta,\quad \mu(y)+r>2\delta,\quad \mu(xay)+r<2\delta$$

and $\mu(xay)<\delta<r$. Hence $[x;r]q_\delta\mu,[y;r]q_\delta\mu$ but $[xay;r]\overline{\in_\gamma\vee q_\delta}\mu$, a contradiction. Therefore

$$\max\{\mu(xay),\gamma\}\geq\min\{\mu(x),\mu(y),\delta\}$$

for all $x,y\in S$. Similarly, we can prove that for all $x,y\in S$, $\max\{\mu(x),\gamma\}\geq\min\{\mu(y),\delta\}$, with $x\leq y$. □

3.12. Theorem

A fuzzy subset μ of S is an $(\in_\gamma,\in_\gamma\vee q_\delta)$-fuzzy subsemigroup of S if and only if the following condition holds:

$$(\forall x,y\in S)(\max\{\mu(xy),\gamma\}\geq\min\{\mu(x),\mu(y),\delta\})$$

Proof. Assume that μ is an $(\in_\gamma,\in_\gamma\vee q_\delta)$-fuzzy subsemigroup of S. If there exist $x,y\in S$ such that $\max\{\mu(xy),\gamma\}<r\leq\min\{\mu(x),\mu(y),\delta\}$. Then

$$\mu(x)\geq r>\gamma,\quad \mu(y)\geq r>\gamma,\quad \mu(xy)<r$$

and $\mu(xy)+r<2r\leq 2\delta$, i.e., $[x;r]\in_\gamma\mu,[y;r]\in_\gamma\mu$ but $[xy;r]\overline{\in_\gamma\vee q_\delta}\mu$, a contradiction. Hence $\max\{\mu(xy),\gamma\}\geq\min\{\mu(x),\mu(y),\delta\}$ for all $x,y\in S$.

Conversely, assume that there exist $x,y\in S$ and $r,t\in(\gamma,1]$ such that $[x;r]\in_\gamma\mu,[y;t]\in_\gamma\mu$ but $[xy;\min\{r,t\}]\overline{\in_\gamma\vee q_\delta}\mu$, then

$$\mu(x)\geq r>\gamma,\quad \mu(y)\geq t>\gamma,\quad \mu(xy)<\min\{r,t\}$$

and $\mu(xy)+\min\{r,t\}\leq 2\delta$. It follows that $\mu(xy)<\delta$ and so $\max\{\mu(xy),\gamma\}<\min\{\mu(x),\mu(y),\delta\}$, a contradiction. Hence $[xy;r]\in_\gamma\vee q_\delta\mu$, consequently, μ is an $(\in_\gamma,\in_\gamma\vee q_\delta)$-fuzzy subsemigroup. □

3.13. Theorem

A fuzzy subset μ of S is an $(\in_\gamma,\in_\gamma\vee q_\delta)$-fuzzy bi-ideal of S if and only if the following condition hold
(B10) $(\forall x,y\in S)(\max\{\mu(x),\gamma\}\geq\min\{\mu(y),\delta\}$ with $x\leq y)$,
(B11) $(\forall x,y\in S)(\max\{\mu(xy),\gamma\}\geq\min\{\mu(x),\mu(y),\delta\})$,
(B12) $(\forall x,a,y\in S)(\max\{\mu(xay),\gamma\}\geq\min\{\mu(x),\mu(y),\delta\})$.

Proof. $(B7)\Longrightarrow(B10)$. If there exist $x,y\in S$ with $x\leq y$ such that $\max\{\mu(x),\gamma\}<t\leq\min\{\mu(y),\delta\}$. Then $\mu(y)\geq t>\gamma$, $\mu(x)<t$ and $\mu(x)+t<2t\leq 2\delta$, i.e., $[y;t]\in_\gamma\mu$ but $[x;t]\overline{\in_\gamma\vee q_\delta}\mu$, a contradiction. Hence $(B10)$ is valid.

$(B10)\Longrightarrow(B7)$. Assume that there exist $x,y\in S$ with $x\leq y$ and $t\in(\gamma,1]$ such that $[y;t]\in_\gamma\mu$ but $[x;t]\overline{\in_\gamma\vee q_\delta}\mu$, then $\mu(y)\geq t>\gamma$, $\mu(x)<t$ and $\mu(x)+t<2\delta$. It follows that $\mu(x)<\delta$ and so $\max\{\mu(x),\gamma\}<\min\{t,\delta\}\leq\min\{\mu(y),\delta\}$, a contradiction. Hence $(B7)$ is valid.

$(B8)\Longrightarrow(B11)$. If there exist $x,y\in S$ such that $\max\{\mu(xy),\gamma\}<r\leq\min\{\mu(x),\mu(y),\delta\}$. Then

$$\mu(x)\geq r>\gamma,\quad \mu(y)\geq r>\gamma,\quad \mu(xy)<r$$

and $\mu(xy)+r<2r\leq 2\delta$, i.e., $[x;r]\in_\gamma\mu,[y;r]\in_\gamma\mu$ but $[xy;r]\overline{\in_\gamma\vee q_\delta}\mu$, a contradiction. Hence $\max\{\mu(xy),\gamma\}\geq\min\{\mu(x),\mu(y),\delta\}$ for all $x,y\in S$.

$(B11)\Longrightarrow(B8)$. Assume that there exist $x,y\in S$ and $r,t\in(\gamma,1]$ such that $[x;r]\in_\gamma\mu,[y;t]\in_\gamma\mu$ but $[xy;\min\{r,t\}]\overline{\in_\gamma\vee q_\delta}\mu$, then

$$\mu(x)\geq r>\gamma,\quad \mu(y)\geq t>\gamma,$$
$$\mu(xy)<\min\{\mu(x),\mu(y),\delta\}$$

and $\mu(xy)+\min\{r,t\}\leq 2\delta$. It follows that $\mu(xy)<\delta$ and so $\max\{\mu(xy),\gamma\}<\min\{\mu(x),\mu(y),\delta\}$, a contradiction. Hence $(B8)$ is valid.

$(B9)\Longrightarrow(B12)$. If there exist $x,a,y\in S$ such that $\max\{\mu(xay),\gamma\}<r\leq\min\{\mu(x),\mu(y),\delta\}$. Then

$$\mu(x)\geq r>\gamma,\quad \mu(y)\geq r>\gamma,\quad \mu(xay)<r$$

and $\mu(xay)+r<2r\leq 2\delta$, i.e., $[x;r]\in_\gamma\mu,[y;r]\in_\gamma\mu$ but $[xay;r]\overline{\in_\gamma\vee q_\delta}$, a contradiction. Hence $\max\{\mu(xay),\gamma\}\geq\min\{\mu(x),\mu(y),\delta\}$ for all $x,y\in S$.

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$(B12) \Longrightarrow (B9)$. Assume that there exist $x, a, y \in S$ and $r, t \in (\gamma, 1]$ such that $[x; r] \in_\gamma \mu, [y; t] \in_\gamma \mu$ but $[xay; \min\{r, t\}] \overline{\in_\gamma \vee q_\delta} \mu$, then

$$\mu(x) \geq r > \gamma, \quad \mu(y) \geq t > \gamma,$$
$$\mu(xay) < \min\{\mu(x), \mu(y), \delta\}$$

and $\mu(xay) + \min\{r, t\} \leq 2\delta$. It follows that $\mu(xay) < \delta$ and so $\max\{\mu(xay), \gamma\} < \min\{\mu(x), \mu(y), \delta\}$, a contradiction. Hence $(B9)$ is valid. □

As a direct consequence of Theorems 3.11 and 3.13, we have the following result.

3.14. Proposition

Every $(q_\delta, \in_\gamma \vee q_\delta)$-fuzzy bi-ideal of S is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy bi-ideal of S.

The following example shows that the converse of Proposition 3.14 is not true.

3.15. Example

Consider the ordered semigroup and a fuzzy subset μ as given in Example 3.6. Then μ is an $(\in_{0.3}, \in_{0.3} \vee q_{0.6})$-fuzzy bi-ideal of S, but it is not a $(q_{0.6}, \in_{0.3} \vee q_{0.6})$-fuzzy bi-ideal, since $[a; 0.55] q_{0.6} \mu$ and $[b; 0.68] q_{0.6} \mu$ but $[ab; \min\{0.55, 0.68\}] = [d; 0.55] \overline{\in_{0.3} \vee q_{0.6}} \mu$.

For any $\mu \in \mathbb{F}(S)$, where $\mathbb{F}(S)$ denotes the set of all fuzzy subsets of S, we define $\mu_r = \{x \in S \mid [x; r] \in_\gamma \mu\}$, $\mu_r^\delta = \{x \in S \mid [x; r] q_\delta \mu\}$ and $[\mu]_r^\delta = \{x \in S \mid [x; r] \in_\gamma \vee q_\delta \mu\}$ for all $r \in (\gamma, 1]$. It follows that $[\mu]_r^\delta = \mu_r \cup \mu_r^\delta$.

The next theorem provides the relationship between $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy subsemigroups, $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy bi-ideals of S and crisp subsemigroups and bi-ideals of S.

3.16. Theorem

Let $\mu \in \mathbb{F}(S)$.
(1) μ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy subsemigroup of S if and only if $\mu_r (\neq \emptyset)$ is a subsemigroup of S for all $r \in (\gamma, \delta]$.
(2) If $2\delta = 1 + \gamma$, then μ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy subsemigroup if and only if $\mu_r^\delta (\neq \emptyset)$ is a subsemigroup of S.
(3) If $2\delta = 1 + \gamma$, then μ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy subsemigroup if and only if $[\mu]_r^\delta (\neq \emptyset)$ is a subsemigroup of S.

Proof. We prove only (3). Let μ be an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy subsemigroup of S and let $x, y \in [\mu]_r^\delta$ for some $r \in (\gamma, 1]$. Then $[x; r] \in_\gamma \vee q_\delta \mu$ and $[y; r] \in_\gamma \vee q_\delta \mu$, i.e., $\mu(x) \geq r$ or $\mu(x) > 2\delta - r \geq 2\delta - 1 = \gamma$, and $\mu(y) \geq r$ or

$$\mu(y) > 2\delta - r \geq 2\delta - 1 = \gamma$$

Since μ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy subsemigroup of S, we have $\mu(xy) \geq \min\{\mu(x), \mu(y), \delta\}$. We consider the following cases:
Case 1: $r \in (\gamma, \delta]$. Since $r \in (\gamma, \delta]$, we have $2\delta - r \geq \delta \geq r$. It follows that $\mu(x) \geq r$ and $\mu(y) \geq r$. Hence

$$\mu(xy) \geq \min\{\mu(x), \mu(y), \delta\} \geq \min\{r, r, r\} = r$$

and so $[xy; r] \in_r \mu$.
Case 2: $r \in (\delta, 1]$. Since $r \in (\delta, 1]$, we have $2\delta - r < \delta < r$. It follows that $\mu(x) > 2\delta - r$ and $\mu(y) > 2\delta - r$. Hence

$$\begin{aligned} \mu(xy) &\geq \min\{\mu(x), \mu(y), \delta\} \\ &> \min\{2\delta - r, 2\delta - r, 2\delta - r\} = 2\delta - r \end{aligned}$$

and so $[xy; r] q_\delta \mu$. Thus, in any case, we have $[xy; r] \in_\gamma \vee q_\delta \mu$. Therefore $[\mu]_r^\delta$ is a subsemigroup of S.

Conversely, assume that the given conditions hold. Let $x, y \in S$. If $\max\{\mu(xy), \gamma\} < r = \min\{\mu(x), \mu(y), \delta\}$, then $[x; r] \in_\gamma \mu$ and $[y; r] \in_\gamma \mu$, but $[xy; r] \overline{\in_\gamma \vee q_\delta} \mu$, i.e., $x, y \in [\mu]_r^\delta$, but $xy \notin [\mu]_r^\delta$, a contradiction. Therefore, $\max\{\mu(xy), \gamma\} \geq \min\{\mu(x), \mu(y), \delta\}$ for all $x, y \in S$. Consequently, μ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy subsemigroup. □

Similarly we have the following Theorem:

3.17. Theorem

Let $\mu \in \mathbb{F}(S)$.
(1) μ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy bi-ideal of S if and only if $\mu_r (\neq \emptyset)$ is a bi-ideal of S for all $r \in (\gamma, \delta]$.
(2) If $2\delta = 1 + \gamma$, then μ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy bi-ideal if and only if $\mu_r^\delta (\neq \emptyset)$ is a bi-ideal of S.
(3) If $2\delta = 1 + \gamma$, then μ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy bi-ideal if and only if $[\mu]_r^\delta (\neq \emptyset)$ is a bi-ideal of S.

As a direct consequence of Theorem 3.17, we have the following corollaries:

3.18. Corollary

Let $\gamma, \gamma', \delta, \delta' \in [0, 1]$ be such that $\gamma < \delta, \gamma' < \delta'$, and $\delta' < \delta$. Then every $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy subsemigroup of S is an $(\in_{\gamma'}, \in_{\gamma'} \vee q_{\delta'})$-fuzzy subsemigroup of S.

3.19. Corollary

Let $\gamma, \gamma', \delta, \delta' \in [0, 1]$ be such that $\gamma < \delta, \gamma' < \delta'$, and $\delta' < \delta$. Then every $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy bi-ideal of S is an $(\in_{\gamma'}, \in_{\gamma'} \vee q_{\delta'})$-fuzzy bi-ideal of S.

The following example shows that the converse of Corollary 3.18 and Corollary 3.19 may not true in general, as shown in the following example.

3.20. Example

Consider the ordered semigroup and fuzzy subset μ as shown in Example 3.6, the μ is an $(\in_{0.3}, \in_{0.3} \vee q_{0.4})$-fuzzy bi-ideal (subsemigroup) but not an $(\in_{0.3}, \in_{0.3} \vee q_{0.9})$-fuzzy bi-ideal (subsemigroup) of S.

If we take $\gamma = 0$ and $\delta = 0.5$ in Theorem 3.17, then we have the following result.

3.21. Corollary

Let $\mu \in \mathbb{F}(S)$.
(1) μ is an $(\in, \in \vee q)$-fuzzy bi-ideal of S if and only $U(\mu; r) (\neq \emptyset)$ is a bi-ideal of S for all $r \in (0, 0.5]$, where $U(\mu; r) = \{x \in S \mid [x; r] \in \mu\}$ (see Jun et al. 2009).

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(2) μ is an $(\in, \in \vee q)$-fuzzy bi-ideal of S if and only $Q(\mu; r)(\neq \emptyset)$ is a bi-ideal of S for all $r \in (0.5, 1]$, where $Q(\mu; r) = \{x \in S \mid [x; r]q\mu\}$ (see Jun et al. 2009).
(3) μ is an $(\in, \in \vee q)$-fuzzy bi-ideal of S if and only $[\mu]_r(\neq \emptyset)$ is a bi-ideal of S for all $r \in (0, 1]$, where $[\mu]_r = \{x \in S \mid [x; r]q\mu\}$ (see Jun et al. 2009).

4. $(\overline{\beta}, \overline{\alpha})$-FUZZY BI-IDEALS

In this section, we define and investigate $(\overline{\beta}, \overline{\alpha})$-fuzzy subsemigroups and $(\overline{\beta}, \overline{\alpha})$-fuzzy bi-ideals of ordered semigroups, where $\overline{\alpha}, \overline{\beta} \in \{\overline{\in_\gamma}, \overline{q_\delta}, \overline{\in_\gamma} \wedge \overline{q_\delta}, \overline{\in_\gamma} \vee \overline{q_\delta}\}$.

4.1. Definition

A fuzzy subset μ of S is called $(\overline{\beta}, \overline{\alpha})$-fuzzy bi-ideal of S if it satisfies the following conditions:

(B13) $(\forall x, y \in S)(\forall r, t \in (\gamma, 1])([xy; \min\{r, t\}]\overline{\beta}\mu \Longrightarrow [x; r]\overline{\alpha}\mu$ or $[y; t]\overline{\alpha}\mu)$,

(B14) $(\forall x, a, y \in S)(\forall r, t \in (\gamma, 1])([xay; \min\{r, t\}]\overline{\beta}\mu \Longrightarrow [x; r]\overline{\alpha}\mu$ or $[y; t]\overline{\alpha}\mu)$.

(B15) $(\forall x, y \in S)(\forall r \in (\gamma, 1])([x; r]\overline{\beta}\mu \Longrightarrow [y; r]\overline{\alpha}\mu$ with $x \leq y)$.

The case when $\overline{\beta} = \overline{\in_\gamma} \wedge \overline{q_\delta}$ can be omitted since for a fuzzy subset μ of S such that $\mu(x) \geq \delta$ for any $x \in S$ in the case $[x; r]\overline{\in_\gamma} \wedge \overline{q_\delta}\mu$ we have $\mu(x) < r$ and $\mu(x) + r < 2\delta$. Thus $\mu(x) + \mu(x) < \mu(x) + r \leq 2\delta$, which implies $\mu(x) < \delta$. This means that $\{[x; r] : [x; r]\overline{\in_\gamma} \wedge \overline{q_\delta}\mu\} = \emptyset$.

As it is not difficult to see that every $(\overline{\beta}, \overline{\alpha})$-fuzzy bi-ideal of S is a $(\overline{\beta}, \overline{\in_\gamma} \vee \overline{q_\delta})$-fuzzy bi-ideal of S. Hence in the theory of $(\overline{\beta}, \overline{\alpha})$-fuzzy bi-ideals the central role is played by $(\overline{\beta}, \overline{\in_\gamma} \vee \overline{q_\delta})$-fuzzy bi-ideals and we are interesting to investigate only the properties of $(\overline{\beta}, \overline{\in_\gamma} \vee \overline{q_\delta})$-fuzzy bi-ideals of S.

4.2. Example

Consider an ordered semigroup $S = \{a, b, c, d, e\}$ with the following multiplication table and order relation:

$\cdot$	a	b	c	d	e
a	a	d	a	d	d
b	a	b	a	d	d
c	a	d	c	d	e
d	a	d	a	d	d
e	a	d	c	d	e

$$\leq = \{(a, a), (a, c), (a, e), (b, b), (b, e),$$
$$(c, c), (c, e), (d, d), (d, e), (e, e)\}$$

Define a fuzzy subset $\mu : S \longrightarrow [0, 1]$ as follows:
$\mu(a) = 0.6, \mu(b) = 0.5, \mu(c) = 0.7, \mu(d) = 0.8, \mu(e) = 0.9$.

Then by a routine calculation, μ is an $(\overline{\in_{0.3}}, \overline{\in_{0.3}} \vee \overline{q_{0.6}})$-fuzzy bi-ideal of S.

4.3. Theorem

Let μ be a $(\overline{\beta}, \overline{\in_\gamma} \vee \overline{q_\delta})$-fuzzy subsemigroup of S. Then the set $\mu_{\overline{\delta}}$ is a subsemigroup of S, where $\mu_{\overline{\delta}} = \{x \in S \mid \mu(x) > \delta\}$.

Proof. Assume that μ is a $(\overline{\beta}, \overline{\in_\gamma} \vee \overline{q_\delta})$-fuzzy subsemigroup of S. Let $x, y \in \mu_{\overline{\delta}}$. Then $\mu(x) > \delta$ and $\mu(y) > \delta$. If $\mu(xy) < \delta$, we have the following two cases:
Case 1: $\overline{\beta} \in \{\overline{\in_\gamma}, \overline{\in_\gamma} \vee \overline{q_\delta}\}$. Then for all r such that $\delta < r < \min\{\mu(x), \mu(y)\}$, $[xy; r]\overline{\beta}\mu$. By (B13), $[x; r] \in_\gamma \vee q_\delta\mu$ or $[y; r] \in_\gamma \vee q_\delta\mu$. It follows that $\mu(x) < r$ or $\mu(x) + r \leq 2\delta$, or $\mu(y) < r$ or $\mu(y) + r \leq 2\delta$, a contradiction.
Case 2: $\overline{\beta} = \overline{q_\delta}$. Then $[xy; \delta]\overline{\beta}\mu$. By (B13), $[x; \delta]\overline{\in_\gamma} \vee \overline{q_\delta}$ or $[y; \delta]\overline{\in_\gamma} \vee \overline{q_\delta}$. It follows that $\mu(x) < \delta$ or $\mu(x) + \delta \leq 2\delta$, or $\mu(y) < \delta$ or $\mu(x) + \delta \leq 2\delta$, which is a contradiction to $\mu(x) > \delta$ and $\mu(y) > \delta$. Thus, in any case, $\mu(xy) > \delta$ and hence in case, $\mu(xy) > \delta$, and so $xy \in \mu_{\overline{\delta}}$. Therefore, $\mu_{\overline{\delta}}$ is a subsemigroup of S. $\square$

4.4. Theorem

Let μ be a $(\overline{\beta}, \overline{\in_\gamma} \vee \overline{q_\delta})$-fuzzy bi-ideal of S. Then the set $\mu_{\overline{\delta}}$ is a bi-ideal of S, where $\mu_{\overline{\delta}} = \{x \in S \mid \mu(x) > \delta\}$.

Proof. Assume that μ is an $(\overline{\beta}, \overline{\in_\gamma} \vee \overline{q_\delta})$-fuzzy bi-ideal of S. Let $x, b, y \in \mu_{\overline{\delta}}$ and $a \in S$ be such that $a \leq b$. Then $\mu(x) > \delta$, $\mu(y) > \delta$ and $\mu(b) > \delta$. If $\mu(xay) \leq \delta$, we have the following two cases:
Case 1: $\overline{\beta} \in \{\overline{\in_\gamma}, \overline{\in_\gamma} \vee \overline{q_\delta}\}$. Then for all r such that $\delta < r < \min\{\mu(x), \mu(y)\}$, $[xay; r]\overline{\beta}\mu$. By (B14), $[x; r]\overline{\in_\gamma} \vee \overline{q_\delta}\mu$ or $[y; r]\overline{\in_\gamma} \vee \overline{q_\delta}\mu$. It follows that $\mu(x) < r$ or $\mu(x) + r \leq 2\delta$, or $\mu(y) < r$ or $\mu(y) + r \leq 2\delta$, a contradiction.
Case 2: $\overline{\beta} = \overline{q_\delta}$. Then $[xay; \delta]\overline{\beta}\mu$. By (B14), $[x; \delta]\overline{\in_\gamma} \vee \overline{q_\delta}$ or $[y; \delta]\overline{\in_\gamma} \vee \overline{q_\delta}$. It follows that $\mu(x) < \delta$ or $\mu(x) + \delta \leq 2\delta$, or $\mu(y) < \delta$ or $\mu(x) + \delta \leq 2\delta$, which is a contradiction to $\mu(x) > \delta$ and $\mu(y) > \delta$. Thus, in any case, $\mu(xay) > \delta$ and so $xay \in \mu_{\overline{\delta}}$. Similarly, we may show that $a \in \mu_{\overline{\delta}}$. The remaining proof follows from Theorem 4.3. Therefore, $\mu_{\overline{\delta}}$ is a bi-ideal of S. $\square$

4.5. Theorem

Let A be a non-empty subset of S. Then A is a subsemigroup of S if and only if the fuzzy subset μ of S such that $\mu(x) = 1$ for all $x \in A$ and $\mu(x) = \delta$ otherwise is a $(\overline{\beta}, \overline{\in_\gamma} \vee \overline{q_\delta})$-fuzzy subsemigroup of S.

Proof. Assume that A is a subsemigroup of S. Let $x, y \in S$ and $r, t \in (\gamma, 1]$ be such that $[xy; \min\{r, t\}]\overline{\beta}\mu$. Then we have the following three cases:
Case 1: $[xy; \min\{r, t\}]\overline{\in_\gamma}\mu$. Then $\mu(xy) < \min\{r, t\} \leq 1$ and so $\mu(xy) = \delta < \min\{r, t\}$, i.e., $xy \notin A$. It follows that $x \notin A$ or $y \notin A$, and so $\mu(x) = \delta$ or $\mu(y) = \delta$. Hence $[x; r]\overline{\in_\gamma}\mu$ or $[y; t]\overline{\in_\gamma}\mu$, i.e., $[x; r]\overline{\in_\gamma} \vee \overline{q_\delta}\mu$ or $[y; t]\overline{\in_\gamma} \vee \overline{q_\delta}\mu$.

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Case 2. $[xy;\min\{r,t\}]\overline{q_\delta}\mu$. Then $\mu(xy)+\min\{r,t\}\leq 2\delta$. If $\mu(xy)=\delta$, analogous to the proof of case 1, we have $[x;r]\overline{\in_\gamma}\vee\overline{q_\delta}\mu$ or $[y;t]\overline{\in_\gamma}\vee\overline{q_\delta}\mu$. If $\mu(xy)=1$, then

$$\begin{aligned}\max\{\mu(x),\mu(y)\}+\min\{r,t\} &\leq 1+\min\{r,t\}\\ &= \mu(xy)+\min\{r,t\}\leq 2\delta\end{aligned}$$

It follows that $\mu(x)+r\leq 2\delta$ or $\mu(y)+t\leq 2\delta$. Hence $[x;r]\overline{q_\delta}\mu$ or $[y;t]\overline{q_\delta}\mu$, i.e., $[x;r]\overline{\in_\gamma}\vee\overline{q_\delta}\mu$ or $[y;t]\overline{\in_\gamma}\vee\overline{q_\delta}\mu$.
Case 3. $[xy;\min\{r,t\}]\overline{\in_\gamma}\vee\overline{q_\delta}\mu$. Then $[xy;\min\{r,t\}]\overline{\in_\gamma}\mu$ or $[xy;\min\{r,t\}]\overline{q_\delta}\mu$. Hence $[x;r]\overline{\in_\gamma}\vee\overline{q_\delta}\mu$ or $[y;t]\overline{\in_\gamma}\vee\overline{q_\delta}\mu$ as in cases 1 and 2.

Conversely, assume that μ is a $(\overline{\in_\gamma},\overline{\in_\gamma}\vee\overline{q_\delta})$-fuzzy subsemigroup of S. It is easy to see that $A=\mu_{\overline{\delta}}$. Hence by Theorem 4.3, A is a subsemigroup of S. $\square$

4.6. Theorem

Let A be a non-empty subset of S. Then A is a bi-ideal of S if and only if the fuzzy subset μ of S such that $\mu(x)=1$ for all $x\in A$ and $\mu(x)=\delta$ otherwise is a $(\overline{\beta},\overline{\in_\gamma}\vee\overline{q_\delta})$-fuzzy bi-ideal of S.

Proof. Assume that A is a bi-ideal of S. Let $x,a,y\in S$ and $r,t\in(\gamma,1]$ be such that $[xay;\min\{r,t\}]\overline{\beta}\mu$. Then we have the following three cases:
Case 1: $[xay;\min\{r,t\}]\overline{\in_\gamma}\mu$. Then $\mu(xay)<\min\{r,t\}\leq 1$ and so $\mu(xay)=\delta<\min\{r,t\}$, i.e., $xay\notin A$. It follows that $x\notin A$ or $y\notin A$, and so $\mu(x)=\delta$ or $\mu(y)=\delta$. Hence $[x;r]\overline{\in_\gamma}\mu$ or $[y;t]\overline{\in_\gamma}\mu$, i.e., $[x;r]\overline{\in_\gamma}\vee\overline{q_\delta}\mu$ or $[y;t]\overline{\in_\gamma}\vee\overline{q_\delta}\mu$.
Case 2. $[xay;\min\{r,t\}]\overline{q_\delta}\mu$. Then $\mu(xay)+\min\{r,t\}\leq 2\delta$. If $\mu(xay)=\delta$, analogous to the proof of case 1, we have $[x;r]\overline{\in_\gamma}\vee\overline{q_\delta}\mu$ or $[y;t]\overline{\in_\gamma}\vee\overline{q_\delta}\mu$. If $\mu(xay)=1$, then $\max\{\mu(x),\mu(y)\}+\min\{r,t\}\leq 1+\min\{r,t\}=\mu(xay)+\min\{r,t\}\leq 2\delta$. It follows that $\mu(x)+r\leq 2\delta$ or $\mu(y)+t\leq 2\delta$. Hence $[x;r]\overline{q_\delta}\mu$ or $[y;t]\overline{q_\delta}\mu$, i.e., $[x;r]\overline{\in_\gamma}\vee\overline{q_\delta}\mu$ or $[y;t]\overline{\in_\gamma}\vee\overline{q_\delta}\mu$.
Case 3. $[xay;\min\{r,t\}]\overline{\in_\gamma}\vee\overline{q_\delta}\mu$. Then $[xay;\min\{r,t\}]\overline{\in_\gamma}\mu$ or $[xay;\min\{r,t\}]\overline{q_\delta}\mu$. Hence $[x;r]\overline{\in_\gamma}\vee\overline{q_\delta}\mu$ or $[y;t]\overline{\in_\gamma}\vee\overline{q_\delta}\mu$ as in cases 1 and 2.

In a similar way we can show that $[x;r]\overline{\beta}\mu$ implies that $[y;r]\overline{\in_\gamma}\vee\overline{q_\delta}\mu$ for all $x,y\in S$ with $x\leq y$.

Conversely, assume that μ is a $(\overline{\in_\gamma},\overline{\in_\gamma}\vee\overline{q_\delta})$-fuzzy bi-ideal of S. It is easy to see that $A=\mu_{\overline{\delta}}$. The remaining proof is a consequence of Theorem 4.5. Hence by Theorem 4.4, A is a bi-ideal of S. $\square$

4.7. Proposition

Every $(\overline{\in_\gamma}\vee\overline{q_\delta},\overline{\in_\gamma}\vee\overline{q_\delta})$-fuzzy subsemigroup (resp. $(\overline{\in_\gamma}\vee\overline{q_\delta},\overline{\in_\gamma}\vee\overline{q_\delta})$-fuzzy bi-ideal)of S is a $(\overline{\in_\gamma},\overline{\in_\gamma}\vee\overline{q_\delta})$-fuzzy subsemigroup (resp. $(\overline{\in_\gamma},\overline{\in_\gamma}\vee\overline{q_\delta})$-fuzzy bi-ideal) of S.

Proof. It is straightforward since $[x;r]\overline{\in_\gamma}\mu$ implies $[x;r]\overline{\in_\gamma}\vee\overline{q_\delta}\mu$ for all $x\in S$ and $r\in(\gamma,1]$. $\square$

4.8. Proposition

Every $(\overline{\in_\gamma},\overline{\in_\gamma})$-fuzzy subsemigroup (resp. $(\overline{\in_\gamma},\overline{\in_\gamma})$-fuzzy bi-ideal) of S is a $(\overline{\in_\gamma},\overline{\in_\gamma}\vee\overline{q_\delta})$-fuzzy subsemigroup (resp. $(\overline{\in_\gamma},\overline{\in_\gamma}\vee\overline{q_\delta})$-fuzzy bi-ideal) of S.

Proof. It is straightforward. $\square$

The converses of Proposition 4.8 and Proposition 4.9 are not true in general as shown in the following example.

4.9. Example

Consider the ordered semigroup as given in Example 4.2, and define a fuzzy subset μ by $\mu(a)=0.6$, $\mu(b)=0.5$, $\mu(c)=0.7$, $\mu(d)=0.8$, $\mu(e)=0.9$. Then
(1) μ is an $(\overline{\in_{0.3}},\overline{\in_{0.3}}\vee\overline{q_{0.6}})$-fuzzy subsemigroup (resp. $(\overline{\in_{0.3}},\overline{\in_{0.3}}\vee\overline{q_{0.6}})$-fuzzy bi-ideal) of S.
(2) μ is not an $(\overline{\in_{0.3}},\overline{\in_{0.3}})$-fuzzy subsemigroup (resp. $(\overline{\in_{0.3}},\overline{\in_{0.3}})$-fuzzy bi-ideal) of S since $[dc;\min\{0.78,0.68\}]=[a;0.68]\overline{\in_{0.3}}\mu$ but $[d;0.68]\in_{0.3}\mu$ and $[c;0.68]\in_{0.3}\mu$.
(3) μ is not an $(\overline{\in_{0.3}},\overline{\in_{0.3}}\vee\overline{q_{0.6}})$-fuzzy subsemigroup (resp. $(\overline{\in_{0.3}},\overline{\in_{0.3}}\vee\overline{q_{0.6}})$-fuzzy bi-ideal) of S since $[e;0.78]\overline{\in_{0.3}}\vee\overline{q_{0.6}}\mu$ and $a\leq e$ but $[a;0.78]\overline{\overline{\in_{0.3}}\vee\overline{q_{0.6}}}\mu$.

4.10. Theorem

Let μ be a fuzzy subset of S. If μ is a $(\overline{q_\delta},\overline{\in_\gamma}\vee\overline{q_\delta})$-fuzzy subsemigroup of S, then the following condition hold:
$(\forall x,y\in S)(\max\{\mu(xy),\delta\}\geq\min\{\mu(x),\mu(y)\})$.

Proof. Assume that μ is a $(\overline{q_\delta},\overline{\in_\gamma}\vee\overline{q_\delta})$-fuzzy subsemigroup of S. If there exist $x,y\in S$ such that $\max\{\mu(xy),\delta\}<\min\{\mu(x),\mu(y)\}$. Then for all r such that 2δ-$\max\{\mu(xy),\delta\}>r>2\delta-\min\{\mu(x),\mu(y)\}$, we have

$$\min\{2\delta-\mu(xy),\delta\}>r>\max\{2\delta-\mu(x),2\delta-\mu(y)\}$$

and so

$$\mu(xy)+r<2\delta,\quad \mu(x)+r>2\delta>2r$$

and $\mu(y)+r>2\delta>2r$. Hence $[xy;r]\overline{q_\delta}\mu$ but $[x;r]\overline{\overline{\in_\gamma}\vee\overline{q_\delta}}\mu$, $[y;r]\overline{\overline{\in_\gamma}\vee\overline{q_\delta}}\mu$, contradiction. Hence $\max\{\mu(xy),\delta\}\geq\min\{\mu(x),\mu(y)\}$ for all $x,y\in S$. $\square$

4.11. Theorem

Let μ be a fuzzy subset of S. If μ is a $(\overline{q_\delta},\overline{\in_\gamma}\vee\overline{q_\delta})$-fuzzy bi-ideal of S, then the following conditions hold, respectively.
(1) $(\forall x,y\in S)(\max\{\mu(xy),\delta\}\geq\min\{\mu(x),\mu(y)\})$,
(2) $(\forall x,a,y\in S)(\max\{\mu(xay),\delta\}\geq\min\{\mu(x),\mu(y)\})$
(3) $(\forall x,y\in S)(\max\{\mu(x),\delta\}\geq\mu(y)$ with $x\leq y)$.

Proof. We only give the proof of (2) and (3), the proof of (1) follows from Theorem 4.10. Assume that μ is a $(\overline{q_\delta},\overline{\in_\gamma}\vee\overline{q_\delta})$-fuzzy bi-ideal of S. If there exist $x,a,y\in S$ such that $\max\{\mu(xay),\delta\}<\min\{\mu(x),\mu(y)\}$.

Then for all r such that $2\delta - \max\{\mu(xay), \delta\} > r > 2\delta - \min\{\mu(x), \mu(y)\}$, we have $\min\{2\delta - \mu(xay), \delta\} > r > \max\{2\delta - \mu(x), 2\delta - \mu(y)\}$ and so $\mu(xay) + r < 2\delta, \mu(x) + r > 2\delta > 2r$ and $\mu(y) + r > 2\delta > 2r$. Hence $[xay; r]\overline{q_\delta}\mu$ but $[x; r]\overline{\overline{\in_\gamma} \vee \overline{q_\delta}}\mu$, $[y; r]\overline{\overline{\in_\gamma} \vee \overline{q_\delta}}\mu$, contradiction. Hence $\max\{\mu(xay), \delta\} \geq \min\{\mu(x), \mu(y)\}$ for all $x, a, y \in S$. Similarly, we can prove that $\max\{\mu(x), \delta\} \geq \mu(y)$ for all $x, y \in S$ with $x \leq y$. $\square$

4.12. Theorem

A fuzzy subset μ of S is an $(\overline{\in_\gamma}, \overline{\in_\gamma} \vee \overline{q_\delta})$-fuzzy bi-ideal of S if and only if the following conditions hold:

(B16) $(\forall x, y \in S)(\max\{\mu(xy), \delta\} \geq \min\{\mu(x), \mu(y)\})$,

(B17) $(\forall x, a, y \in S)(\max\{\mu(xay), \delta\} \geq \min\{\mu(x), \mu(y)\})$,

(B18) $(\forall x, y \in S)(\max\{\mu(x), \delta\} \geq \mu(y)$ with $x \leq y)$.

Proof. (B13)$\Longrightarrow$(B16). Assume that there exist $x, y \in S$ such that $\max\{\mu(xy), \delta\} < r = \min\{\mu(x), \mu(y)\}$. Then $\mu(xy) < r$, $\mu(x) \geq r > \gamma, \mu(y) \geq r > \gamma, \mu(x) + r \geq 2r > 2\delta$ and $\mu(y) + r \geq 2r > 2\delta$, i.e., $[xy; r]\overline{\in_\gamma}\mu$ but $[x; r]\overline{\overline{\in_\gamma} \vee \overline{q_\delta}}\mu$ and $[y; r]\overline{\overline{\in_\gamma} \vee \overline{q_\delta}}\mu$, a contradiction. Hence (B16) is valid.

(B16)$\Longrightarrow$(B13). Assume that there exist $x, y \in S$ and $r, t \in (\gamma, 1]$ such that $[xy; \min\{r, t\}]\overline{\in_\gamma}\mu$ but $[x; r]\overline{\overline{\in_\gamma} \vee \overline{q_\delta}}\mu$ and $[y; r]\overline{\overline{\in_\gamma} \vee \overline{q_\delta}}\mu$, then

$$\mu(xy) < \min\{r, t\}, \quad \mu(x) \geq r$$
$$\mu(y) \geq t, \quad \mu(x) + r > 2\delta$$

and $\mu(y) + r > 2\delta$. It follows that $\mu(x) > 2\delta$ and $\mu(y) > 2\delta$, and so

$$\min\{\mu(x), \mu(y)\} \geq \max\{\min\{r, t\}, \delta\} > \max\{\mu(xy), \delta\}$$

a contradiction. Hence (B13) is satisfied.

(B14)$\Longrightarrow$(B17). Assume that there exist $x, a, y \in S$ such that $\max\{\mu(xay), \delta\} < r = \min\{\mu(x), \mu(y)\}$. Then $\mu(xay) < r$,

$$\mu(x) \geq r > \gamma, \quad \mu(y) \geq r > \gamma, \quad \mu(x) + r \geq 2r > 2\delta$$

and $\mu(y) + r \geq 2r > 2\delta$, i.e., $[xay; r]\overline{\in_\gamma}\mu$ but $[x; r]\overline{\overline{\in_\gamma} \vee \overline{q_\delta}}\mu$ and $[y; r]\overline{\overline{\in_\gamma} \vee \overline{q_\delta}}\mu$, a contradiction. Hence (B17) is valid.

(B17)$\Longrightarrow$(B14). Assume that there exist $x, a, y \in S$ and $r, t \in (\gamma, 1]$ such that $[xay; \min\{r, t\}]\overline{\in_\gamma}\mu$ but $[x; r]\overline{\overline{\in_\gamma} \vee \overline{q_\delta}}\mu$ and $[y; r]\overline{\overline{\in_\gamma} \vee \overline{q_\delta}}\mu$, then

$$\mu(xay) < \min\{r, t\}, \quad \mu(x) \geq r,$$
$$\mu(y) \geq t, \quad \mu(x) + r > 2\delta$$

and $\mu(y) + r > 2\delta$. It follows that $\mu(x) > 2\delta$ and $\mu(y) > 2\delta$, and so

$$\min\{\mu(x), \mu(y)\} \geq \max\{\min\{r, t\}, \delta\} > \max\{\mu(xay), \delta\}$$

a contradiction. Hence (B14) is satisfied.

(B15)$\Longrightarrow$(B18). If there exist $x, y \in S$ with $x \leq y$ such that $\max\{\mu(x), \delta\} < r = \mu(y)$. Then $\mu(x) < r, \mu(y) \geq r > \gamma$ and $\mu(y) + r \geq 2r > 2\delta$, i.e., $[x; r]\overline{\in_\gamma}\mu$ but $[y; r]\overline{\overline{\in_\gamma} \vee \overline{q_\delta}}\mu$, a contradiction. Hence $\max\{\mu(x), \delta\} \geq \mu(y)$ for all $x, y \in S$ with $x \leq y$.

(B18)$\Longrightarrow$(B15). Assume that there exist $x, y \in S$ and $r \in (\gamma, 1]$ such that $[x; r]\overline{\in_\gamma}\mu$ but $[y; r]\overline{\overline{\in_\gamma} \vee \overline{q_\delta}}\mu$ then

$$\mu(x) < r, \quad \mu(x) \geq r, \quad \mu(y) \geq t, \quad \mu(x) + r > 2\delta$$

and $\mu(y) + r > 2\delta$. It follows that $\mu(x) > 2\delta$ and $\mu(y) > 2\delta$, and so

$$\min\{\mu(x), \mu(y)\} \geq \max\{\min\{r, t\}, \delta\} > \max\{\mu(xay), \delta\}$$

a contradiction. Hence (B15) is satisfied. $\square$

As a direct consequence of Theorem 4.11 and Theorem 4.12, we have the following Proposition:

4.13. Proposition

Every $(\overline{q_\delta}, \overline{\in_\gamma} \vee \overline{q_\delta})$-fuzzy subsemigroup (resp. $(\overline{q_\delta}, \overline{\in_\gamma} \vee \overline{q_\delta})$-fuzzy bi-ideal) of S is an $(\overline{\in_\gamma}, \overline{\in_\gamma} \vee \overline{q_\delta})$-fuzzy subsemigroup (resp. $(\overline{\in_\gamma}, \overline{\in_\gamma} \vee \overline{q_\delta})$-fuzzy bi-ideal) of S.

The following example shows that the converse of Proposition 4.13 is not true in general.

4.14. Example

Let μ be same as in Example 4.2. Then μ is an $(\overline{\in_{0.3}}, \overline{\in_{0.3}} \vee \overline{q_{0.6}})$-fuzzy subsemigroup (resp. $(\overline{\in_{0.3}}, \overline{\in_{0.3}} \vee \overline{q_{0.6}})$-fuzzy bi-ideal) of S but μ is not a $(\overline{q_{0.6}}, \overline{\in_{0.3}} \vee \overline{q_{0.6}})$-fuzzy subsemigroup (resp. $(\overline{q_{0.6}}, \overline{\in_{0.3}} \vee \overline{q_{0.6}})$-fuzzy bi-ideal) of S since $[a; 0.58]\overline{q_\delta}\mu$ and $a \leq e$ but $[e; 0.58]\overline{\overline{\in_\gamma} \vee \overline{q_\delta}}\mu$.

4.15. Remark

For any $(\overline{\in_\gamma}, \overline{\in_\gamma} \vee \overline{q_\delta})$-fuzzy bi-ideal of S, we conclude that μ is an $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy bi-ideal of S when $\delta = 0.5$.

The next theorem provides the relationship between $(\overline{\in_\gamma}, \overline{\in_\gamma} \vee \overline{q_\delta})$-fuzzy bi-ideals of S and crisp bi-ideals of S.

4.16. Theorem

Let $\mu \in \mathbb{F}(S)$.

(1) μ is an $(\overline{\in_\gamma}, \overline{\in_\gamma} \vee \overline{q_\delta})$-fuzzy bi-ideal of S if and only if $\mu_r (\neq \emptyset)$ is a bi-ideal of S for all $r \in (\delta, 1]$.

(2) μ is an $(\overline{\in_\gamma}, \overline{\in_\gamma} \vee \overline{q_\delta})$-fuzzy bi-ideal of S if and only if $\mu_r^\delta (\neq \emptyset)$ is a bi-ideal of S for all $r \in (\gamma, \delta]$.

Proof.

(1) Let μ be an $(\overline{\in_\gamma}, \overline{\in_\gamma} \vee \overline{q_\delta})$-fuzzy bi-ideal of S and $x, y \in \mu_r$ for some $r \in (\gamma, 1]$. Then $[x; r]\overline{\in_\gamma}\mu, [y; r]\overline{\in_\gamma}\mu$, i.e., $\mu(x) \geq r$ and $\mu(y) \geq r$. Since μ an $(\overline{\in_\gamma}, \overline{\in_\gamma} \vee \overline{q_\delta})$-fuzzy bi-ideal of S, we have

$$\max\{\mu(xy), \delta\} \geq \min\{\mu(x), \mu(y)\} \geq r$$

It follows from $r \in (\delta, 1]$ that $\mu(xy) \geq r > \gamma$. Hence $xy \in \mu_r$. If $a \in S$ and $x, y \in \mu_r$ for some $r \in (\gamma, 1]$. Then

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$[x; r]\overline{\in_\gamma}\mu$, $[y; r]\overline{\in_\gamma}\mu$, i.e., $\mu(x) \geq r$ and $\mu(y) \geq r$, and we have $\max\{\mu(xay), \delta\} \geq \min\{\mu(x), \mu(y)\} \geq r$. It follows from $r \in (\delta, 1]$ that $\mu(xay) \geq r > \gamma$. Hence $xay \in \mu_r$. Similarly, we can show that for $y \in \mu_r$ and $x \leq y$ implies $x \in \mu_r$. Therefore μ_r is a bi-ideal of S.

Conversely, assume that the given conditions hold. Let $x, a, y \in S$. If $\max\{\mu(xy), \delta\} < r = \min\{\mu(x), \mu(y)\}$, then $r > \delta, [x; r] \in_\gamma \mu, [y; r] \in_\gamma \mu$ but $[xy; r]\overline{\in_\gamma}\mu$, i.e., $x, y \in \mu_r$ but $xy \notin \mu_r$, contradiction. Therefore

$$\max\{\mu(xy), \delta\} \geq \min\{\mu(x), \mu(y)\}$$

for all $x, y \in S$. If

$$\max\{\mu(xay), \delta\} < r = \min\{\mu(x), \mu(y)\}$$

then $r > \delta, [x; r] \in_\gamma \mu, [y; r] \in_\gamma \mu$ but $[xay; r]\overline{\in_\gamma}\mu$, i.e., $x, y \in \mu_r$ but $xay \notin \mu_r$, contradiction. Therefore $\max\{\mu(xay), \delta\} \geq \min\{\mu(x), \mu(y)\}$ for all $x, a, y \in S$. Similarly, we can show that (B18) is true. Therefore μ is an $(\overline{\in_\gamma}, \overline{\in_\gamma} \vee \overline{q_\delta})$-fuzzy bi-ideal of S, by Theorem 4.12.

(2) Let μ be an $(\overline{\in_\gamma}, \overline{\in_\gamma} \vee \overline{q_\delta})$-fuzzy bi-ideal of S and $x, y \in \mu_r^\delta$ for some $r \in (\gamma, \delta]$. Then $[x; r]q_\delta\mu$, $[y; r]q_\delta\mu$, i.e., $\mu(x)+r > 2\delta$, $\mu(y)+r > 2\delta$. Since μ is an $(\overline{\in_\gamma}, \overline{\in_\gamma} \vee \overline{q_\delta})$-fuzzy bi-ideal of S, we have

$$\begin{aligned}\max\{\mu(xy), \delta\} &\geq \min\{\mu(x), \mu(y)\} \\ &> \min\{2\delta - r, 2\delta - r\} = 2\delta - r\end{aligned}$$

It follows from $r \in (\gamma, \delta]$ that $\delta \leq 2\delta - r$, and so $\mu(xy) > 2\delta - r$, i.e., $[xy; r]\overline{q_\delta}\mu$. Hence $xy \in \mu_r^\delta$. If $x, y \in \mu_r^\delta, a \in S$ for some $r \in (\gamma, \delta]$. Then $[x; r]q_\delta\mu$, $[y; r]q_\delta\mu$, i.e., $\mu(x)+r > 2\delta$, $\mu(y)+r > 2\delta$ and we have

$$\begin{aligned}\max\{\mu(xay), \delta\} &\geq \min\{\mu(x), \mu(y)\} \\ &> \min\{2\delta - r, 2\delta - r\} = 2\delta - r\end{aligned}$$

It follows from $r \in (\gamma, \delta]$ that $\delta \leq 2\delta - r$, and so $\mu(xay) > 2\delta - r$, i.e., $[xay; r]\overline{q_\delta}\mu$. Hence $xay \in \mu_r^\delta$. Similarly, we can show that for $y \in \mu_r^\delta$ and $x \leq y$ implies $x \in \mu_r^\delta$. Therefore μ_r^δ is a bi-ideal of S.

Conversely, suppose that the given conditions hold. Let $x, a, y \in S$. If $\max\{\mu(xy), \delta\} < r = \min\{\mu(x), \mu(y)\}$, then $r > \delta, [x; r]q_\delta\mu, [y; r]q_\delta\mu$ but $[xy; r]\overline{q_\delta}\mu$, i.e., $x, y \in \mu_r^\delta$ but $xy \notin \mu_r^\delta$, contradiction. Therefore

$$\max\{\mu(xy), \delta\} \geq \min\{\mu(x), \mu(y)\}$$

for all $x, y \in S$. If

$$\max\{\mu(xay), \delta\} < r = \min\{\mu(x), \mu(y)\}$$

then $r > \delta, [x; r]q_\delta\mu, [y; r]q_\delta\mu$ but $[xay; r]\overline{q_\delta}\mu$, i.e., $x, y \in \mu_r^\delta$ but $xay \notin \mu_r^\delta$, contradiction. Therefore $\max\{\mu(xay), \delta\} \geq \min\{\mu(x), \mu(y)\}$ for all $x, a, y \in S$. Similarly, we can show that (B18) is true. Hence μ is an $(\overline{\in_\gamma}, \overline{\in_\gamma} \vee \overline{q_\delta})$-fuzzy bi-ideal of S, by Theorem 4.12. □

As a direct consequence of Theorem 4.16, we have the following result.

4.17. Corollary

Let $\gamma, \gamma', \delta, \delta' \in [0, 1]$ be such that $\gamma < \delta, \gamma' < \delta'$ and $\delta' < \delta$. Then every $(\overline{\in_{\gamma'}}, \overline{\in_{\gamma'}} \vee \overline{q_{\delta'}})$-fuzzy bi-ideal of S is an $(\overline{\in_\gamma}, \overline{\in_\gamma} \vee \overline{q_\delta})$-fuzzy bi-ideal of S.

If we take $\delta = 0.5$ in Theorem 4.16, then we have the following result.

4.18. Corollary

Let $\mu \in \mathbb{F}(S)$.

(1) μ is an $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy bi-ideal of S if and only if $U(\mu; r)(\neq \emptyset)$ is a bi-ideal of S for all $r \in (0.5, 1]$.

(2) μ is an $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy bi-ideal of S if and only if $Q(\mu; r)(\neq \emptyset)$ is a bi-ideal of S for all $r \in (0, 0.5]$.

5. CONCLUSION

Ordered semigroups arise by considering different numerical semigroups, semigroups of functions and binary relations, semigroups of subsets (or subsystems of different algebraic systems, for example ideals in rings and semigroups), etc. Every ordered semigroup is isomorphic to a certain semigroup of binary relations, considered as an ordered semigroup, where the order is set-theoretic inclusion. The classical example of a lattice-ordered semigroup is the semigroup of all binary relations on an arbitrary set. In this paper, we study the generalization of $(\in, \in \vee q)$-fuzzy bi-ideals and give different characterization theorems of ordered semigroups in terms of this notion. In particular, if $J = \{r \mid r \in (0, 1]$ and $U(\mu; r)$ is an empty set or a bi-ideal of $S\}$, we give answer of the following questions:

(1) If $J = (\gamma, \delta]$, what kind of fuzzy bi-ideals of S will be μ?

(2) If $J = (\delta, 1]$, what kind of fuzzy bi-ideals of S will be μ?

(3) If $J = (r, t], (r, t \in (\gamma, \delta])$ what kind of fuzzy bi-ideals of S will be μ?

In our future work, we want to study those ordered semigroups for which each (α, β)-fuzzy bi-ideal and $(\overline{\beta}, \overline{\alpha})$-fuzzy bi-ideal are idempotent. We also want to define prime (α, β)-fuzzy bi-ideals and prime $(\overline{\beta}, \overline{\alpha})$-fuzzy bi-ideals and establish a generalized fuzzy spectrum of ordered semigroups.

Hopefully, the research along this direction will be continued and our results presented in this paper will constitute a platform for further development of ordered semigroups and their applications in other branches of algebra.

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