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The Schur Multiplier of Pairs of Groups of Order p^3q

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Abstract. Let (G,N) be a pair of groups in which N is a normal subgroup of $G.$ Then, the Schur multiplier of pairs of groups (G,N), denoted by $M(G,N)$, is an extension of the Schur multiplier of a group G, which is a functorial abelian group. In this research, the Schur multiplier of pairs of all groups of order p^3q where p is an odd prime and $p<q$ is determined.

INTRODUCTION

The Schur multiplier of a group G, denoted as $M(G)$, was introduced by Schur [1] while studying projective representations of groups in 1904. The Schur multiplier of a group G is defined as the second cohomology group $H_2(G,\mathbb{C}^*)$ where the modular multiplication acts identically: $g\cdot c=c$ for $g\in G$; $c\in\mathbb{C}^*$ and $\mathbb{C}^*$ represents the nonzero complex numbers. In [2], Schur stated that for a group G with a free presentation $1\to R\to F\to G\to 1,$ the Schur multiplier of G is isomorphic to $\left(R\cap[F,F]\right)/[F,R]$ where F is a free group, the group R of relators is the kernel of the surjective homomorphism $F\to G$ and $[F,R]$ is the group generated by all elements of the form $frf^{-1}r^{-1}$ for $f\in F$ and $r\in R.$

Schur computed the Schur multiplier for many different kinds of group such as alternating groups, symmetric groups and dihedral groups. All results of his computations can be found in [3]. In [4], Rashid computed the Schur multiplier of nonabelian groups of order p^3q for distinct primes p and q where $p<q$ by using the classification of nonabelian groups of order p^3q given by Western in [5]. The result shows that the Schur multiplier of nonabelian groups of order p^3q is either trivial, cyclic or elementary abelian.

In 1998, Ellis [6] defined the notion of the Schur multiplier of a pair of groups as follows:

Definition 1 [6] Let (G,N) be an arbitrary pair of finite groups where N is a normal subgroup of $G.$ Then the Schur multiplier of the pair, $M(G,N)$ is a functorial abelian group whose principal feature is a natural exact sequence

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$$H_3(G)\xrightarrow{\eta}H_3\left(G/N\right)\rightarrow M(G,N)\rightarrow M(G)\xrightarrow{\mu}M\left(G/N\right)\rightarrow N/[N,G]\rightarrow (G)^{ab}\xrightarrow{\alpha}\left(G/N\right)^{ab}\rightarrow 1 \tag{1}$$

in which $H_3(-)$ denotes some finiteness-preserving functor from groups to abelian groups (to be precise, $H_3(-)$ is the third homology of a group with integer coefficients). The homomorphisms η,μ,α are those due to the functorial of $H_3(-)$, $M(-)$ and $(-)^{ab}$. He also gave a group theoric definition of $M(G,N)$. The theoric definition is given in the following theorem.

Theorem 1 [6] For any pair of groups (G,N) there is an isomorphism $M(G,N)\cong\ker(\partial)$ where $\partial: N\wedge G\rightarrow G$, a map from a nonabelian exterior product of *N* and *G* to the group *G*.

In [6], Ellis also showed that the order Schur multiplier of (G,N) is bounded by $p^{\frac{1}{2}n(2m+n-1)}$ if G is a finite *p*-group with a normal subgroup N of order p^n and its quotient of order p^m. So there exists a non-negative integer $t(G,N)$ such that $|M(G,N)| = p^{\frac{1}{2}n(2m+n-1)-t(G,N)}$. In [7], Moghaddam *et al*. determined all pairs of finite *p*-groups (G,N), which satisfy the equality for $t(G,N)=0,1,2$. Besides, Moghaddam *et al*. in [8] showed that if *S* is a normal subgroup of F such that $N\cong S/R$ then $M(G,N)\cong (R\cap[S,F])/[F,R]$.

In our previous research, the commutator subgroup and centre of groups of order p^3q, where *p* and *q* are distinct primes and $p<q$, and the Schur multiplier of pairs of groups of order p^2q where *p* and *q* are prime numbers that have been determined in [9] and [10] respectively. In this research, the Schur multiplier of pairs of all groups of order p^3q where p is an odd prime and $p<q$ is determined. Note that throughout this paper, we denote the trivial group as 1.

PRELIMINARIES

This section includes some preliminary results that are used in proving our main theorem. The definition of normal Hall subgroup is given below.

Definition 2 [3] A normal subgroup *N* of *G* is called a normal Hall subgroup of *G* if the order of *N* is coprime to its index in *G*.

The classification of nonabelian groups of order p^3q, where *p* and *q* are distinct primes and $p<q$, given in [5], are listed in the following.

Theorem 2 [5] Let *G* be a nonabelian group of order p^3q, where *p* and *q* are distinct primes and $p<q$. Then exactly one of the following holds:

The case $p=2$ (The first nine groups exist for all values of $q\left(q>2\right)$)

$G_1\cong\left\langle a,b,c \mid a^4=b^2=c^q=1, bab=a^{-1}, ac=ca, bc=cb\right\rangle.$

$G_2\cong\left\langle a,b,c \mid a^4=b^4=c^q=1, b^2=a^2, b^{-1}ab=a^{-1}, ac=ca, bc=cb\right\rangle.$

$G_3\cong\left\langle a,b \mid a^8=b^q=1, a^{-1}ba=b^{-1}\right\rangle.$

$G_4\cong\left\langle a,b,c \mid a^4=b^2=c^q=1, ab=ba, ac=ca, bcb=c^{-1}\right\rangle.$

$G_5\cong\left\langle a,b,c \mid a^4=b^2=c^q=1, ab=ba, a^{-1}ca=c^{-1}, bc=cb\right\rangle.$

$G_6\cong\left\langle a,b,c,d \mid a^2=b^2=c^2=d^q=1, ab=ba, ac=ca, bc=cb, ad=da, bd=db, cdc=d^{-1}\right\rangle.$

$G_7\cong\left\langle a,b,c \mid a^4=b^2=c^q=1, bab=a^{-1}, ac=ca, bcb=c^{-1}\right\rangle.$

$G_8\cong\left\langle a,b,c \mid a^4=b^2=c^q=1, bab=a^{-1}, a^{-1}ca=c^{-1}, bc=cb\right\rangle.$

$G_9 \cong \left\langle a,b,c \mid a^4=b^4=c^q=1, b^2=a^2, b^{-1}ab=a^{-1}, ac=ca, b^{-1}cb=c^{-1} \right\rangle$.

$G_{10} \cong \left\langle a,b \mid a^8=b^q=1, a^{-1}ba=b^m \right\rangle$ where m is any primitive root of $m^4 \equiv 1 \pmod{q}$ and $q \equiv 1 \pmod{4}$.

$G_{11} \cong \left\langle a,b,c \mid a^4=b^2=c^q=1, ab=ba, a^{-1}ca=c^m, bc=cb \right\rangle$, where m is any primitive root of $m^4 \equiv 1 \pmod{q}$ and $q \equiv 1 \pmod{4}$.

$G_{12} \cong \left\langle a,b \mid a^8=b^q=1, a^{-1}ba=b^m \right\rangle$ where m is any primitive root of $m^8 \equiv 1 \pmod{q}$ and $q \equiv 1 \pmod{8}$.

The case $p=2$ $(q=3)$

$G_{13} \cong \left\langle a,b,c,d \mid a^2=b^2=c^2=d^3=1, ab=ba, ac=ca, bc=cb, ad=da, d^{-1}bd=c, d^{-1}cd=bc \right\rangle$.

$G_{14} \cong \left\langle a,b,c \mid a^4=b^4=c^3=1, a^2=b^2, b^{-1}ab=a^{-1}, c^{-1}ac=b, c^{-1}bc=ab \right\rangle$.

$G_{15} \cong \left\langle a,b,c \mid a^4=b^2=c^3=1, bab=a^{-1}, c^{-1}a^2b=b, c^{-1}bc=a^2b, a^{-1}ca=c^2a^2b \right\rangle$.

The case $p=2$ $(q=7)$

$G_{16} \cong \left\langle a,b,c,d \mid a^2=b^2=c^2=d^7=1, ab=ba, ac=ca, bc=cb, d^{-1}ad=b, d^{-1}bd=c, d^{-1}cd=ab \right\rangle$.

The case p is odd

$G_{17} \cong \left\langle a,b,c \mid a^{p^2}=b^p=c^q=1, b^{-1}ab=a^{p+1}, ac=ca, bc=cb \right\rangle$.

$G_{18} \cong \left\langle a,b,c,d \mid a^p=b^p=c^p=d^q=1, ab=ba, ac=ca, c^{-1}bc=ab, ad=da, bd=db, cd=dc \right\rangle$.

$G_{19} \cong \left\langle a,b \mid a^{p^3}=b^q=1, a^{-1}ba=b^m \right\rangle$, where m is any primitive root of $m^p \equiv 1 \pmod{q}$ and $q \equiv 1 \pmod{p}$.

$G_{20} \cong \left\langle a,b,c \mid a^{p^2}=b^p=c^q=1, ab=ba, ac=ca, b^{-1}cb=c^m \right\rangle$, where m is any primitive root of $m^p \equiv 1 \pmod{q}$ and $q \equiv 1 \pmod{p}$.

$G_{21} \cong \left\langle a,b,c \mid a^{p^2}=b^p=c^q=1, ab=ba, a^{-1}ca=c^m, bc=cb \right\rangle$, where m is any primitive root of $m^p \equiv 1 \pmod{q}$ and $q \equiv 1 \pmod{p}$.

$G_{22} \cong \left\langle a,b,c,d \mid a^p=b^p=c^p=d^q=1, ab=ba, ac=ca, bc=cb, ad=da, bd=db, c^{-1}dc=d^m \right\rangle$, where m is any primitive root of $m^p \equiv 1 \pmod{q}$ and $q \equiv 1 \pmod{p}$.

$G_{23} \cong \left\langle a,b,c \mid a^{p^2}=b^p=c^q=1, b^{-1}ab=a^{p+1}, ac=ca, b^{-1}cb=c^n \right\rangle$, where m is any primitive root of $m^p \equiv 1 \pmod{q}$, $q \equiv 1 \pmod{p}$. and $n=m, m^2, \ldots, m^{p-1}$.

$G_{24} \cong \left\langle a,b,c,d \mid a^p=b^p=c^p=d^q=1, ab=ba, ac=ca, ad=da, bd=db, c^{-1}bc=ab, c^{-1}dc=d^m \right\rangle$, where m is any primitive root of $m^p \equiv 1 \pmod{q}$ and $q \equiv 1 \pmod{p}$.

$G_{25} \cong \left\langle a,b \mid a^{p^3}=b^q=1, a^{-1}ba=b^m \right\rangle$, where m is any primitive root of $m^{p^2} \equiv 1 \pmod{q}$ and $q \equiv 1 \pmod{p^2}$.

$G_{26} \cong \left\langle a,b,c \mid a^{p^2}=b^p=c^q=1, ab=ba, a^{-1}ca=c^m, bc=cb \right\rangle$, where m is any primitive root of $m^{p^2} \equiv 1 \pmod{q}$ and $q \equiv 1 \pmod{p^2}$.

$G_{27} \cong \left\langle a,b \mid a^{p^3}=b^q=1, a^{-1}ba=b^m \right\rangle$, where m is any primitive root of $m^{p^3} \equiv 1 \pmod{q}$ and $q \equiv 1 \pmod{p^3}$.

Note that:

$G_{18} \cong \mathbb{Z}_q \times \left(\left(\mathbb{Z}_p \times \mathbb{Z}_p\right) \rtimes \mathbb{Z}_p\right);$

$G_{20} \cong \mathbb{Z}_{p^2} \times \left(\mathbb{Z}_q \rtimes \mathbb{Z}_p\right);$

$G_{21} \cong \mathbb{Z}_p \times \left(\mathbb{Z}_q \rtimes \mathbb{Z}_{p^2}\right);$

$G_{22} \cong \mathbb{Z}_p \times \mathbb{Z}_p \times \left(\mathbb{Z}_q \rtimes \mathbb{Z}_p\right);$

$G_{24} \cong \left(\mathbb{Z}_{pq} \times \mathbb{Z}_p\right) \rtimes \mathbb{Z}_p;$

$G_{26} \cong \mathbb{Z}_p \times \left(\mathbb{Z}_q \rtimes \mathbb{Z}_{p^2}\right).$

The commutator subgroup and center of groups of order p^3q where $p < q$ are given in the following theorem.

Theorem 3 [4] Let G be a nonabelian group of order p^3q where p and q are primes and $p < q$. Then

(i) for the commutator subgroup of G exactly one of the following holds:

$$G' = \begin{cases} \mathbb{Z}_2; & G \text{ is of type } G_1 \text{ and } G_2, \\ \mathbb{Z}_q; & G \text{ is of type } G_3 \text{ to } G_6, G_{10} \text{ to } G_{12}, G_{19} \text{ to } G_{22} \text{ and } G_{25} \text{ to } G_{27}, \\ \mathbb{Z}_{2q}; & G \text{ is of type } G_7 \text{ to } G_9, \\ \left(\mathbb{Z}_2\right)^2; & G \text{ is of type } G_{13}, \\ \mathbb{Q}_2; & G \text{ is of type } G_{14}, \\ A_4; & G \text{ is of type } G_{15}, \\ \left(\mathbb{Z}_2\right)^3; & G \text{ is of type } G_{16}, \\ \mathbb{Z}_p; & G \text{ is of type } G_{17} \text{ and } G_{18}, \\ \mathbb{Z}_{pq}; & G \text{ is of type } G_{23} \text{ and } G_{24}. \end{cases}$$

(ii) for the center of G exactly one of the following holds:

$$Z(G) = \begin{cases} 1; & G \text{ is of type } G_{12}, G_{15}, G_{16} \text{ and } G_{27}, \\ \mathbb{Z}_2; & G \text{ is of type } G_7 \text{ to } G_{11}, G_{13} \text{ and } G_{14}, \\ \mathbb{Z}_p; & G \text{ is of type } G_{23} \text{ to } G_{26}, \\ \mathbb{Z}_{2q}; & G \text{ is of type } G_1 \text{ and } G_2, \\ \mathbb{Z}_q; & G \text{ is of type } G_3 \text{ and } G_4, \\ \mathbb{Z}_{p^2}; & G \text{ is of type } G_{19} \text{ and } G_{20}, \\ \mathbb{Z}_{pq}; & G \text{ is of type } G_{17} \text{ and } G_{18}, \\ \mathbb{Z}_p \times \mathbb{Z}_p; & G \text{ is of type } G_{21} \text{ and } G_{22}. \\ \mathbb{Z}_2 \times \mathbb{Z}_2; & G \text{ is of type } G_5 \text{ and } G_6. \end{cases}$$

Some results that are essential to compute the Schur multiplier and the nonabelian tensor product are stated below.

Theorem 4 [11] The factor group G/G' is abelian. If K is a normal subgroup of G such that G/K is abelian, then $G' \subseteq K$.

Theorem 5 [12] Let $G \cong \mathbb{Z}_m$ and $H \cong \mathbb{Z}_n$ be cyclic groups that act trivially on each other. Then $G \otimes H \cong \mathbb{Z}_{(m,n)}$.

Theorem 6 [13] Let A, B, C be groups, with given actions of A on B and C, and of B and C on A. Suppose that the latter actions

(i) commute: ${}^{bc}a = {}^{cb}a$, so that $B \times C$ acts on A,

(ii) induce the trivial action of B on $A\otimes C$; ${}^{b}(a\otimes c)=a\otimes c$, and

(iii) induce the trivial action of C on $A\otimes B$; ${}^{c}(a\otimes b)=a\otimes b$,

for all $a\in A, b\in B, c\in C$. Then $A\otimes(B\times C)\cong(A\otimes B)\times(A\otimes C)$.

Theorem 7 [2] Let G be a finite group and let $G\cong F/R$ where F is a free group of rank n. Then $M(G)\cong \left(R\cap[F,F]\right)/[F,R]$.

Theorem 8 [6] Let (G,N) be a pair of group such that N has a complement in G then $M(G,N)=\ker\left(\mu: M(G)\to M\left(G/N\right)\right)$.

As a consequence of Theorem 7 and Theorem 8, Moghaddam *et al*. [8] obtained the following result.

Proposition 1 [8] Let G be a group with a free presentation $1\to R\to F\to G\to 1$ and let (G,N) be the pairs of groups such that N has a complement in G. If S is a normal subgroup of F such that $N\cong S/R$, then $M(G,N)\cong \left(R\cap[S,F]\right)/[F,R]$.

Theorem 9 [3] Let G be a finite group. Then

(i) $M(G)$ is a finite group whose elements have order dividing the order of G.

(ii) $M(G)=1$ if G is cyclic.

Theorem 10 [3] If the Sylow p-subgroups of G are cyclic for all p divides $|G|$, then $M(G)=1$.

Theorem 11 [3] Let N be a normal Hall subgroup of G and T be a complement of N in G. Then $M(G)\cong M(T)\times M(N)^{T}$.

The Schur multiplier of groups of order p^2q is given below.

Theorem 12 [14] Let G be a group of order p^2q where p and q are distinct primes. Then $M(G)=1$ or $\mathbb{Z}_p$.

Note that for groups of order p^2q where p and q are distinct primes, there are three cases to be considered.

Case 1: Let P be a normal Sylow p-subgroup of G.

Since $\left(|P|,\left|G/P\right|\right) = 1$, thus P is a normal Hall subgroup of G. Therefore, $G\cong P\rtimes T$, where T is a subgroup of G of order q and $P\cong\mathbb{Z}_{p^2}$ or $\mathbb{Z}_p\times\mathbb{Z}_p$. Therefore, by Theorem 12,

$$M(G)=M(T)\times M(P)^{T}=M(P)^{T}=\begin{cases}1; & P=\mathbb{Z}_{p^2},\\ \mathbb{Z}_p; & P=\mathbb{Z}_p\times\mathbb{Z}_p.\end{cases}$$

Case 2: Let Q be a normal Sylow q-subgroup of G.

Since $\left(|Q|,\left|G/Q\right|\right) = 1$, thus Q is a normal Hall subgroup of G. Therefore, $G\cong Q\rtimes T$ where T is a subgroup of G of order p^2. Therefore,

$$M(G)=M(T)\times M(Q)^{T}=M(T)=\begin{cases}1; & T=\mathbb{Z}_{p^2},\\ \mathbb{Z}_p; & T=\mathbb{Z}_p\times\mathbb{Z}_p.\end{cases}$$

Case 3: If $G \cong A_4$, then $M(G)$ has been computed in [13].

The Schur multiplier of nonabelian groups of order p^3q, where p and q are primes and $p < q$, is stated in the following theorem.

Theorem 13 [4] Let G be a nonabelian group of order p^3q where p and q are primes and $p < q$. Then exactly one of the following holds:

$$M(G) = \begin{cases} 1; & G \text{ is of type } G_2, G_3, G_9, G_{10}, G_{12}, G_{14}, G_{16}, G_{17}, G_{19}, G_{23}, G_{25} \text{ or } G_{27}, \\ \mathbb{Z}_2; & G \text{ is of type } G_1, G_4, G_5, G_7, G_8, G_{11}, G_{13} \text{ or } G_{15}, \\ (\mathbb{Z}_2)^3; & G \text{ is of type } G_6, \\ \mathbb{Z}_p; & G \text{ is of type } G_{20}, G_{21} \text{ or } G_{26}, \\ \mathbb{Z}_p \times \mathbb{Z}_p; & G \text{ is of type } G_{18} \text{ or } G_{24}, \\ (\mathbb{Z}_p)^3; & G \text{ is of type } G_{22}. \end{cases}$$

The following theorems are some of the basic results of the Schur multiplier of a pair deduced by Ellis [6].

Theorem 14 [6] Let $N = 1$, then $M(G,N) = 1$.

Theorem 15 [6] Let $N = G$, then $M(G,G) = M(G)$.

Theorem 16 [6] Suppose that G is a finite group. Let the order of the normal subgroup N be coprime to its index in G and T a complement of N in G. Then $G \cong N \rtimes T$ and $M(G,N) \cong M(N)^T$.

The structure for the Schur multiplier of a direct product of finite groups given by Karpilovsky in [3] is shown as follows:

Theorem 17 [3] If G_1 and G_2 are finite groups, then $M(G_1 \times G_2) = M(G_1) \times M(G_2) \times (G_1^{ab} \otimes G_2^{ab})$.

As a consequence of the above fact, Mohammadzadeh *et al.* [13] gave the following result.

Theorem 18 [15] Let (G,N) be a pair of groups and K be the complement of N in G. Then $|M(G,N)| = |M(N)||N^{ab} \otimes K^{ab}|$.

MAIN RESULTS

In the following theorem, the Schur multiplier of pairs of groups of order p^3q, where p and q are distinct odd primes and $p < q$, is stated and proved.

Theorem 19 Let G be a nonabelian group of order p^3q, where p and q are distinct odd primes and $p < q$. Then exactly one of the following holds:

$$
M(G,N)=\begin{cases}
1; & \text{if } \left(G=G_{17},G_{19},G_{23},G_{25},G_{27}\right) \text{ or } \left(G=G_{18},G_{20},G_{21},G_{22},G_{24},G_{26} \text{ when } N=1,\mathbb{Z}_q\right),\\
 & \text{if } \left(G=G_{20} \text{ when } N=\mathbb{Z}_p,\mathbb{Z}_{p^2},\mathbb{Z}_{p^2}\rtimes\mathbb{Z}_p,\mathbb{Z}_{p^2}\times\mathbb{Z}_p,\mathbb{Z}_q\rtimes\mathbb{Z}_p,\mathbb{Z}_q\rtimes\mathbb{Z}_{p^2},\mathbb{Z}_{p^2q},\mathbb{Z}_p\times\left(\mathbb{Z}_q\rtimes\mathbb{Z}_p\right),G\right),\\
\mathbb{Z}_p; & \left(G=G_{21} \text{ when } N=\mathbb{Z}_p,\left(\mathbb{Z}_p\right)^2,\mathbb{Z}_{p^2}\rtimes\mathbb{Z}_p,\mathbb{Z}_{p^2}\times\mathbb{Z}_p,\mathbb{Z}_{pq},\mathbb{Z}_q\rtimes\mathbb{Z}_{p^2},\mathbb{Z}_{pq}\rtimes\mathbb{Z}_p,G\right),\\
 & \left(G=G_{26} \text{ when } N=\mathbb{Z}_p,\left(\mathbb{Z}_p\right)^2,\mathbb{Z}_{p^2}\rtimes\mathbb{Z}_p,\mathbb{Z}_{p^2}\times\mathbb{Z}_p,\mathbb{Z}_q\rtimes\mathbb{Z}_p,\mathbb{Z}_p\times\left(\mathbb{Z}_q\rtimes\mathbb{Z}_p\right),\mathbb{Z}_q\rtimes\mathbb{Z}_{p^2},G\right)\\
\left(\mathbb{Z}_p\right)^2; & \text{if } \left(G=G_{18} \text{ when } N=\mathbb{Z}_p,\left(\mathbb{Z}_p\right)^2,\left(\mathbb{Z}_p\right)^2\rtimes\mathbb{Z}_p,\mathbb{Z}_{pq},\mathbb{Z}_{pq}\times\mathbb{Z}_p,G\right),\left(G=G_{22} \text{ when } N=\mathbb{Z}_p,\mathbb{Z}_q\rtimes\mathbb{Z}_p,\right),\\
 & \left(G=G_{24} \text{ when } N=\mathbb{Z}_p,\left(\mathbb{Z}_p\right)^2,\left(\mathbb{Z}_p\right)^2\rtimes\mathbb{Z}_p,\mathbb{Z}_p\times\left(\mathbb{Z}_q\rtimes\mathbb{Z}_p\right),\mathbb{Z}_{pq}\times\mathbb{Z}_p,G\right) \text{or } \left(G=G_{26} \text{ when } N=\mathbb{Z}_{pq}\right),\\
\left(\mathbb{Z}_p\right)^3; & \text{if } \left(G=G_{22} \text{ when } N=\left(\mathbb{Z}_p\right)^2,\left(\mathbb{Z}_p\right)^2\rtimes\mathbb{Z}_p,\left(\mathbb{Z}_p\right)^3,\mathbb{Z}_p\times\left(\mathbb{Z}_q\rtimes\mathbb{Z}_p\right),\mathbb{Z}_{pq}\times\mathbb{Z}_p,G\right).
\end{cases}
$$

Proof

Let G be a nonabelian group of order p^3q where p and q are distinct odd primes and $p<q$. Suppose $N \triangleleft G$, then the Schur multiplier of pairs of G is computed below by using the classification in Theorem 2. First, we have the following:

(i) If $N=1$, then by Theorem 14, $M(G,N)=1$.

(ii) If $N=G$, then by Theorem 15, $M(G,N)=M(G)$.

Next, we have the following cases:

Case 1: Let $G=G_{17},G_{19},G_{23},G_{25}$ or G_{27}. Then by Theorem 13, $M(G)=1$. Since we have $M(G)\cong {(R\cap[F,F])}/{[F,R]}$ and $M(G,N)\cong {(R\cap[S,F])}/{[F,R]}$ where S is a normal subgroup of F such that $N\cong S/R$ in Theorem 7 and Proposition 1 respectively, thus $M(G,N)\le M(G)$. So for all normal subgroups N of G, $M(G,N)\le M(G)=1$.

Therefore, for $G=G_{17},G_{19},G_{23},G_{25}$ or G_{27}, $M(G,N)=1$.

Case 2: Let $G=G_{18}$ which is $G\cong\mathbb{Z}_q\times\left(\left(\mathbb{Z}_p\times\mathbb{Z}_p\right)\rtimes\mathbb{Z}_p\right)$. Then by Theorem 3 and Theorem 13, $G'=\mathbb{Z}_p$, $Z(G)=\mathbb{Z}_{pq}$ and $M(G)=\mathbb{Z}_p\times\mathbb{Z}_p$.

(i) If $N=\mathbb{Z}_p=G'$ then $\left|G/N\right|=\left|G/G'\right|=p^2q$ which implies $G/N=G/G'\cong\mathbb{Z}_{pq}\times\mathbb{Z}_p$. Then we have the complement of N, $K\cong G/N\cong\mathbb{Z}_{pq}\times\mathbb{Z}_p$. By Theorem 18, Theorem 9, Theorem 6 and Theorem 5,

$$
\begin{aligned}
\left|M(G,N)\right| &= \left|M\left(\mathbb{Z}_p\right)\right|\left|\left(\mathbb{Z}_p\right)^{ab}\otimes\left(\mathbb{Z}_{pq}\times\mathbb{Z}_p\right)^{ab}\right|\\
&=(1)\left|\left(\mathbb{Z}_p\right)\otimes\left(\mathbb{Z}_{pq}\times\mathbb{Z}_p\right)\right|\\
&=\left|\left(\mathbb{Z}_p\otimes\mathbb{Z}_{pq}\right)\times\left(\mathbb{Z}_p\otimes\mathbb{Z}_p\right)\right|\\
&=\left|\left(\mathbb{Z}_{(p,pq)}\right)\times\left(\mathbb{Z}_{(p,p)}\right)\right|\\
&=\left|\mathbb{Z}_p\times\mathbb{Z}_p\right|.
\end{aligned}
$$

Therefore, $M(G,N)\cong\mathbb{Z}_p\times\mathbb{Z}_p$.

(ii) If $N=\mathbb{Z}_p\times\mathbb{Z}_p$ then $\left|G/N\right|=pq$ which implies $G/N\cong\mathbb{Z}_{pq}$ or $\mathbb{Z}_p\rtimes\mathbb{Z}_q$.

a) If $G/N \cong \mathbb{Z}_{pq}$ then by Theorem 9, $M\left(G/N\right) \cong 1$. Thus the exact sequence $M(G,N) \to M(G) \to M\left(G/N\right) = 1$ shows that $M(G,N)/\kappa \cong 1$ where κ is the kernel of homomorphism $M(G,N)$ to $M(G)$. Thus, $M(G,N) \cong \mathbb{Z}_p \times \mathbb{Z}_p$.

b) If $G/N \cong \mathbb{Z}_p \rtimes \mathbb{Z}_q$ then by Sylow's theorems, $\mathbb{Z}_p$ is the Sylow p-subgroup of $G/N \cong \mathbb{Z}_p \rtimes \mathbb{Z}_q$ and $\mathbb{Z}_q$ is the Sylow q-subgroup of $G/N \cong \mathbb{Z}_p \rtimes \mathbb{Z}_q$. Since $\mathbb{Z}_p$ and $\mathbb{Z}_q$ are cyclic then Theorem 10, $M\left(G/N\right) \cong 1$. Thus the exact sequence $M(G,N) \to M(G) \to M\left(G/N\right) = 1$ shows that $M(G,N)/\kappa \cong 1$ where κ is the kernel of homomorphism $M(G,N)$ to $M(G)$. Thus, $M(G,N) \cong \mathbb{Z}_p \times \mathbb{Z}_p$.

(iii) If $N = \left(\mathbb{Z}_p \times \mathbb{Z}_p\right) \rtimes \mathbb{Z}_p$ then $\left|G/N\right| = q$ which implies $G/N \cong \mathbb{Z}_q$. By Theorem 9, $M\left(G/N\right) \cong 1$. Thus the exact sequence $M(G,N) \to M(G) \to M\left(G/N\right) = 1$ shows that $M(G,N)/\kappa \cong 1$ where κ is the kernel of homomorphism $M(G,N)$ to $M(G)$. Thus, $M(G,N) \cong \mathbb{Z}_p \times \mathbb{Z}_p$.

(iv) If $N = \mathbb{Z}_q$ then by Definition 2, N is a normal Hall subgroup of G since $|N| = q$ and $\left|G/N\right| = p^3$ are coprime. By Theorem 16, $M(G,N) = M\left(\mathbb{Z}_q\right)^H = 1$ since $M\left(\mathbb{Z}_q\right) = 1$.

(v) If $N = \mathbb{Z}_{pq} = Z(G)$ then $\left|G/N\right| = \left|G/Z(G)\right| = p^2$ which implies $G/N = G/Z(G) \cong \mathbb{Z}_p \times \mathbb{Z}_p$. Then we have the complement of N, $K \cong G/N \cong \mathbb{Z}_p \times \mathbb{Z}_p$. By Theorem 18, Theorem 9, Theorem 6 and Theorem 5,

$$
\begin{aligned}
|M(G,N)| &= \left|M\left(\mathbb{Z}_{pq}\right)\right|\left|\left(\mathbb{Z}_{pq}\right)^{ab} \otimes \left(\mathbb{Z}_p \times \mathbb{Z}_p\right)^{ab}\right| \\
&= (1)\left|\left(\mathbb{Z}_{pq}\right) \otimes \left(\mathbb{Z}_p \times \mathbb{Z}_p\right)\right| \\
&= \left|\left(\mathbb{Z}_{pq} \otimes \mathbb{Z}_p\right) \times \left(\mathbb{Z}_{pq} \otimes \mathbb{Z}_p\right)\right| \\
&= \left|\left(\mathbb{Z}_{(pq,p)}\right) \times \left(\mathbb{Z}_{(pq,p)}\right)\right| \\
&= \left|\mathbb{Z}_p \times \mathbb{Z}_p\right|.
\end{aligned}
$$

Therefore, $M(G,N) \cong \mathbb{Z}_p \times \mathbb{Z}_p$.

(vi) If $N \cong \mathbb{Z}_{pq} \times \mathbb{Z}_p$ then $\left|G/N\right| = p$ which implies $G/N \cong \mathbb{Z}_p$. By Theorem 9, $M\left(G/N\right) \cong 1$. Thus the exact sequence $M(G,N) \to M(G) \to M\left(G/N\right) = 1$ shows that $M(G,N)/\kappa \cong 1$ where κ is the kernel of homomorphism $M(G,N)$ to $M(G)$. Thus, $M(G,N) \cong \mathbb{Z}_p \times \mathbb{Z}_p$.

Case 3: Let $G = G_{20}$ which is $G \cong \mathbb{Z}_{p^2} \times \left(\mathbb{Z}_q \rtimes \mathbb{Z}_p\right)$. Then by Theorem 3 and Theorem 13, $G' = \mathbb{Z}_q$, $Z(G) = \mathbb{Z}_{p^2}$ and $M(G) = \mathbb{Z}_p$.

(i) If $N = \mathbb{Z}_p$ then $\left|G/N\right| = p^2 q$ which implies $G/N \cong \mathbb{Z}_q \rtimes \mathbb{Z}_{p^2}$ or $\mathbb{Z}_{p^2} \rtimes \mathbb{Z}_q$. (If G/N is abelian then by Theorem 4 $G' \subseteq N$; that is $\mathbb{Z}_q \subseteq \mathbb{Z}_p$ and this statement is a contradiction.) Then by Theorem 12, $M\left(G/N\right) \cong 1$. Thus the exact sequence $M(G,N) \to M(G) \to M\left(G/N\right) = 1$ shows that $M(G,N)/\kappa \cong 1$ where κ is the kernel of homomorphism $M(G,N)$ to $M(G)$. Thus, $M(G,N) \cong \mathbb{Z}_p$.

(ii) If $N=\mathbb{Z}_{p^2}=Z(G)$ then $\left|G/N\right|=\left|G/Z(G)\right|=pq$ which implies $G/N=G/Z(G)\cong\mathbb{Z}_q\rtimes\mathbb{Z}_p$. Then by Sylow's theorems, $\mathbb{Z}_p$ is the Sylow p-subgroup of $G/N\cong\mathbb{Z}_q\rtimes\mathbb{Z}_p$ and $\mathbb{Z}_q$ is the Sylow q-subgroup of $G/N\cong\mathbb{Z}_q\rtimes\mathbb{Z}_p$. Since $\mathbb{Z}_p$ and $\mathbb{Z}_q$ are cyclic then Theorem 10, $M\left(G/N\right)\cong 1$. Thus the exact sequence $M(G,N)\to M(G)\to M\left(G/N\right)=1$ shows that $M(G,N)/\kappa\cong 1$ where κ is the kernel of homomorphism $M(G,N)$ to $M(G)$. Thus, $M(G,N)\cong\mathbb{Z}_p$.

(iii) If $N=\mathbb{Z}_{p^2}\rtimes\mathbb{Z}_p$ or $\mathbb{Z}_{p^2}\times\mathbb{Z}_p$ then $\left|G/N\right|=q$ which implies $G/N\cong\mathbb{Z}_q$. By Theorem 9, $M\left(G/N\right)\cong 1$. Thus the exact sequence $M(G,N)\to M(G)\to M\left(G/N\right)=1$ shows that $M(G,N)/\kappa\cong 1$ where κ is the kernel of homomorphism $M(G,N)$ to $M(G)$. Thus, $M(G,N)\cong\mathbb{Z}_p$.

(iv) If $N=\mathbb{Z}_q=G'$ then by Definition 2, N is a normal Hall subgroup of G since $|N|=q$ and $\left|G/N\right|=p^3$ are coprime. By Theorem 16, $M(G,N)=M(\mathbb{Z}_q)^H=1$ since $M(\mathbb{Z}_q)=1$.

(v) If $N=\mathbb{Z}_q\rtimes\mathbb{Z}_p$ then $\left|G/N\right|=p^2$ which implies $G/N\cong\mathbb{Z}_{p^2}$. By Theorem 9, $M\left(G/N\right)\cong 1$. Thus the exact sequence $M(G,N)\to M(G)\to M\left(G/N\right)=1$ shows that $M(G,N)/\kappa\cong 1$ where κ is the kernel of homomorphism $M(G,N)$ to $M(G)$. Thus, $M(G,N)\cong\mathbb{Z}_p$.

(vi) If $N\cong\mathbb{Z}_q\rtimes\mathbb{Z}_{p^2}$, $\mathbb{Z}_{p^2q}$ or $\mathbb{Z}_p\times(\mathbb{Z}_q\rtimes\mathbb{Z}_p)$, then $\left|G/N\right|=p$ which implies $G/N\cong\mathbb{Z}_p$. By Theorem 9, $M\left(G/N\right)\cong 1$. Thus the exact sequence $M(G,N)\to M(G)\to M\left(G/N\right)=1$ shows that $M(G,N)/\kappa\cong 1$ where κ is the kernel of homomorphism $M(G,N)$ to $M(G)$. Thus, $M(G,N)\cong\mathbb{Z}_p$.

Case 4: Let $G=G_{21}$ which is $G\cong\mathbb{Z}_p\times(\mathbb{Z}_q\rtimes\mathbb{Z}_{p^2})$. Then by Theorem 3 and Theorem 13, $G'=\mathbb{Z}_q$, $Z(G)=\mathbb{Z}_p\times\mathbb{Z}_p$ and $M(G)=\mathbb{Z}_p$.

(i) If $N=\mathbb{Z}_p$ then $\left|G/N\right|=p^2q$ which implies $G/N\cong\mathbb{Z}_q\rtimes\mathbb{Z}_{p^2}$ or $\mathbb{Z}_{p^2}\rtimes\mathbb{Z}_q$. Thus, by similar way as in case 3(i), $M(G,N)\cong\mathbb{Z}_p$.

(ii) If $N=\mathbb{Z}_p\times\mathbb{Z}_p=Z(G)$ then $\left|G/N\right|=\left|G/Z(G)\right|=pq$ which implies $G/N=G/Z(G)\cong\mathbb{Z}_q\rtimes\mathbb{Z}_p$. Thus, by similar way as in case 3(ii), $M(G,N)\cong\mathbb{Z}_p$.

(iii) If $N=\mathbb{Z}_{p^2}\rtimes\mathbb{Z}_p$ or $\mathbb{Z}_{p^2}\times\mathbb{Z}_p$ then $\left|G/N\right|=q$ which implies $G/N\cong\mathbb{Z}_q$. Thus, by similar way as in case 3(iii), $M(G,N)\cong\mathbb{Z}_p$.

(iv) If $N=\mathbb{Z}_q=G'$ then by Definition 2, N is a normal Hall subgroup of G since $|N|=q$ and $\left|G/N\right|=p^3$ are coprime. Thus, by similar way as in case 3(iv), $M(G,N)\cong 1$.

(v) If $N=\mathbb{Z}_{pq}$ then $\left|G/N\right|=p^2$ which implies $G/N\cong\mathbb{Z}_{p^2}$. Thus, by similar way as in case 3(v), $M(G,N)\cong\mathbb{Z}_p$.

(vi) If $N \cong \mathbb{Z}_q \rtimes \mathbb{Z}_{p^2}$ or $\mathbb{Z}_{pq} \times \mathbb{Z}_p$, then $\left| G/N \right| = p$ which implies $G/N \cong \mathbb{Z}_p$. Thus, by similar way as in case 3(v), $M(G,N) \cong \mathbb{Z}_p$.

Case 5: Let $G = G_{22}$ which is $G \cong \mathbb{Z}_p \times \mathbb{Z}_p \times (\mathbb{Z}_q \rtimes \mathbb{Z}_p)$. Then by Theorem 3 and Theorem 13, $G' = \mathbb{Z}_q$, $Z(G) = \mathbb{Z}_p \times \mathbb{Z}_p$ and $M(G) = (\mathbb{Z}_p)^3$.

(i) If $N = \mathbb{Z}_p$ then $\left| G/N \right| = p^2 q$ which implies $G/N \cong \mathbb{Z}_p \times (\mathbb{Z}_q \rtimes \mathbb{Z}_p)$. Then we have the complement of N, $K \cong G/N \cong \mathbb{Z}_p \times (\mathbb{Z}_q \rtimes \mathbb{Z}_p)$. By Theorem 18, Theorem 9, Theorem 6 and Theorem 5,

$$\begin{aligned}
\left| M(G,N) \right| &= \left| M(\mathbb{Z}_p) \right| \left| (\mathbb{Z}_p)^{ab} \otimes \left(\mathbb{Z}_p \times (\mathbb{Z}_q \rtimes \mathbb{Z}_p) \right)^{ab} \right| \\
&= (1) \left| (\mathbb{Z}_p) \otimes \left((\mathbb{Z}_p)^{ab} \times (\mathbb{Z}_q \rtimes \mathbb{Z}_p)^{ab} \right) \right| \\
&= \left| (\mathbb{Z}_p) \times (\mathbb{Z}_p \otimes \mathbb{Z}_p) \right| \\
&= \left| (\mathbb{Z}_{(p,p)}) \times (\mathbb{Z}_{(p,p)}) \right| \\
&= \left| \mathbb{Z}_p \times \mathbb{Z}_p \right|.
\end{aligned}$$

Therefore, $M(G,N) \cong \mathbb{Z}_p \times \mathbb{Z}_p$.

(ii) If $N = \mathbb{Z}_p \times \mathbb{Z}_p = Z(G)$ then $\left| G/N \right| = \left| G/Z(G) \right| = pq$ which implies $G/N = G/Z(G) \cong \mathbb{Z}_q \rtimes \mathbb{Z}_p$. Thus, by similar way as in case 3(ii), $M(G,N) = (\mathbb{Z}_p)^3$.

(iii) If $N = (\mathbb{Z}_p \times \mathbb{Z}_p) \rtimes \mathbb{Z}_p$ or $\mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p$ then $\left| G/N \right| = q$ which implies $G/N \cong \mathbb{Z}_q$. Thus, by similar way as in case 3(iii), $M(G,N) = (\mathbb{Z}_p)^3$.

(iv) If $N = \mathbb{Z}_q = G'$ then by Definition 2, N is a normal Hall subgroup of G since $|N| = q$ and $\left| G/N \right| = p^3$ are coprime. Thus, by similar way as in case 3(iv), $M(G,N) = 1$.

(v) If $N = \mathbb{Z}_q \rtimes \mathbb{Z}_p$ then $\left| G/N \right| = p^2$ which implies $G/N \cong \mathbb{Z}_p \times \mathbb{Z}_p$. Then we have the complement of N, $K \cong G/N \cong \mathbb{Z}_p \times \mathbb{Z}_p$. By Theorem 18, Theorem 10, Theorem 6 and Theorem 5,

$$\begin{aligned}
\left| M(G,N) \right| &= \left| M(\mathbb{Z}_q \rtimes \mathbb{Z}_p) \right| \left| (\mathbb{Z}_q \rtimes \mathbb{Z}_p)^{ab} \otimes (\mathbb{Z}_p \times \mathbb{Z}_p)^{ab} \right| \\
&= (1) \left| (\mathbb{Z}_p) \otimes (\mathbb{Z}_p \times \mathbb{Z}_p) \right| \\
&= \left| (\mathbb{Z}_p \otimes \mathbb{Z}_p) \times (\mathbb{Z}_p \otimes \mathbb{Z}_p) \right| \\
&= \left| (\mathbb{Z}_{(p,p)}) \times (\mathbb{Z}_{(p,p)}) \right| \\
&= \left| \mathbb{Z}_p \times \mathbb{Z}_p \right|.
\end{aligned}$$

Therefore, $M(G,N) \cong \mathbb{Z}_p \times \mathbb{Z}_p$.

(vi) If $N \cong \mathbb{Z}_p \times (\mathbb{Z}_q \rtimes \mathbb{Z}_p)$ or $\mathbb{Z}_{pq} \times \mathbb{Z}_p$, then $\left| G/N \right| = p$ which implies $G/N \cong \mathbb{Z}_p$. By Theorem 9, $M\left(G/N \right) \cong 1$. Thus the exact sequence $M(G,N) \to M(G) \to M\left(G/N \right) = 1$ shows that $M(G,N)/\kappa \cong 1$ where κ is the kernel of homomorphism $M(G,N)$ to $M(G)$. Thus, $M(G,N) = (\mathbb{Z}_p)^3$.

Case 6: Let $G = G_{24}$ which is $G \cong (\mathbb{Z}_{pq} \times \mathbb{Z}_p) \rtimes \mathbb{Z}_p$. Then by Theorem 3 and Theorem 13, $G' = \mathbb{Z}_{pq}$, $Z(G) = \mathbb{Z}_p$ and $M(G) = \mathbb{Z}_p \times \mathbb{Z}_p$.

(i)If $N = \mathbb{Z}_p = Z(G)$ then $\left| G/N \right| = \left| G/Z(G) \right| = p^2 q$ which implies $G/N = G/Z(G) \cong \mathbb{Z}_p \times (\mathbb{Z}_q \rtimes \mathbb{Z}_p)$. Therefore, by similar way as in case 5(i), $M(G,N) \cong \mathbb{Z}_p \times \mathbb{Z}_p$.

(ii) If $N = \mathbb{Z}_p \times \mathbb{Z}_p$ then $\left| G/N \right| = pq$ which implies $G/N \cong \mathbb{Z}_q \rtimes \mathbb{Z}_p$. (If G/N is abelian then by Theorem 4 $G' \subseteq N$; that is $\mathbb{Z}_{pq} \subseteq \mathbb{Z}_p \times \mathbb{Z}_p$ and this statement is a contradiction.) Thus, by similar as in case 3(ii), $M(G,N) = \mathbb{Z}_p \times \mathbb{Z}_p$.

(iii) If $N = (\mathbb{Z}_p \times \mathbb{Z}_p) \rtimes \mathbb{Z}_p$ then $\left| G/N \right| = q$ which implies $G/N \cong \mathbb{Z}_q$. By Theorem 9, $M(G/N) \cong 1$. Thus the exact sequence $M(G,N) \to M(G) \to M(G/N) = 1$ shows that $M(G,N)/\kappa \cong 1$ where κ is the kernel of homomorphism $M(G,N)$ to $M(G)$. Thus, $M(G,N) = \mathbb{Z}_p \times \mathbb{Z}_p$.

(iv) If $N = \mathbb{Z}_q$ then by Definition 2, N is a normal Hall subgroup of G since $|N| = q$ and $\left| G/N \right| = p^3$ are coprime. Thus, by similar way as in case 3 (iv), $M(G,N) = 1$.

If $N = \mathbb{Z}_{pq} = G'$ then $\left| G/N \right| = \left| G/G' \right| = p^2$ which implies $G/N = G/G' \cong \mathbb{Z}_p \times \mathbb{Z}_p$. Therefore, by similar way as in case 5 (v), $M(G,N) \cong \mathbb{Z}_p \times \mathbb{Z}_p$.

If $N \cong \mathbb{Z}_p \times (\mathbb{Z}_q \rtimes \mathbb{Z}_p)$ or $\mathbb{Z}_{pq} \times \mathbb{Z}_p$, then $\left| G/N \right| = p$ which implies $G/N \cong \mathbb{Z}_p$. Thus, by similar way as in

(v) case 5 (vi), $M(G,N) = \mathbb{Z}_p \times \mathbb{Z}_p$.

(vi)

Case 7: Let $G = G_{26}$ which is $G \cong \mathbb{Z}_p \times (\mathbb{Z}_q \rtimes \mathbb{Z}_{p^2})$. Then by Theorem 3 and Theorem 13, $G' = \mathbb{Z}_q$, $Z(G) = \mathbb{Z}_p$ and $M(G) = \mathbb{Z}_p$.

(i) If $N = \mathbb{Z}_p = Z(G)$ then $\left| G/N \right| = \left| G/Z(G) \right| = p^2 q$ which implies $G/N = G/Z(G) \cong \mathbb{Z}_q \rtimes \mathbb{Z}_{p^2}$. Then by Theorem 12, $M(G/N) \cong 1$. Thus the exact sequence $M(G,N) \to M(G) \to M(G/N) = 1$ shows that $M(G,N)/\kappa \cong 1$ where κ is the kernel of homomorphism $M(G,N)$ to $M(G)$. Thus, $M(G,N) \cong \mathbb{Z}_p$.

(ii) If $N = \mathbb{Z}_p \times \mathbb{Z}_p$ then $\left| G/N \right| = pq$ which implies G/N is a nonabelian group of order pq. (If G/N is abelian then by Theorem 4 $G' \subseteq N$; that is $\mathbb{Z}_q \subseteq \mathbb{Z}_p \times \mathbb{Z}_p$ and this statement is a contradiction.) By Theorem 10, $M(G/N) \cong 1$. Thus the exact sequence $M(G,N) \to M(G) \to M(G/N) = 1$ shows that $M(G,N)/\kappa \cong 1$ where κ is the kernel of homomorphism $M(G,N)$ to $M(G)$. Thus, $M(G,N) \cong \mathbb{Z}_p$.

(iii) If $N = \mathbb{Z}_{p^2} \rtimes \mathbb{Z}_p$ or $\mathbb{Z}_{p^2} \times \mathbb{Z}_p$ then $\left| G/N \right| = q$ which implies $G/N \cong \mathbb{Z}_q$. By Theorem 9, $M(G/N) \cong 1$. Thus the exact sequence $M(G,N) \to M(G) \to M(G/N) = 1$ shows that $M(G,N)/\kappa \cong 1$ where κ is the kernel of homomorphism $M(G,N)$ to $M(G)$. Thus, $M(G,N) \cong \mathbb{Z}_p$.

(iv) If $N = \mathbb{Z}_q = G'$ then by Definition 2, N is a normal Hall subgroup of G since $|N| = q$ and $\left| G/N \right| = p^3$ are coprime. By Theorem 16, $M(G,N) = M(\mathbb{Z}_q)^H = 1$ since $M(\mathbb{Z}_q) = 1$.

(v) If $N = \mathbb{Z}_q \rtimes \mathbb{Z}_p$ then $\left|{}^{G}\!/_{N}\right| = p^2$ which implies ${}^{G}\!/_{N} \cong \mathbb{Z}_{p^2}$. By Theorem 9, $M\left({}^{G}\!/_{N}\right) \cong 1$. Thus the exact sequence $M(G,N) \to M(G) \to M\left({}^{G}\!/_{N}\right) = 1$ shows that ${}^{M(G,N)}\!/_{\kappa} \cong 1$ where κ is the kernel of homomorphism $M(G,N)$ to $M(G)$. Thus, $M(G,N) \cong \mathbb{Z}_p$.

(vi) If $N = \mathbb{Z}_{pq}$ then $\left|{}^{G}\!/_{N}\right| = p^2$ which implies ${}^{G}\!/_{N} \cong \mathbb{Z}_p \times \mathbb{Z}_p$. Then we have the complement of N, $K \cong {}^{G}\!/_{N} \cong \mathbb{Z}_p \times \mathbb{Z}_p$. By Theorem 18, Theorem 9, Theorem 6 and Theorem 5,

$$\begin{aligned}
\left|M(G,N)\right| &= \left|M\left(\mathbb{Z}_{pq}\right)\right|\left|\left(\mathbb{Z}_{pq}\right)^{ab} \otimes \left(\mathbb{Z}_p \times \mathbb{Z}_p\right)^{ab}\right| \\
&= (1)\left|\left(\mathbb{Z}_{pq}\right) \otimes \left(\mathbb{Z}_p \times \mathbb{Z}_p\right)\right| \\
&= \left|\left(\mathbb{Z}_{pq} \otimes \mathbb{Z}_p\right) \times \left(\mathbb{Z}_{pq} \otimes \mathbb{Z}_p\right)\right| \\
&= \left|\left(\mathbb{Z}_{(pq,p)}\right) \times \left(\mathbb{Z}_{(pq,p)}\right)\right| \\
&= \left|\mathbb{Z}_p \times \mathbb{Z}_p\right|.
\end{aligned}$$

Therefore, $M(G,N) \cong \mathbb{Z}_p \times \mathbb{Z}_p$.

(vii) If $N \cong \mathbb{Z}_p \times \left(\mathbb{Z}_q \rtimes \mathbb{Z}_p\right)$ or $\mathbb{Z}_q \rtimes \mathbb{Z}_{p^2}$, then $\left|{}^{G}\!/_{N}\right| = p$ which implies ${}^{G}\!/_{N} \cong \mathbb{Z}_p$. By Theorem 9, $M\left({}^{G}\!/_{N}\right) \cong 1$. Thus the exact sequence $M(G,N) \to M(G) \to M\left({}^{G}\!/_{N}\right) = 1$ shows that ${}^{M(G,N)}\!/_{\kappa} \cong 1$ where κ is the kernel of homomorphism $M(G,N)$ to $M(G)$. Thus, $M(G,N) \cong \mathbb{Z}_p$.

CONCLUSION

There are twenty seven nonabelian groups of order p^3q where p and q are distinct primes and $p < q$. In this paper, we focus only on the eleven nonabelian groups of order p^3q where p and q are distinct odd primes and $p < q$ and we determined the Schur multiplier of pairs of groups of the groups mentioned. Our proofs show that $M(G,N)$ for those groups are equal to $1, \mathbb{Z}_p, \left(\mathbb{Z}_p\right)^2$ or $\left(\mathbb{Z}_p\right)^3$ depending on their normal subgroups.

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