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Classification of Ordered Semigroups in Terms of Generalized Interval-Valued Fuzzy Interior Ideals

DOI 10.1515/jisys-2015-0035
Received April 20, 2015.

Abstract: Several applied fields dealing with decision-making process may not be successfully modeled by ordinary fuzzy sets. In such a situation, the interval-valued fuzzy set theory is more applicable than the fuzzy set theory. Using a new approach of "quasi-coincident with relation", which is a central focused idea for several researchers, we introduced the more general form of the notion of (α, β)-fuzzy interior ideal. This new concept is called interval-valued $(\in, \in \vee q_{\tilde{k}})$ -fuzzy interior ideal of ordered semigroup. As an attempt to investigate the relationships between ordered semigroups and fuzzy ordered semigroups, it is proved that in regular ordered semigroups, the interval-valued $(\in, \in \vee q_{\tilde{k}})$ -fuzzy ideals and interval-valued $(\in, \in \vee q_{\tilde{k}})$ -fuzzy interior ideals coincide. It is also shown that the intersection of non-empty class of interval-valued $(\in, \in \vee q_{\tilde{k}})$ -fuzzy interior ideals of an ordered semigroup is also an interval-valued $(\in, \in \vee q_{\tilde{k}})$ -fuzzy interior ideal.

Keywords: Interior ideal, ordered semigroup, fuzzy set, interval-valued fuzzy set, interval-valued $(\in, \in \vee q_{\tilde{k}})$ -fuzzy interior ideal.

1 Introduction

A precise way for to represent knowledge in decision making is to use intervals instead of ordinary points. This major advancement in the fascinating world of interval-valued fuzzy sets started with the work of renowned scientist Zadeh in 1975, which opened a new era of research around the globe. Interval-valued fuzzy sets provide a more adequate description of uncertainty than traditional fuzzy sets. A fuzzy set with an interval-valued membership function is called an interval-valued fuzzy set [16]. The concept of "belongs to relation" $(\in)$ and "quasi-coincident with relation" (q) of a fuzzy point with a fuzzy set was introduced by Pu and Liu [13] and has boosted the significance of algebraic structures. Moreover, Bhakat and Das [1] gave a remarkable generalization of Rosenfeld's fuzzy subgroup [14] and presented the notion of $(\in, \in \vee q)$-fuzzy subgroups. Besides, this generalization of Rosenfeld's fuzzy subgroup appealed many researchers and opened new ways for future researchers in this field of algebra. Furthermore, Jun [4] generalized the concept of "quasi-coincident with relation" (q) of a fuzzy point with a fuzzy set and gave the idea of $(\in, \in \vee q_k)$-fuzzy sub-algebras of BCK/BCI-algebra, where $k \in [0, 1)$. Additionally, Narayanan and Manikantan [11] introduced the notions of interval-valued fuzzy left (right, two-sided, interior, bi-) ideal generated by an interval-valued fuzzy subset in semigroups. Likewise, Shabir and Khan [15] extended the idea of [11] and defined an interval-valued fuzzy left (right, two-sided, interior, bi-)ideal generated by an interval-valued fuzzy subset in ordered semigroups.

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Further, Davvaz et al. [3] considered ordered semigroup and produced more generalized form of fuzzy filters. Such types of generalizations catch the attention of other scientists from applied fields, and they successfully used these new generalized structures in their respective fields.

It is now natural to investigate similar generalizations as given in [1, 3, 7] of the existing interval-valued fuzzy subsystems. In this regard, Khan et al. [8] initiated a new sort of interval-valued fuzzy bi-ideals known as interval-valued $(\in, \in \vee q)$-fuzzy bi-ideals. Further, Khan et al. [9] introduced a more general form and defined interval-valued $(\in, \in \vee q_{\tilde{k}})$-fuzzy generalized bi-ideal of ordered semigroups and characterized ordered semigroups in terms interval-valued $(\in, \in \vee q_{\tilde{k}})$-fuzzy generalized bi-ideals.

In this paper, new types of interval-valued (α, β)-fuzzy interior ideals in ordered semigroups are introduced, which are the generalization of both interval-valued fuzzy interior ideals [11] and (α, β)-fuzzy interior ideals [7]. Several useful characterizations of interval-valued $(\in, \in \vee q_{\tilde{k}})$-fuzzy interior ideals of ordered semigroup are also provided. It is investigated that in regular ordered semigroups, the interval-valued $(\in, \in \vee q_{\tilde{k}})$-fuzzy ideal and interval-valued $(\in, \in \vee q_{\tilde{k}})$-fuzzy interior ideal coincides. We have also shown that the intersection of non-empty class of interval-valued $(\in, \in \vee q_{\tilde{k}})$-fuzzy interior ideals of an ordered semigroup is also an interval-valued $(\in, \in \vee q_{\tilde{k}})$-fuzzy interior ideal.

2 Preliminaries

By an *ordered semigroup* (or *po-semigroup*), we mean a structure $(S, \cdot, \leq)$ in which the following are satisfied:

(i) $(S, \cdot)$ is a semigroup,
(ii) $(S, \leq)$ is a poset,
(iii) $(\forall x, a, b \in S)\ (a \leq b \Rightarrow a \cdot x \leq b \cdot x, x \cdot a \leq x \cdot b)$.

In what follows, $x \cdot y$ is simply denoted by xy for all $x, y \in S$.

For any two ordered semigroups, $(S_1, \cdot, \leq_{S_1})$ and $(S_2, \cdot, \leq_{S_2})$, the Cartesian product $S_1 \times S_2$ forms a semigroup under the coordinate-wise multiplication [12].

Changphas [2] defined an ordered relation $\leq_{S_1 \times S_2}$ on $S_1 \times S_2$ by $(a, b) \leq_{S_1 \times S_2} (c, d)$ if and only if $a \leq_{S_1} c$ and $b \leq_{S_2} d$ for all $(a, b), (c, d) \in S_1 \times S_2$, and hence, $S_1 \times S_2$ is an ordered semigroup under the ordered relation $\leq_{S_1 \times S_2}$.

2.1 Definition [5]

An ordered semigroup S is called regular ordered semigroup if for all $a \in S$ there exists $x \in S$, such that $a \leq axa$.

2.2 Definition [6]

A non-empty subset A of an ordered semigroup S is called an interior ideal of S if the following conditions hold:

(I_1) $SAS \subseteq A$, $(\forall\, x, y \in A, a \in A)$
(I_2) $(\forall\, a \in S, b \in A)\ (a \leq b \rightarrow a \in A)$.
(I_3) $A^2 \subseteq A$.

By a fuzzy subset of an ordered semigroup S, we mean a mapping

$$\mu: S \rightarrow [0, 1].$$

2.3 Definition [15]

An interval-valued fuzzy subset $\tilde{\mu}$ of an ordered semigroup S is called an interval-valued fuzzy interior ideal of S if the following three conditions hold for all $x, y, z, \in S$:

(I_4) $\tilde{\mu}(xyz) \geq \tilde{\mu}(y)$,
(I_5) $(x \leq y \Rightarrow \tilde{\mu}(x) \geq \tilde{\mu}(y)$,
(I_6) $(\tilde{\mu}(xy) \geq \min\{\tilde{\mu}(x), \tilde{\mu}(y)\}$.

By an interval number $\tilde{a}$, we mean an interval $[a^-, a^+]$ where $0 \leq a^- \leq a^+ \leq 1$ and the set of all closed subinterval numbers is denoted by $D[0, 1]$ The interval $[a, a]$ can be simply identified by the number $a \in [0, 1]$.

For the interval numbers $\tilde{a}_i = [a_i^-, a_i^+]$, $\tilde{b}_i = [b_i^-, b_i^+] \in D[0, 1]$, $i \in I$, we define

$$(\forall i \in I)(\mathrm{r}\max\{\tilde{a}_i, \tilde{b}_i\} = [\max(a_i^-, b_i^-),\ \max(a_i^+, b_i^+)]),$$

$$(\forall i \in I)(\mathrm{r}\min\{\tilde{a}_i, \tilde{b}_i\} = [\min(a_i^-, b_i^-),\ \min(a_i^+, b_i^+)]),$$

$\mathrm{r}\inf\tilde{a}_i = \left[\bigwedge_{i\in I} a_i^-, \bigwedge_{i\in I} a_i^+\right]$, $\mathrm{r}\sup\tilde{a}_i = \left[\bigvee_{i\in I} a_i^-, \bigvee_{i\in I} a_i^+\right]$ and

- $\tilde{a}_1 \leq \tilde{a}_2 \Leftrightarrow a_1^- \leq a_2^-$ and $a_1^+ \leq a_2^+$,
- $\tilde{a}_1 = \tilde{a}_2 \Leftrightarrow a_1^- = a_2^-$ and $a_1^+ = a_2^+$,
- $\tilde{a}_1 < \tilde{a}_2 \Leftrightarrow \tilde{a}_1 \leq \tilde{a}_2$ and $\tilde{a}_1 \neq \tilde{a}_2$,
- $k\tilde{a}_i = [ka_i^-, ka_i^+]$, whenever $0 \leq k \leq 1$.

Then, it is clear that $(D[0, 1], \leq, \vee, \wedge)$ forms a complete lattice with $\tilde{0} = [0, 0]$ as its least element and $\tilde{1} = [1, 1]$ as its greatest element.

The interval-valued fuzzy subsets provide a more adequate description of uncertainty than the traditional fuzzy subsets; it is therefore important to use interval-valued fuzzy subsets in applications. One of the main applications of fuzzy subsets is fuzzy control, and one of the most computationally intensive parts of fuzzy control is the "defuzzification". As transition to interval-valued fuzzy subsets usually increases the amount of computations, it is virtually important to design faster algorithms for the corresponding defuzzification.

2.4 Definition [16]

An interval-valued fuzzy subset $\tilde{\mu}: X \to D[0, 1]$ of X is the set

$$\tilde{\mu} = \{x \in X | (x, [\mu^-(x), \mu^+(x)]) \in D[0, 1]\},$$

where μ^- and μ^+ are two fuzzy subsets, such that $\mu^-(x) \leq \mu^+(x)$ for all $x \in X$. Let $\tilde{\mu}$ be an interval-valued fuzzy subset of X. Then, for every $[0, 0] < \tilde{t} \leq 1, 1]$, the crisp set $U(\tilde{\mu}; \tilde{t}) = \{x \in X | \tilde{\mu}(x) \geq \tilde{t}\}$ is called the *level set* of $\tilde{\mu}$.

Note that because every $a \in [0, 1]$ is in correspondence with the interval $[a, a] \in D[0, 1]$; hence a fuzzy set is a particular case of the interval-valued fuzzy sets.

For any $\tilde{\mu} = [\mu^-, \mu^+]$ and $\tilde{t} = [t^-, t^+]$, $\tilde{\mu}(x) + \tilde{t} = [\mu^-(x) + t^-, \mu^+(x) + t^+]$ for all $x \in X$. In particular, if $\mu^-(x) + t^- > 1$ and $\mu^+(x) + t^+ > 1$, then we write $\tilde{\mu}(x) + \tilde{t} > \tilde{1}$.

2.5 Definition [10]

An interval-valued fuzzy subset $\tilde{\mu}$ of a set S of the form

$$\tilde{\mu}(y) := \begin{cases} \tilde{t} \in D(0, 1] & \text{if } y = x, \\ [0, 0] & \text{if } y \neq x, \end{cases}$$

is called an *interval-valued fuzzy point* with support x at value $\tilde{t}$ and is denoted by $x_{\tilde{t}}$.

For an interval-valued fuzzy subset $\tilde{\mu}$ of a set S, we say that an interval-valued fuzzy point $x_{\tilde{t}}$ is

(I_7) *contained in* $\tilde{\mu}$, denoted by $x_{\tilde{t}} \in \tilde{\mu}$, if $\tilde{\mu}(x) \geq \tilde{t}$.

(I_8) *quasi-coincident* with $\tilde{\mu}$, denoted by $x_{\tilde{t}} q \tilde{\mu}$, if $\tilde{\mu}(x) + \tilde{t} > \tilde{1} = [1, 1]$.

For an interval-valued fuzzy point $x_{\tilde{t}}$ and an interval-valued fuzzy subset $\tilde{\mu}$ of a set S, we say that

(I_9) $x_{\tilde{t}} \in \vee q \tilde{\mu}$ if $x_{\tilde{t}} \in \tilde{\mu}$ or $x_{\tilde{t}} q \tilde{\mu}$.

(I_{10}) $x_{\tilde{t}} \overline{\alpha} \tilde{\mu}$ if $x_{\tilde{t}} \alpha \tilde{\mu}$ does not hold for $\alpha \in \{\in, q, \in \vee q\}$.

3 Interval-valued $(\in, \in \vee q_{\tilde{k}})$-fuzzy Interior Ideals

In what follows, let S be an ordered semigroup and let $\tilde{k} = [k^-, k^+]$ denote an arbitrary element of $D[0, 1)$ unless otherwise specified. For an interval-valued fuzzy point $x_{\tilde{t}}$ and an interval-valued fuzzy subset $\tilde{\mu}$ of S, we say that

(i) $x_{\tilde{t}} q_{\tilde{k}} \tilde{\mu}$ if $\tilde{\mu}(x) + \tilde{t} + \tilde{k} > \tilde{1}$, where $\mu^- + t^- + k^- > 1$ and $\mu^+ + t^+ + k^+ > 1$.

(ii) $x_{\tilde{t}} \in \vee q_{\tilde{k}} \tilde{\mu}$ if $x_{\tilde{t}} \in \tilde{\mu}$ or $x_{\tilde{t}} q_{\tilde{k}} \tilde{\mu}$.

(iii) $x_{\tilde{t}} \overline{\alpha} \tilde{\mu}$ if $x_{\tilde{t}} \alpha \tilde{\mu}$ does not hold for $\alpha \in \{q_{\tilde{k}}, \in \vee q_{\tilde{k}}\}$.

3.1 Definition

An interval-valued fuzzy subset $\tilde{\mu}$ of S is called an interval-valued $(\in, \in \vee q_{\tilde{k}})$-fuzzy *interior ideal* of S if the following conditions are satisfied for all $x, a, y \in S$ and $\tilde{t}, \tilde{t}_1, \tilde{t}_2 \in D(0, 1]$:

(c_1) $x \leq y,\ y_{\tilde{t}} \in \tilde{\mu} \Rightarrow x_{\tilde{t}} \in \vee q_{\tilde{k}} \tilde{\mu}$,

(c_2) $x_{\tilde{t}_1} \in \tilde{\mu},\ y_{\tilde{t}_2} \in \tilde{\mu} \Rightarrow (xy)_{r\min\{\tilde{t}_1, \tilde{t}_2\}} \in \vee q_{\tilde{k}} \tilde{\mu}$,

(c_3) $a_{\tilde{t}} \in \tilde{\mu} \Rightarrow (xay)_{\tilde{t}} \in \vee q_{\tilde{k}} \tilde{\mu}$.

3.2 Example

Consider the ordered semigroup $S = \{a, b, c, d, e\}$ with order relations $a \leq c \leq e$, $a \leq d \leq e$, $b \leq d$, and $b \leq e$ and the multiplication given in Table 1.

Define an interval-valued fuzzy subset $\tilde{\mu}: S \rightarrow [0, 1]$ by

$$\tilde{\mu}(x) = \begin{cases} [0.50, 0.55] & \text{if } x = a, \\ [0.45, 0.50] & \text{if } x = b, \\ [0.65, 0.70] & \text{if } x = c, \\ [0.55, 0.60] & \text{if } x = d, \\ [0.40, 0.45] & \text{if } x = e. \end{cases}$$

Then $\tilde{\mu}$ is an interval-valued $(\in, \in \vee q_{[0.20, 0.30]})$-fuzzy interior ideals of S.

Table 1:

.	a	b	c	d	e
a	a	d	a	d	d
b	a	b	a	d	d
c	a	d	c	d	e
d	a	d	a	d	d
e	a	d	c	d	e

Caption for Table 1 (Multiplication table of S={a,b,c,d,e})

3.3 Theorem

For any interval-valued fuzzy subset $\tilde{\mu}$ of S, the following assertions are equivalent:

(1) $\left(\forall \tilde{t} \in D\left(0, \frac{1-k}{2}\right]\right)\left(U(\tilde{\mu}; \tilde{t}) \neq \varnothing \Rightarrow U(\tilde{\mu}; \tilde{t})\right.$ is an interior ideal of S).

(2) $\tilde{\mu}$ satisfies the following assertions for all $x, a, y \in S$:

(2.1) $x \leq y \Rightarrow \tilde{\mu}(x) \geq \mathrm{r}\min\left\{\tilde{\mu}(y), \left[\frac{1-k^{+}}{2}, \frac{1-k^{-}}{2}\right]\right\}$,

(2.2) $\tilde{\mu}(xy) \geq \mathrm{r}\min\left\{\tilde{\mu}(x), \tilde{\mu}(y), \left[\frac{1-k^{+}}{2}, \frac{1-k^{-}}{2}\right]\right\}$,

(2.3) $\tilde{\mu}(xay) \geq \mathrm{r}\min\left\{\tilde{\mu}(a), \left[\frac{1-k^{+}}{2}, \frac{1-k^{-}}{2}\right]\right\}$.

Proof 1. Assume that the non-empty $(\in)$-level subset $U(\tilde{\mu}; \tilde{t})$ is an interior ideal of S for all $\tilde{t} \in D\left(0, \frac{1-k}{2}\right]$. We claim that Condition (2.1) is true. If not, then there exist $a, b \in S$, such that $a \leq b$ and

$$\tilde{\mu}(a) < \tilde{t} \leq \mathrm{r}\min\left\{\tilde{\mu}(b), \left[\frac{1-k^{+}}{2}, \frac{1-k^{-}}{2}\right]\right\},$$

for some $\tilde{t} \in D\left(0, \frac{1-k}{2}\right]$. In which it follows that $b \in U(\tilde{\mu}; \tilde{t})$ but $a \overline{\in} U(\tilde{\mu}; \tilde{t})$, a contradiction, and hence (2.1) is valid for all $a, b \in S$ with $a \leq b$. Again, let us suppose that Condition (2.2) is not true, and hence,

$$\tilde{\mu}(ab) < \mathrm{r}\min\left\{\tilde{\mu}(a), \tilde{\mu}(b), \left[\frac{1-k^{+}}{2}, \frac{1-k^{-}}{2}\right]\right\},$$

for some $a, b \in S$. Then there exist $\tilde{s} \in D\left(0, \frac{1-k}{2}\right]$, such that

$$\tilde{\mu}(ab) < \tilde{s} \leq \mathrm{r}\min\left\{\tilde{\mu}(a), \tilde{\mu}(b), \left[\frac{1-k^{+}}{2}, \frac{1-k^{-}}{2}\right]\right\}.$$

This implies $a, b \in U(\tilde{\mu}; \tilde{t})$ but $ab \overline{\in} U(\tilde{\mu}; \tilde{t})$. Again, a contradiction, and therefore, it is concluded that (2.2) holds for all $a, b \in S$. Next, assume that

$$\tilde{\mu}(acb) < \mathrm{r}\min\left\{\tilde{\mu}(c), \left[\frac{1-k^{+}}{2}, \frac{1-k^{-}}{2}\right]\right\},$$

for some $a, b, c \in S$, then there exists $\tilde{t} \in D\left(0, \frac{1-k}{2}\right]$, such that

$$\tilde{\mu}(acb) < \tilde{t} \leq \mathrm{r}\min\left\{\tilde{\mu}(c), \left[\frac{1-k^{+}}{2}, \frac{1-k^{-}}{2}\right]\right\}.$$

Follows that $c \in U(\tilde{\mu}; \tilde{t})$ and $acb \overline{\in} U(\tilde{\mu}; \tilde{t})$ contradicting the definition of interior ideal, and hence,

$$\tilde{\mu}(xay) \geq \mathrm{r}\min\left\{\tilde{\mu}(a), \left[\frac{1-k^{-}}{2}, \frac{1-k^{+}}{2}\right]\right\}$$

for all $x, a, y \in S$.

Conversely, assume that $\tilde{\mu}$ satisfies (2.1), (2.2), and (2.3) and $U(\tilde{\mu};\,\tilde{t})\neq\varnothing$ for all $\tilde{t}\in D\left(0,\,\frac{1-k}{2}\right]$. If a, $b\in S$, such that $a\leq b$ and $b\in U(\tilde{\mu};\,\tilde{t})$, then $\tilde{\mu}(b)\geq\tilde{t}$, and hence, by (2.1),

$$\begin{aligned}\tilde{\mu}(a)&\geq \mathrm{r\,min}\left\{\tilde{\mu}(b),\left[\frac{1-k^{+}}{2},\frac{1-k^{-}}{2}\right]\right\},\\ &\geq \mathrm{r\,min}\left\{\tilde{t},\left[\frac{1-k^{+}}{2},\frac{1-k^{-}}{2}\right]\right\},\\ &=\tilde{t}.\end{aligned}$$

This implies $a\in U(\tilde{\mu};\,\tilde{t})$. If $a,\,b\in U(\tilde{\mu};\,\tilde{t})$, then by (2.2),

$$\begin{aligned}\tilde{\mu}(ab)&\geq \mathrm{r\,min}\left\{\tilde{\mu}(a),\tilde{\mu}(b),\left[\frac{1-k^{+}}{2},\frac{1-k^{-}}{2}\right]\right\},\\ &\geq \mathrm{r\,min}\left\{\tilde{t},\tilde{t},\left[\frac{1-k^{+}}{2},\frac{1-k^{-}}{2}\right]\right\},\\ &=\tilde{t}.\end{aligned}$$

It follows that $ab\in U(\tilde{\mu};\,\tilde{t})$. Take $x,\,a,\,y\in S$, such that $a\in U(\tilde{\mu};\,\tilde{t})$. Then using (2.3), we have

$$\begin{aligned}\tilde{\mu}(xay)&\geq \mathrm{r\,min}\left\{\tilde{\mu}(a),\left[\frac{1-k^{+}}{2},\frac{1-k^{-}}{2}\right]\right\},\\ &\geq \mathrm{r\,min}\left\{\tilde{t},\left[\frac{1-k^{+}}{2},\frac{1-k^{-}}{2}\right]\right\},\\ &=\tilde{t},\end{aligned}$$

it follows that $xay\in U(\tilde{\mu};\,\tilde{t})$. The above discussion shows that the non-empty $(\in)$-level subset $(U(\tilde{\mu};\,\tilde{t}))$ is an interior ideal of S for all $\tilde{t}\in D\left(0,\,\frac{1-k}{2}\right]$. □

By taking $\tilde{k}=[0,\,0]$ Theorem 3.3 reduces to the following corollary.

3.4 Corollary

Let $\tilde{\mu}$ be an interval-valued fuzzy subset of S. Then, Conditions (1) and (2) are equivalent:

(1) $(\forall\tilde{t}\in D(0,\,0.5](U(\tilde{\mu};\,\tilde{t})\neq\varnothing\Rightarrow U(\tilde{\mu};\,\tilde{t})$ is an interior ideal of S).

(2) $\tilde{\mu}$ satisfies the following assertions:

(2.1) $x\leq y\Rightarrow\tilde{\mu}(x)\geq \mathrm{r\,min}\{\tilde{\mu}(y),[0.5,\,0.5]\}$,

(2.2) $\tilde{\mu}(xy)\geq \mathrm{r\,min}\{\tilde{\mu}(x),\tilde{\mu}(y),[0.5,\,0.5]\}$,

(2.3) $\tilde{\mu}(xay)\geq \mathrm{r\,min}\{\tilde{\mu}(a),[0.5,0.5]\}$.

The following result provides necessary and sufficient conditions for an interval-valued fuzzy subset to be an interval-valued $(\in,\,\in\vee q_{\tilde{k}})$-fuzzy interior ideal.

3.5 Theorem

An interval-valued fuzzy subset $\tilde{\mu}$ of S is an interval-valued $(\in,\,\in\vee q_{\tilde{k}})$-fuzzy interior ideal of S if and only if the following conditions hold for all $x,\,a,\,y\in S$:

$(c_4)\ x \le y \Rightarrow \tilde{\mu}(x) \ge \mathrm{r}\min\left\{\tilde{\mu}(y), \left[\frac{1-k^+}{2}, \frac{1-k^-}{2}\right]\right\},$

$(c_5)\ \tilde{\mu}(xy) \ge \mathrm{r}\min\left\{\tilde{\mu}(x), \tilde{\mu}(y), \left[\frac{1-k^+}{2}, \frac{1-k^-}{2}\right]\right\},$

$(c_6)\ \tilde{\mu}(xay) \ge \mathrm{r}\min\left\{\tilde{\mu}(a), \left[\frac{1-k^+}{2}, \frac{1-k^-}{2}\right]\right\}.$

Proof 2. Suppose that $\tilde{\mu}$ is an interval-valued $(\in, \in\vee \mathrm{q}_{\tilde{k}})$-fuzzy interior ideal of S and $a, b \in S$, such that $a \le b$. If $\tilde{\mu}(a) < \tilde{\mu}(b)$, then $\tilde{\mu}(a) < \tilde{t} \le \tilde{\mu}(b)$ for some $\tilde{t} \in \overline{D}\left(0, \frac{1-k}{2}\right]$. It follows that $b_{\tilde{t}} \in \tilde{\mu}$ and $a_{\tilde{t}} \overline{\in} \tilde{\mu}$, also $\tilde{\mu}(a) + \tilde{t} < 2\tilde{t} \le \tilde{1} - \tilde{k}$, i.e. $a_{\tilde{t}} \overline{q}_{\tilde{k}} \tilde{\mu}$. Therefore, $a_{\tilde{t}} \overline{\in \vee \mathrm{q}_{\tilde{k}}} \tilde{\mu}$, a contradiction. Hence, $\tilde{\mu}(a) \ge \tilde{\mu}(b)$. Now, if $\tilde{\mu}(b) \ge \left[\frac{1-k^+}{2}, \frac{1-k^-}{2}\right]$, then $b_{\left[\frac{1-k^-}{2}, \frac{1-k^+}{2}\right]} \in \tilde{\mu}$, and so $a_{\left[\frac{1-k^+}{2}, \frac{1-k^-}{2}\right]} \in \vee q_{\tilde{k}} \tilde{\mu}$; this implies $\tilde{\mu}(a) \ge \left[\frac{1-k^+}{2}, \frac{1-k^-}{2}\right]$ or $\tilde{\mu}(a) + \left[\frac{1-k^+}{2}, \frac{1-k^-}{2}\right] > \tilde{1} - \tilde{k}$. Hence, $\tilde{\mu}(a) \ge \left[\frac{1-k^+}{2}, \frac{1-k^-}{2}\right]$; otherwise, $\tilde{\mu}(a) + \left[\frac{1-k^+}{2}, \frac{1-k^-}{2}\right] < \tilde{1} - \tilde{k}$, a contradiction. Consequently, $\tilde{\mu}(x) \ge \mathrm{r}\min\left\{\tilde{\mu}(y), \left[\frac{1-k^+}{2}, \frac{1-k^-}{2}\right]\right\}$ for all $x, y \in S$ with $x \le y$. Let $a, b \in S$ be such that $\mathrm{r}\min\{\tilde{\mu}(a), \tilde{\mu}(b)\} \le \left[\frac{1-k^+}{2}, \frac{1-k^-}{2}\right]$. We claim that $\tilde{\mu}(ab) \ge \mathrm{r}\min\{\tilde{\mu}(a), \tilde{\mu}(b)\}$. If not, then $\tilde{\mu}(ab) < \tilde{t} \le \mathrm{r}\min\{\tilde{\mu}(a), \tilde{\mu}(b)\}$ for some $\tilde{t} \in D\left(0, \frac{1-k}{2}\right]$. In which it follows that $a_{\tilde{t}} \in \tilde{\mu}$, $b_{\tilde{t}} \in \tilde{\mu}$, but $(ab)_{\tilde{t}} \overline{\in} \tilde{\mu}$ and $(ab)_{\tilde{t}} \overline{\mathrm{q}}_{\tilde{k}} \tilde{\mu}$, a contradiction, and hence, $\tilde{\mu}(ab) \ge \mathrm{r}\min\{\tilde{\mu}(a), \tilde{\mu}(b)\}$. If $\mathrm{r}\min\{\tilde{\mu}(a), \tilde{\mu}(b)\} > \left[\frac{1-k^+}{2}, \frac{1-k^-}{2}\right]$, then $a_{\left[\frac{1-k^+}{2}, \frac{1-k^-}{2}\right]} \in \tilde{\mu}$ and $b_{\left[\frac{1-k^+}{2}, \frac{1-k^-}{2}\right]} \in \tilde{\mu}$, and using Definition 3.1 (c_2), we have

$$(ab)_{\left[\frac{1-k^+}{2}, \frac{1-k^-}{2}\right]} = ab_{\mathrm{r}\min\left\{\left[\frac{1-k^+}{2}, \frac{1-k^-}{2}\right], \left[\frac{1-k^+}{2}, \frac{1-k^-}{2}\right]\right\}} \in \vee \mathrm{q}_{\tilde{k}} \tilde{\mu},$$

it follows that

$$\tilde{\mu}(ab) \ge \left[\frac{1-k^+}{2}, \frac{1-k^-}{2}\right]$$

or

$$\tilde{\mu}(ab) + \left[\frac{1-k^+}{2}, \frac{1-k^-}{2}\right] > \tilde{1} - \tilde{k}.$$

However, if $\tilde{\mu}(ab) < \left[\frac{1-k^+}{2}, \frac{1-k^-}{2}\right]$, then

$$\tilde{\mu}(ab) + \left[\frac{1-k^+}{2}, \frac{1-k^-}{2}\right] < \tilde{1} - \tilde{k},$$

a contradiction, and thus, $\tilde{\mu}(ab) \ge \left[\frac{1-k^+}{2}, \frac{1-k^-}{2}\right]$. Consequently,

$$\tilde{\mu}(xy) \geq \mathrm{r}\min\left\{\tilde{\mu}(x), \tilde{\mu}(y), \left[\frac{1-k^+}{2}, \frac{1-k^-}{2}\right]\right\} \text{ for all } x, y \in S.$$

Assume that there exist $x, a, y \in S$, such that $\tilde{\mu}(xay) < \mathrm{r}\min\left\{\tilde{\mu}(a), \left[\frac{1-k^+}{2}, \frac{1-k^-}{2}\right]\right\}$. Then there exist some $\tilde{t} \in D\left(0, \frac{1-k}{2}\right]$, such that $\tilde{\mu}(xay) < \tilde{t} \leq \mathrm{r}\min\left\{\tilde{\mu}(a), \left[\frac{1-k^+}{2}, \frac{1-k^-}{2}\right]\right\}$. This implies, $a_{\tilde{t}} \in \tilde{\mu}$ but $(xay)_{\tilde{t}} \overline{\in} \tilde{\mu}$ and $(xay)_{\tilde{t}} \overline{\mathrm{q}}_{\tilde{k}} \tilde{\mu}$, that is $(xay)_{\tilde{t}} \overline{\in \vee \mathrm{q}_{\tilde{k}}} \tilde{\mu}$, a contradiction. Hence, $\tilde{\mu}(xay) \geq \mathrm{r}\min\left\{\tilde{\mu}(a), \left[\frac{1-k^+}{2}, \frac{1-k^-}{2}\right]\right\}$ for all $x, a, y \in S$.

Conversely, let $\tilde{\mu}$ be an interval-valued fuzzy subset of S satisfying (c_4), (c_5), and (c_6). Let $a, b \in S$ $(a \leq b)$, $\tilde{t} \in D(0, 1]$, and $b_{\tilde{t}} \in \tilde{\mu}$. Then by ($c_4$), we have

$$\begin{aligned}\tilde{\mu}(a) &\geq \mathrm{r}\min\left\{\tilde{\mu}(b), \left[\frac{1-k^+}{2}, \frac{1-k^-}{2}\right]\right\},\\ &\geq \mathrm{r}\min\left\{\tilde{t}, \left[\frac{1-k^+}{2}, \frac{1-k^-}{2}\right]\right\},\\ &= \begin{cases} \tilde{t}, & \text{if } \tilde{t} \leq \left[\frac{1-k^+}{2}, \frac{1-k^-}{2}\right], \\ \left[\frac{1-k^+}{2}, \frac{1-k^-}{2}\right], & \text{if } \tilde{t} > \left[\frac{1-k^+}{2}, \frac{1-k^-}{2}\right]. \end{cases}\end{aligned}$$

It follows that $a_{\tilde{t}} \in \tilde{\mu}$ or $a_{\tilde{t}} q_{\tilde{k}} \tilde{\mu}$, and hence, $a_{\tilde{t}} \in \vee q_{\tilde{k}} \tilde{\mu}$.

Let $a, b \in S$ and $\tilde{t}_1, \tilde{t}_2 \in D(0, 1]$, such that $a_{\tilde{t}_1} \in \tilde{\mu}$ and $b_{\tilde{t}_2} \in \tilde{\mu}$. Then, $\tilde{\mu}(a) \geq \tilde{t}_1$ and $\tilde{\mu}(b) \geq \tilde{t}_2$. From ($c_5$)

$$\begin{aligned}\tilde{\mu}(ab) &\geq \mathrm{r}\min\left\{\tilde{\mu}(a), \tilde{\mu}(b), \left[\frac{1-k^+}{2}, \frac{1-k^-}{2}\right]\right\},\\ &\geq \mathrm{r}\min\left\{\tilde{t}_1, \tilde{t}_2, \left[\frac{1-k^+}{2}, \frac{1-k^-}{2}\right]\right\},\\ &= \begin{cases} \mathrm{r}\min\{\tilde{t}_1, \tilde{t}_2\}, & \text{if } \mathrm{r}\min\{\tilde{t}_1, \tilde{t}_2\} \leq \left[\frac{1-k^+}{2}, \frac{1-k^-}{2}\right], \\ \left[\frac{1-k^+}{2}, \frac{1-k^-}{2}\right], & \text{if } \mathrm{r}\min\{\tilde{t}_1, \tilde{t}_2\} > \left[\frac{1-k^+}{2}, \frac{1-k^-}{2}\right], \end{cases}\end{aligned}$$

In which it follows that $(ab)_{\mathrm{r}\min\{\tilde{t}_1, \tilde{t}_2\}} \in \vee q_{\tilde{k}} \tilde{\mu}$.

Let $a, b, c \in S$ and $\tilde{t} \in D(0, 1]$ be such that $c_{\tilde{t}} \in \tilde{\mu}$, then using ($c_6$), we have

$$\begin{aligned}\tilde{\mu}(acb) &\geq \mathrm{r}\min\left\{\tilde{\mu}(c), \left[\frac{1-k^+}{2}, \frac{1-k^-}{2}\right]\right\},\\ &\geq \mathrm{r}\min\left\{\tilde{t}, \left[\frac{1-k^+}{2}, \frac{1-k^-}{2}\right]\right\},\\ &= \begin{cases} \tilde{t}, & \text{if } \tilde{t} \leq \left[\frac{1-k^+}{2}, \frac{1-k^-}{2}\right], \\ \left[\frac{1-k^+}{2}, \frac{1-k^-}{2}\right], & \text{if } \tilde{t} > \left[\frac{1-k^+}{2}, \frac{1-k^-}{2}\right]. \end{cases}\end{aligned}$$

In which it follows that $(acb)_{\tilde{t}} \in \vee q_{\tilde{k}}\tilde{\mu}$. Hence, $\tilde{\mu}$ is an interval-valued $(\in, \in \vee \mathrm{q}_{\tilde{k}})$-fuzzy interior ideal of S. □

The following corollary comes by taking $\tilde{k} = [0, 0]$ in Theorem 3.5.

3.6 Corollary

An interval-valued fuzzy subset $\tilde{\mu}$ of S is an interval-valued $(\in, \in \vee \mathrm{q})$-fuzzy interior ideal of S if and only if
(i) $x \le y \Rightarrow \tilde{\mu}(x) \ge \mathrm{r}\min\{\tilde{\mu}(y), [0.5, 0.5]\}$,
(ii) $\tilde{\mu}(xy) \ge \mathrm{r}\min\{\tilde{\mu}(x), \tilde{\mu}(y), [0.5, 0.5]\}$,
(iii) $\tilde{\mu}(xray) \ge \mathrm{r}\min\{\tilde{\mu}(a), [0.5, 0.5]\}$.

3.7 Theorem

For an interval-valued fuzzy subset $\tilde{\mu}$ of S, the following are equivalent:
(1) $\tilde{\mu}$ is an interval-valued $(\in, \in \vee q_{\tilde{k}})$-fuzzy interior ideal of S.
(2) $U(\tilde{\mu}; \tilde{t}) \neq \emptyset \Rightarrow U(\tilde{\mu}; \tilde{t})$ is an interior of S for all $\tilde{t} \in D\left(0, \frac{1-k}{2}\right]$.

Proof 3. The proof follows from Theorem 3.3. □

Taking $\tilde{k} = [0, 0]$ in Theorem 3.7 induces the following corollary.

3.8 Corollary

For an interval-valued fuzzy subset $\tilde{\mu}$ of S, the following are equivalent:
(1) $\tilde{\mu}$ is an interval-valued $(\in, \in \vee q)$-fuzzy interior ideal of S.
(2) $U(\tilde{\mu}; \tilde{t}) \neq \emptyset \Rightarrow U(\tilde{\mu}; \tilde{t})$ is an interior of S for all $\tilde{t} \in D(0, 0.5]$.

3.9 Definition

An interval-valued fuzzy subset $\tilde{\mu}$ of S is called an interval-valued $(\in, \in)$-fuzzy *interior ideal* of S if the following conditions are satisfied for all $x, a, y \in S$ and $\tilde{t}, \tilde{t}_1, \tilde{t}_2 \in D(0, 1]$:

(c_7) $x \le y,\ y_{\tilde{t}} \in \tilde{\mu} \Rightarrow x_{\tilde{t}} \in \tilde{\mu}$,
(c_8) $x_{\tilde{t}_1} \in \tilde{\mu},\ y_{\tilde{t}_2} \in \tilde{\mu} \Rightarrow (xy)_{\mathrm{r}\min\{\tilde{t}_1, \tilde{t}_2\}} \in \tilde{\mu}$,
(c_9) $a_{\tilde{t}} \in \tilde{\mu} \Rightarrow (xay)_{\tilde{t}} \in \tilde{\mu}$.

3.10 Theorem

Every interval-valued fuzzy interior ideal of an ordered semigroup S is an interval-valued $(\in, \in)$-fuzzy interior ideal of S.

Proof 4. Let $\tilde{\mu}$ is an interval-valued fuzzy interior ideal of S and $a, b \in S$, such that $b_{\tilde{t}} \in \tilde{\mu}$ with $a \le b$. Then $\tilde{\mu}(b) \ge \tilde{t}$ and by Definition 2.3 (I_5) $\tilde{\mu}(a) \ge \tilde{\mu}(b) \ge \tilde{t}$, it follows that $a_{\tilde{t}} \in \tilde{\mu}$. If $a_{\tilde{t}_1} \in \tilde{\mu}$, $b_{\tilde{t}_2} \in \tilde{\mu}$, then $\tilde{\mu}(a) \ge \tilde{t}_1$ and $\tilde{\mu}(b) \ge \tilde{t}_2$, and by Definition 2.3 (I_6)

$$\begin{aligned} \tilde{\mu}(ab) &\ge \min\{\tilde{\mu}(a), \tilde{\mu}(b)\}, \\ &\ge \min\{\tilde{t}_1, \tilde{t}_2\}, \end{aligned}$$

hence, $(ab)_{\min\{\tilde{t}_1,\tilde{t}_2\}} \in \tilde{\mu}$. Finally, let us suppose that $c_{\tilde{t}} \in \tilde{\mu}$, then $\tilde{\mu}(c) \geq \tilde{t}$, and by Definition 2.3 (I_4) $\tilde{\mu}(acb) \geq \tilde{\mu}(c) \geq \tilde{t}$, i.e. $(acb)_{\tilde{t}} \in \tilde{\mu}$. Consequently, $\tilde{\mu}$ is an interval-valued $(\in, \in)$-fuzzy interior ideal of S. □

3.11 Proposition

Every interval-valued $(\in, \in)$-fuzzy interior ideal of an ordered semigroup S is an interval-valued $(\in, \in \vee q_{\tilde{k}})$-fuzzy interior ideal of S.

Proof 5. It is straight forward. □

3.12 Remark

From Theorem (3.10) and Proposition (3.11), it is concluded that every interval-valued fuzzy interior ideal of an ordered semigroup is an interval-valued $(\in, \in \vee q_{\tilde{k}})$-fuzzy interior ideal. Thus, we have the following proposition.

3.13 Proposition

Every interval-valued fuzzy interior ideal of an ordered semigroup S is an interval-valued $(\in, \in \vee q_{\tilde{k}})$-fuzzy interior ideal of S.

3.14 Remark

The converse of Proposition (3.11) is not true in general.

3.15 Example

Consider the ordered semigroup of Example (3.2) and define an interval-valued fuzzy subset $\tilde{\mu}: S \to [0, 1]$ by

$$\tilde{\mu}: S \to [0,1] \mid \tilde{\mu}(x) = \begin{cases} [0.50, 0.55] \text{ if } x = a, \\ [0.45, 0.50] \text{ if } x = b, \\ [0.65, 0.70] \text{ if } x = c, \\ [0.55, 0.60] \text{ if } x = d, \\ [0.40, 0.45] \text{ if } x = e. \end{cases}$$

Then, clearly, $\tilde{\mu}$ is an interval-valued $(\in, \in \vee q_{[0.20,\, 0.30]})$-fuzzy interior ideal of S, but not an interval-valued $(\in, \in)$-fuzzy interior ideal of S, because for $a \leq d$, $d_{[0.50,\, 0.60]} \in \tilde{\mu}$ but $a_{[0.50,\, 0.60]} \overline{\in} \tilde{\mu}$.

3.16 Theorem

If $\tilde{\mu}$ is an interval-valued $(\in, \in \vee q_{\tilde{k}})$-fuzzy interior ideal of S, then the set $Q^{\tilde{k}}(\tilde{\mu}; \tilde{t})$ (where $Q^{\tilde{k}}(\tilde{\mu}; \tilde{t}) \neq \varnothing$) is an interior ideal of S for all $\tilde{t} \in D\left(0, \frac{1-k}{2}\right]$.

Proof 6. Assume that $\tilde{\mu}$ is an interval-valued $(\in, \in \vee q_{\tilde{k}})$-fuzzy interior ideal of S. Let $y \in Q^{\tilde{k}}(\tilde{\mu}; \tilde{t})$, where $\tilde{t} \in D\left(0, \frac{1-k}{2}\right]$ and $y \in S$ be such that $x \leq y$ Then, $\tilde{\mu}(y) + \tilde{t} > \tilde{1} - \tilde{k}$. Using (c_4), we have

$$\begin{aligned}\tilde{\mu}(x) &\geq \text{r}\min\left\{\tilde{\mu}(y), \left[\frac{1-k^+}{2}, \frac{1-k^-}{2}\right]\right\},\\ &> \text{r}\min\left\{\tilde{1} - \tilde{t} - \tilde{k}, \left[\frac{1-k^+}{2}, \frac{1-k^-}{2}\right]\right\},\\ &= \tilde{1} - \tilde{t} - \tilde{k},\end{aligned}$$

and so $x \in Q^{\tilde{k}}(\tilde{\mu}; \tilde{t})$. Let $x, y \in Q^{\tilde{k}}(\tilde{\mu}; \tilde{t})$. Then, $\tilde{\mu}(x) + \tilde{t} > \tilde{1} - \tilde{k}$ and then $\tilde{\mu}(y) + \tilde{t} > \tilde{1} - \tilde{k}$. It follows from (c_5) that

$$\begin{aligned}\tilde{\mu}(xy) &\geq \text{r}\min\left\{\tilde{\mu}(x), \tilde{\mu}(y), \left[\frac{1-k^+}{2}, \frac{1-k^-}{2}\right]\right\},\\ &> \text{r}\min\left\{\tilde{1} - \tilde{t} - \tilde{k}, \tilde{1} - \tilde{t} - \tilde{k}, \left[\frac{1-k^+}{2}, \frac{1-k^-}{2}\right]\right\},\\ &= \tilde{1} - \tilde{t} - \tilde{k}\end{aligned}$$

and so, $xy \in Q^{\tilde{k}}(\tilde{\mu}; \tilde{t})$.

If $x, y, z \in S$, such that $y \in Q^{\tilde{k}}(\tilde{\mu}; \tilde{t})$, then $\tilde{\mu}(y) + \tilde{t} > \tilde{1} - \tilde{k}$, and so by (c_6),

$$\begin{aligned}\tilde{\mu}(xyz) &\geq \text{r}\min\left\{\tilde{\mu}(y), \left[\frac{1-k^+}{2}, \frac{1-k^-}{2}\right]\right\},\\ &> \text{r}\min\left\{\tilde{1} - \tilde{t} - \tilde{k}, \left[\frac{1-k^+}{2}, \frac{1-k^-}{2}\right]\right\},\\ &= \tilde{1} - \tilde{t} - \tilde{k}.\end{aligned}$$

Hence, $xyz \in Q^{\tilde{k}}(\tilde{\mu}; \tilde{t})$. Therefore, $Q^{\tilde{k}}(\tilde{\mu}; \tilde{t})$ is an interior ideal of S. □

3.17 Corollary

If $\tilde{\mu}$ is an interval-valued $(\in, \in \vee q)$-fuzzy interior ideal of S, then the set $Q(\tilde{\mu}; \tilde{t})$ (where $Q(\tilde{\mu}; \tilde{t}) \neq \emptyset$) is an interior ideal of S for all $\tilde{t} \in D(0, 0.5]$.

3.18 Theorem

An interval-valued fuzzy subset $\tilde{\mu}$ of S is an interval-valued $(\in, \in \vee q_{\tilde{k}})$-fuzzy interior ideal of S if and only if $[\tilde{\mu}]_{\tilde{t}}^{\tilde{k}}$ is an interior ideal of S for all $\tilde{t} \in D(0, 1]$, where $[\tilde{\mu}]_{\tilde{t}}^{\tilde{k}} \neq \emptyset$.

Proof 7. Assume that $\tilde{\mu}$ is an interval-valued $(\in, \in \vee q_{\tilde{k}})$-fuzzy interior ideal of S, and let $\tilde{t} \in D(0, 1]$, such that $[\tilde{\mu}]_{\tilde{t}}^{\tilde{k}} \neq \emptyset$. Let $y \in [\tilde{\mu}]_{\tilde{t}}^{\tilde{k}}$ and $x \in S$ be such that $x \leq y$. Then, $y \in U(\tilde{\mu}; \tilde{t})$ or $y \in Q^{\tilde{k}}(\tilde{\mu}; \tilde{t})$, i.e. $\tilde{\mu}(y) \geq \tilde{t}$ or $\tilde{\mu}(y) + \tilde{t} > \tilde{1} - \tilde{k}$. Using (c_4), we get

$$\tilde{\mu}(x) \geq \text{r}\min\left\{\tilde{\mu}(y), \left[\frac{1-k^+}{2}, \frac{1-k^-}{2}\right]\right\}. \tag{A}$$

We consider the following cases:

Case (i). If $\tilde{\mu}(y) \leq \left[\frac{1-k^+}{2}, \frac{1-k^-}{2}\right]$, then from (A), we have $\tilde{\mu}(x) \geq \tilde{\mu}(y)$. Thus, $\tilde{\mu}(x) \geq \tilde{t}$, if $\tilde{\mu}(y) \geq \tilde{t}$. It follows that $x \in U(\tilde{\mu};\tilde{t}) \subseteq [\tilde{\mu}]_{\tilde{t}}^{\tilde{k}}$. If $\tilde{\mu}(y) + \tilde{t} > \tilde{1} - \tilde{k}$, then $\tilde{\mu}(x) \geq \tilde{\mu}(y) > \tilde{1} - \tilde{k} - \tilde{t}$, and hence $x \in Q^{\tilde{k}}(\tilde{\mu};\tilde{t}) \subseteq [\tilde{\mu}]_{\tilde{t}}^{\tilde{k}}$.

Case (ii). If $\tilde{\mu}(y) > \left[\frac{1-k^+}{2}, \frac{1-k^-}{2}\right]$, then from (A), we have $\tilde{\mu}(x) \geq \left[\frac{1-k^+}{2}, \frac{1-k^-}{2}\right]$. If $\tilde{t} \leq \left[\frac{1-k^+}{2}, \frac{1-k^-}{2}\right]$, then $\tilde{\mu}(x) \geq \tilde{t}$, i.e. $x \in U(\tilde{\mu};\tilde{t}) \subseteq [\tilde{\mu}]_{\tilde{t}}^{\tilde{k}}$. If $\tilde{t} > \left[\frac{1-k^+}{2}, \frac{1-k^-}{2}\right]$, $\tilde{\mu}(x) + \tilde{t} > \tilde{1} - \tilde{k}$. It follows that $x \in Q^{\tilde{k}}(\tilde{\mu};\tilde{t}) \subseteq [\tilde{\mu}]_{\tilde{t}}^{\tilde{k}}$.

Let $x, y \in [\tilde{\mu}]_{\tilde{t}}^{\tilde{k}}$. Then, $x \in U(\tilde{\mu};\tilde{t})$ or $x_{\tilde{t}} q_{\tilde{k}} \tilde{\mu}$ and $y \in U(\tilde{\mu};\tilde{t})$ or $y_{\tilde{t}} q_{\tilde{k}} \tilde{\mu}$, that is $\tilde{\mu}(x) \geq \tilde{t}$ or $\tilde{\mu}(x) + \tilde{t} > \tilde{1} - \tilde{k}$ and $\tilde{\mu}(y) \geq \tilde{t}$ or $\tilde{\mu}(y) + \tilde{t} > \tilde{1} - \tilde{k}$. We consider the following four cases.

(i) If $\tilde{\mu}(x) \geq \tilde{t}$ and $\tilde{\mu}(y) \geq \tilde{t}$.
(ii) If $\tilde{\mu}(x) \geq \tilde{t}$ and $\tilde{\mu}(y) + \tilde{t} > \tilde{1} - \tilde{k}$.
(iii) If $\tilde{\mu}(x) + \tilde{t} > \tilde{1} - \tilde{k}$ and $\tilde{\mu}(y) \geq \tilde{t}$.
(iv) If $\tilde{\mu}(x) + \tilde{t} > \tilde{1} - \tilde{k}$ and $\tilde{\mu}(y) + \tilde{t} > \tilde{1} - \tilde{k}$.

For Case (i), (c_5) implies that

$$\begin{aligned}
\tilde{\mu}(xy) &\geq \mathrm{r\,min}\left\{\tilde{\mu}(x), \tilde{\mu}(y), \left[\frac{1-k^+}{2}, \frac{1-k^-}{2}\right]\right\}, \\
&\geq \mathrm{r\,min}\left\{\tilde{t}, \tilde{t}, \left[\frac{1-k^+}{2}, \frac{1-k^-}{2}\right]\right\}, \\
&= \begin{cases} \tilde{t}, & \text{if } \tilde{t} \leq \left[\frac{1-k^+}{2}, \frac{1-k^-}{2}\right], \\ \left[\frac{1-k^+}{2}, \frac{1-k^-}{2}\right], & \text{if } \tilde{t} > \left[\frac{1-k^+}{2}, \frac{1-k^-}{2}\right], \end{cases}
\end{aligned}$$

so that $xy \in U(\tilde{\mu};\tilde{t})$ or $xy \in Q^{\tilde{k}}(\tilde{\mu};\tilde{t})$. Hence, $xy \in [\tilde{\mu}]_{\tilde{t}}^{\tilde{k}}$. For the second case, using (c_5),

$$\begin{aligned}
\tilde{\mu}(xy) &\geq \mathrm{r\,min}\left\{\tilde{\mu}(x), \tilde{\mu}(y), \left[\frac{1-k^+}{2}, \frac{1-k^-}{2}\right]\right\}, \\
&\geq \mathrm{r\,min}\left\{\tilde{t}, \tilde{1} - \tilde{t} - \tilde{k}, \left[\frac{1-k^+}{2}, \frac{1-k^-}{2}\right]\right\}, \\
&= \begin{cases} \tilde{1} - \tilde{t} - \tilde{k}, & \text{if } \tilde{t} > \left[\frac{1-k^+}{2}, \frac{1-k^-}{2}\right], \\ \tilde{t}, & \text{if } \tilde{t} \leq \left[\frac{1-k^+}{2}, \frac{1-k^-}{2}\right]. \end{cases}
\end{aligned}$$

Thus, $xy \in U(\tilde{\mu};\tilde{t}) \cup Q^{\tilde{k}}(\tilde{\mu};\tilde{t}) = [\tilde{\mu}]_{\tilde{t}}^{\tilde{k}}$. We have similar result for the Case (iii). For the final case, if $\tilde{t} > \left[\frac{1-k^+}{2}, \frac{1-k^-}{2}\right]$, then $\tilde{1} - \tilde{t} - \tilde{k} < \left[\frac{1-k^+}{2}, \frac{1-k^-}{2}\right] < \tilde{t}$. Hence,

$$\begin{aligned}
\tilde{\mu}(xy) &\geq \mathrm{r\,min}\left\{\tilde{\mu}(x), \tilde{\mu}(y), \left[\frac{1-k^+}{2}, \frac{1-k^-}{2}\right]\right\}, \\
&\geq \mathrm{r\,min}\left\{\tilde{1} - \tilde{t} - \tilde{k}, \tilde{1} - \tilde{t} - \tilde{k}, \left[\frac{1-k^+}{2}, \frac{1-k^-}{2}\right]\right\}, \\
&= \tilde{1} - \tilde{t} - \tilde{k}.
\end{aligned}$$

Thus, $xy \in Q^{\tilde{k}}(\tilde{\mu};\tilde{t}) \subseteq [\tilde{\mu}]_{\tilde{t}}^{\tilde{k}}$. If $\tilde{t} \leq \left[\frac{1-k^{+}}{2}, \frac{1-k^{-}}{2}\right]$, then $\tilde{1}-\tilde{t}-\tilde{k} \geq \left[\frac{1-k^{+}}{2}, \frac{1-k^{-}}{2}\right] \geq \tilde{t}$, and by ($c_5$),

$$\begin{aligned}\tilde{\mu}(xy) &\geq \mathrm{r\,min}\left\{\tilde{\mu}(x), \tilde{\mu}(y), \left[\frac{1-k^{+}}{2}, \frac{1-k^{-}}{2}\right]\right\},\\ &\geq \mathrm{r\,min}\left\{\tilde{1}-\tilde{t}-\tilde{k}, \tilde{1}-\tilde{t}-\tilde{k}, \left[\frac{1-k^{+}}{2}, \frac{1-k^{-}}{2}\right]\right\},\\ &= \left[\frac{1-k^{+}}{2}, \frac{1-k^{-}}{2}\right] \geq \tilde{t},\end{aligned}$$

this implies that $xy \in U(\tilde{\mu};\tilde{t}) \subseteq [\tilde{\mu}]_{\tilde{t}}^{\tilde{k}}$. Let $x, y, a \in S$ be such that $a \in [\tilde{\mu}]_{\tilde{t}}^{\tilde{k}}$, then, $\tilde{\mu}(a) \geq \tilde{t}$ or $\tilde{\mu}(a)+\tilde{t} > \tilde{1}-\tilde{k}$. It follows from ($c_6$)

$$\tilde{\mu}(xay) \geq \mathrm{r\,min}\left\{\tilde{\mu}(a), \left[\frac{1-k^{+}}{2}, \frac{1-k^{-}}{2}\right]\right\}. \tag{B}$$

We consider the following two cases:

Case 1. If $\tilde{\mu}(a) \leq \left[\frac{1-k^{+}}{2}, \frac{1-k^{-}}{2}\right]$.

Case 2. If $\tilde{\mu}(a) > \left[\frac{1-k^{+}}{2}, \frac{1-k^{-}}{2}\right]$.

Using Case 1 in (B), we get $\tilde{\mu}(xay) \geq \tilde{\mu}(a)$. Thus, if $\tilde{\mu}(a) \geq \tilde{t}$, then $\tilde{\mu}(xay) \geq \tilde{t}$, and so, $xay \in U(\tilde{\mu};\tilde{t}) \subseteq [\tilde{\mu}]_{\tilde{t}}^{\tilde{k}}$. If $\tilde{\mu}(a)+\tilde{t} > \tilde{1}-\tilde{k}$, then $\tilde{\mu}(xay)+\tilde{t} \geq \tilde{\mu}(a)+\tilde{t} > \tilde{1}-\tilde{k}$, this implies that $(xay)_{\tilde{t}}\, q_{\tilde{k}}\, \tilde{\mu}$, i.e. $xay \in Q^{\tilde{k}}(\tilde{\mu};\tilde{t}) \subseteq [\tilde{\mu}]_{\tilde{t}}^{\tilde{k}}$. Combining Case 2 and (B), we see that $\tilde{\mu}(xay) \geq \left[\frac{1-k^{+}}{2}, \frac{1-k^{-}}{2}\right]$. If $\tilde{t} \leq \left[\frac{1-k^{+}}{2}, \frac{1-k^{-}}{2}\right]$, then $\tilde{\mu}(xay) \geq \tilde{t}$, and hence, $xay \in U(\tilde{\mu};\tilde{t}) \subseteq [\tilde{\mu}]_{\tilde{t}}^{\tilde{k}}$, but if $\tilde{t} > \left[\frac{1-k^{-}}{2}, \frac{1-k^{+}}{2}\right]$, then $\tilde{\mu}(xay)+\tilde{t} > \tilde{1}-\tilde{k}$, which implies that $xay \in Q^{\tilde{k}}(\tilde{\mu};\tilde{t}) \subseteq [\tilde{\mu}]_{\tilde{t}}^{\tilde{k}}$. Therefore, $[\tilde{\mu}]_{\tilde{t}}^{\tilde{k}}$ is interior ideal of S.

Conversely, assume that $[\tilde{\mu}]_{\tilde{t}}^{\tilde{k}}$ is an interior ideal of S.

If $\tilde{\mu}(a) < \mathrm{r\,min}\left\{\tilde{\mu}(b), \left[\frac{1-k^{+}}{2}, \frac{1-k^{-}}{2}\right]\right\}$ for some a, $b \in S$, such that $a \leq b$, then $\tilde{\mu}(a) < \tilde{s} \leq \mathrm{r\,min}\left\{\tilde{\mu}(b), \left[\frac{1-k^{+}}{2}, \frac{1-k^{-}}{2}\right]\right\}$ for some $\tilde{s} \in D\left(0, \frac{1-k}{2}\right]$. This shows that $b \in U(\tilde{\mu};\tilde{s}) \subseteq [\tilde{\mu}]_{\tilde{s}}^{\tilde{k}}$ but $a \overline{\in} U(\tilde{\mu};\tilde{s})$ and $a \overline{\in} Q^{\tilde{k}}(\tilde{\mu};\tilde{s})$, i.e. $a \overline{\in} [\tilde{\mu}]_{\tilde{s}}^{\tilde{k}}$, a contradiction. Hence, $\tilde{\mu}(x) \geq \mathrm{r\,min}\left\{\tilde{\mu}(y), \left[\frac{1-k^{+}}{2}, \frac{1-k^{-}}{2}\right]\right\}$ for all $x, y \in S$ with $x \leq y$.

If $\tilde{\mu}(ab) < \mathrm{r\,min}\left\{\tilde{\mu}(a), \tilde{\mu}(b), \left[\frac{1-k^{+}}{2}, \frac{1-k^{-}}{2}\right]\right\}$ for some a, $b \in S$, then there exists $\tilde{t} \in D\left(0, \frac{1-k}{2}\right]$, such that

$$\tilde{\mu}(ab) < \tilde{t} \leq \mathrm{r\,min}\left\{\tilde{\mu}(a), \tilde{\mu}(b), \left[\frac{1-k^{+}}{2}, \frac{1-k^{-}}{2}\right]\right\},$$

and it follows that $a \in U(\tilde{\mu};\tilde{t}) \subseteq [\tilde{\mu}]_{\tilde{t}}^{\tilde{k}}$ and $b \in U(\tilde{\mu};\tilde{t}) \subseteq [\tilde{\mu}]_{\tilde{t}}^{\tilde{k}}$. Therefore, from Definition 2.2, (I_3) $ab \in [\tilde{\mu}]_{\tilde{t}}^{\tilde{k}}$, i.e. $\tilde{\mu}(ab) \geq \tilde{t}$ or $\tilde{\mu}(ab)+\tilde{t} > \tilde{1}-\tilde{k}$, a contradiction. Therefore, $\tilde{\mu}(xy) \geq \mathrm{r\,min}\left\{\tilde{\mu}(x), \tilde{\mu}(y), \left[\frac{1-k^{+}}{2}, \frac{1-k^{-}}{2}\right]\right\}$ for all $x, y \in S$.

Finally, assuming that $\tilde{\mu}(xay) < \mathrm{r}\min\left\{\tilde{\mu}(a), \left[\frac{1-k^+}{2}, \frac{1-k^-}{2}\right]\right\}$ for some $x, a, y \in S$, then there exist $\tilde{t} \in D\left(0, \frac{1-k}{2}\right]$, such that $\tilde{\mu}(xay) < \tilde{t} \le \mathrm{r}\min\left\{\tilde{\mu}(a), \left[\frac{1-k^+}{2}, \frac{1-k^-}{2}\right]\right\}$. From this, we conclude that $a \in U(\tilde{\mu}; \tilde{t}) \subseteq [\tilde{\mu}]_{\tilde{t}}^{\tilde{k}}$ but $xay \overline{\in} U(\tilde{\mu}; \tilde{t})$ and $xay \overline{\in} Q^{\tilde{k}}(\tilde{\mu}; \tilde{t})$, i.e. $xay \overline{\in} [\tilde{\mu}]_{\tilde{t}}^{\tilde{k}}$, a contradiction. Therefore, $\tilde{\mu}(ab) < \mathrm{r}\min\left\{\tilde{\mu}(a), \tilde{\mu}(b), \left[\frac{1-k^+}{2}, \frac{1-k^-}{2}\right]\right\}$ for all $x, a, y \in S$. Consequently, $[\tilde{\mu}]_{\tilde{t}}^{\tilde{k}}$ is an interval-valued $(\in, \in \vee \mathrm{q}_{\tilde{k}})$-fuzzy interior ideal of S. □

3.19 Corollary

An interval-valued fuzzy subset $\tilde{\mu}$ of S is an interval-valued $(\in, \in \vee \mathrm{q})$-fuzzy interior ideal of S if and only if $[\tilde{\mu}]_{\tilde{t}}$ is an interior ideal of S for all $\tilde{t} \in D(0, 1]$, where $[\tilde{\mu}]_{\tilde{t}} \neq \varnothing$.

3.20 Proposition

If $\{\tilde{\mu}_i\}_{i\in I} \neq \varnothing$ is a collection of interval-valued $(\in, \in \vee \mathrm{q}_{\tilde{k}})$-fuzzy interior ideals of an ordered semigroup S, then $\bigcap_{i\in I} \tilde{\mu}_i$ is an interval-valued $(\in, \in \vee \mathrm{q}_{\tilde{k}})$-fuzzy interior ideal of S.

Proof 8. Let $\tilde{\mu}_i$ is an interval-valued $(\in, \in \vee \mathrm{q}_{\tilde{k}})$-fuzzy interior ideal of S for all $i \in I$ and $a, b \in S$, such that $a \le b$. Then

$$\begin{aligned}
(\bigcap_{i\in I} \tilde{\mu}_i)(a) &= \bigwedge_{i\in I} \mu_i(a), \\
&\ge \bigwedge_{i\in I} \left\{\mathrm{r}\min\left\{\tilde{\mu}_i(b), \left[\frac{1-k^+}{2}, \frac{1-k^-}{2}\right]\right\}\right\}, \\
&= \mathrm{r}\min\left\{\bigwedge_{i\in I} \tilde{\mu}_i(b), \left[\frac{1-k^+}{2}, \frac{1-k^-}{2}\right]\right\}, \\
&= \mathrm{r}\min\left\{(\bigcap_{i\in I} \tilde{\mu}_i)(b), \left[\frac{1-k^+}{2}, \frac{1-k^-}{2}\right]\right\}.
\end{aligned}$$

Let $x, y \in S$, then

$$\begin{aligned}
(\bigcap_{i\in I} \tilde{\mu}_i)(xay) &= \bigwedge_{i\in I} \tilde{\mu}_i(xy), \\
&\ge \bigwedge_{i\in I} \left\{\mathrm{r}\min\left\{\tilde{\mu}_i(x), \tilde{\mu}_i(y), \left[\frac{1-k^+}{2}, \frac{1-k^-}{2}\right]\right\}\right\}, \\
&= \mathrm{r}\min\left\{\bigwedge_{i\in I} \tilde{\mu}_i(x), \bigwedge_{i\in I} \tilde{\mu}_i(y), \left[\frac{1-k^+}{2}, \frac{1-k^-}{2}\right]\right\}, \\
&= \mathrm{r}\min\left\{(\bigcap_{i\in I} \tilde{\mu}_i)(x), (\bigcap_{i\in I} \tilde{\mu}_i)(y), \left[\frac{1-k^+}{2}, \frac{1-k^-}{2}\right]\right\}.
\end{aligned}$$

If $x, y, a \in S$, then

$$\begin{aligned}(\underset{i\in I}{\cap}\tilde{\mu}_i)(xay) &= \underset{i\in I}{\wedge}\tilde{\mu}_i(xay),\\ &\geq \underset{i\in I}{\wedge}\left\{\text{r min}\left\{\tilde{\mu}_i(a), \left[\frac{1-k^+}{2}, \frac{1-k^-}{2}\right]\right\}\right\},\\ &= \text{r min}\left\{\underset{i\in I}{\wedge}\tilde{\mu}_i(a), \left[\frac{1-k^+}{2}, \frac{1-k^-}{2}\right]\right\},\\ &= \text{r min}\left\{(\underset{i\in I}{\cap}\tilde{\mu}_i)(a), \left[\frac{1-k^+}{2}, \frac{1-k^-}{2}\right]\right\}.\end{aligned}$$

Hence, $\underset{i\in I}{\cap}\tilde{\mu}_I$ is an interval-valued $(\in, \in\vee \mathrm{q}_{\tilde{k}})$-fuzzy interior ideal of S. □

Now it is natural to investigate that $\underset{i\in I}{\cup}\tilde{\mu}_i$ is an interval-valued $(\in, \in\vee \mathrm{q}_{\tilde{k}})$-fuzzy interior ideal of S or not for any non-empty $\{\tilde{\mu}_i\}_{i\in I}$ collection of interval-valued $(\in, \in\vee \mathrm{q}_{\tilde{k}})$-fuzzy interior ideals of S. Therefore, the following example is constructed to show that $\underset{i\in I}{\cup}\tilde{\mu}_i$ is not an interval-valued $(\in, \in\vee \mathrm{q}_{\tilde{k}})$-fuzzy interior ideal in general.

3.21 Example

Consider the ordered semigroup $S=\{a, b, c, d\}$ with the multiplication given in the following Table 2 and order relations $a \leq a$, $b \leq b$, $c \leq c$, $d \leq d$, and $a \leq d$.

Define

$$\tilde{\mu}_1(x)=\begin{cases}[0.4, 0.5], & \text{if } x\in\{a, b\},\\ [0.0, 0.0], & \text{if } x\in\{c, d\},\end{cases}$$

and

$$\tilde{\mu}_2(x)=\begin{cases}[0.4, 0.5], & \text{if } x\in\{a, c\},\\ [0.0, 0.0], & \text{if } x\in\{b, d\},\end{cases}$$

then both $\tilde{\mu}_1$ and $\tilde{\mu}_2$ are interval-valued $(\in, \in\vee \mathrm{q}_{\tilde{k}})$-fuzzy interior ideals of S, but $(\tilde{\mu}_1\cup\tilde{\mu}_2)$ is not an interval-valued $(\in, \in\vee \mathrm{q}_{\tilde{k}})$-fuzzy interior ideal. As

$$\begin{aligned}(\tilde{\mu}_1\cup\tilde{\mu}_2)(bc) &= (\tilde{\mu}_1\cup\tilde{\mu}_2)(d)\\ &= \text{r max}\{\tilde{\mu}_1(d)=[0.0, 0.0], \tilde{\mu}_2(d)=[0.0, 0.0]\}\\ &= [0.0, 0.0].\end{aligned}$$

Table 2:

.	a	b	c	d
a	a	a	a	a
b	a	a	d	a
c	a	a	a	a
d	a	a	a	a

Caption for Table 2 (Multiplication table of S={a,b,c,d})

Meanwhile,

$$\mathrm{r\,min}\left\{\begin{matrix}(\tilde{\mu}_1 \cup \tilde{\mu}_2)(b),\\ (\tilde{\mu}_1 \cup \tilde{\mu}_2)(c)\end{matrix}\right\} = \mathrm{r\,min}\left\{\begin{matrix}\mathrm{r\,max}\left\{\begin{matrix}\tilde{\mu}_1(b) = [0.4,\ 0.5],\\ \tilde{\mu}_2(b) = [0.0,\ 0.0]\end{matrix}\right\},\\ \mathrm{r\,max}\left\{\begin{matrix}\tilde{\mu}_1(c) = [0.0,\ 0.0],\\ \tilde{\mu}_2(c) = [0.4,\ 0.5]\end{matrix}\right\}\end{matrix}\right\}$$
$$= \mathrm{r\,min}\{[0.4,\ 0.5],\ [0.4,\ 0.5]\}$$
$$= [0.4,\ 0.5].$$

Hence,

$$(\tilde{\mu}_1 \cup \tilde{\mu}_2)(bc) < \mathrm{r\,min}\{(\tilde{\mu}_1 \cup \tilde{\mu}_2)(b), (\tilde{\mu}_1 \cup \tilde{\mu}_2)(c)\}.$$

3.22 Definition

An interval-valued fuzzy subset $\tilde{\mu}$ of S is called an interval-valued $(\in, \in\vee \mathrm{q}_{\tilde{k}})$-fuzzy left (right) ideal of S if the following conditions are satisfied for all $x, y \in S$ and $\tilde{t}, \tilde{t}_1, \tilde{t}_2 \in D(0,\ 1]$:

(c_{10}) $x \le y,\ y_{\tilde{t}} \in \tilde{\mu} \Rightarrow x_{\tilde{t}} \in\vee \mathrm{q}_{\tilde{k}}\tilde{\mu}$,

(c_{11}) $x_{\tilde{t}_1} \in \tilde{\mu},\ y_{\tilde{t}_2} \in \tilde{\mu} \Rightarrow (xy)_{\mathrm{r\,min}\{\tilde{t}_1, \tilde{t}_2\}} \in\vee \mathrm{q}_{\tilde{k}}\tilde{\mu}$,

(c_{12}) $y_{\tilde{t}} \in \tilde{\mu} \Rightarrow (xy)_{\tilde{t}} \in\vee \mathrm{q}_{\tilde{k}}\tilde{\mu}((yx)_{\tilde{t}} \in\vee \mathrm{q}_{\tilde{k}}\tilde{\mu})$.

An interval-valued fuzzy subset $\tilde{\mu}$ of S is called an interval-valued $(\in, \in\vee \mathrm{q}_{\tilde{k}})$-fuzzy ideal of S if it is both an interval-valued $(\in, \in\vee \mathrm{q}_{\tilde{k}})$-fuzzy left ideal and an interval-valued $(\in, \in\vee \mathrm{q}_{\tilde{k}})$-fuzzy right ideal of S.

3.23 Lemma

An interval-valued fuzzy subset $\tilde{\mu}$ of S is called an interval-valued $(\in, \in\vee \mathrm{q}_{\tilde{k}})$-fuzzy left (right) ideal of S if and only if

(c_{14}) $x \le y \Rightarrow \tilde{\mu}(x) \ge \mathrm{r\,min}\left\{\tilde{\mu}(y), \left[\frac{1-k^+}{2}, \frac{1-k^-}{2}\right]\right\}$,

(c_{15}) $\tilde{\mu}(xy) \ge \mathrm{r\,min}\left\{\tilde{\mu}(x), \tilde{\mu}(y), \left[\frac{1-k^+}{2}, \frac{1-k^-}{2}\right]\right\}$,

(c_{16}) $\tilde{\mu}(xy) \ge \mathrm{r\,min}\left\{\tilde{\mu}(y), \left[\frac{1-k^+}{2}, \frac{1-k^-}{2}\right]\right\}\left(\tilde{\mu}(xy) \ge \mathrm{r\,min}\left\{\tilde{\mu}(x), \left[\frac{1-k^+}{2}, \frac{1-k^-}{2}\right]\right\}\right)$.

Proof 9. Consider $\tilde{\mu}$ be an an interval-valued $(\in, \in\vee \mathrm{q}_{\tilde{k}})$-fuzzy left (right) ideal of S. If there exist $a, b \in S$, such that $a \le b$ and

$$\tilde{\mu}(x) < \mathrm{r\,min}\left\{\tilde{\mu}(y), \left[\frac{1-k^+}{2}, \frac{1-k^-}{2}\right]\right\},$$

then

$$\tilde{\mu}(x) < \tilde{t} \le \mathrm{r\,min}\left\{\tilde{\mu}(y), \left[\frac{1-k^+}{2}, \frac{1-k^-}{2}\right]\right\}$$

for some $\tilde{t} \in D\left(0, \frac{1-k}{2}\right]$. It follows that $y_{\tilde{t}} \in \tilde{\mu}$ but $x_{\tilde{t}} \overline{\in} \vee q_{\tilde{k}} \tilde{\mu}$, a contradiction. Hence,

$$\tilde{\mu}(x) \geq \mathrm{r}\min\left\{\tilde{\mu}(y), \left[\frac{1-k^{+}}{2}, \frac{1-k^{-}}{2}\right]\right\}$$

for all $x, y \in S$ with $x \leq y$.

Let $\tilde{\mu}(ab) < \mathrm{r}\min\left\{\tilde{\mu}(a), \tilde{\mu}(b), \left[\frac{1-k^{+}}{2}, \frac{1-k^{-}}{2}\right]\right\}$ for some $a, b \in S$. Then there exist $\tilde{s} \in D\left(0, \frac{1-k}{2}\right]$, such that

$$\tilde{\mu}(ab) < \tilde{s} \leq \mathrm{r}\min\left\{\tilde{\mu}(a), \tilde{\mu}(b), \left[\frac{1-k^{+}}{2}, \frac{1-k^{-}}{2}\right]\right\}.$$

This implies $a_{\tilde{s}} \in \tilde{\mu}$, $b_{\tilde{s}} \in \tilde{\mu}$, and hence, by ($c_{11}$) $(ab)_{\mathrm{r}\min\{\tilde{s},\tilde{s}\}} \in \vee q_{\tilde{k}} \tilde{\mu}$, i.e. $\tilde{\mu}(ab) \geq \tilde{s}$ or $\tilde{\mu}(ab) + \tilde{s} > \tilde{1} - \tilde{k}$, a contradiction. Hence,

$$\tilde{\mu}(xy) \geq \mathrm{r}\min\left\{\tilde{\mu}(x), \tilde{\mu}(y), \left[\frac{1-k^{+}}{2}, \frac{1-k^{-}}{2}\right]\right\}$$

for all $x, y \in S$.

Lastly, let there exist $a, b \in S$, such that

$$\tilde{\mu}(ab) < \mathrm{r}\min\left\{\tilde{\mu}(b), \left[\frac{1-k^{+}}{2}, \frac{1-k^{-}}{2}\right]\right\},$$

then there exist $\tilde{t} \in D\left(0, \frac{1-k}{2}\right]$, such that

$$\tilde{\mu}(ab) < \tilde{t} \leq \mathrm{r}\min\left\{\tilde{\mu}(b), \left[\frac{1-k^{+}}{2}, \frac{1-k^{-}}{2}\right]\right\}.$$

It follows that $b_{\tilde{t}} \in \tilde{\mu}$ but $(ab)_{\tilde{t}} \overline{\in} \vee q_{\tilde{k}} \tilde{\mu}$, again a contradiction, and hence,

$$\tilde{\mu}(xy) \geq \mathrm{r}\min\left\{\tilde{\mu}(y), \left[\frac{1-k^{+}}{2}, \frac{1-k^{-}}{2}\right]\right\}$$

for all $x, y \in S$.

Conversely, assume that (c_{14}–c_{16}) are valid for all $x, y \in S$. Let $x, y \in S$, such that $x \leq y$ and $y_{\tilde{t}} \in \tilde{\mu}$, where $\tilde{t} \in D(0, 1]$. Then by (c_{14}),

$$\begin{aligned} \tilde{\mu}(x) &\geq \mathrm{r}\min\left\{\tilde{\mu}(y), \left[\frac{1-k^{+}}{2}, \frac{1-k^{-}}{2}\right]\right\}, \\ &\geq \mathrm{r}\min\left\{\tilde{t}, \left[\frac{1-k^{+}}{2}, \frac{1-k^{-}}{2}\right]\right\}, \\ &= \begin{cases} \tilde{t}, & \text{if } \tilde{t} \leq \left[\frac{1-k^{+}}{2}, \frac{1-k^{-}}{2}\right], \\ \left[\frac{1-k^{+}}{2}, \frac{1-k^{-}}{2}\right], & \text{if } \tilde{t} > \left[\frac{1-k^{+}}{2}, \frac{1-k^{-}}{2}\right], \end{cases} \end{aligned}$$

this shows that $x_{\tilde{t}} \in \vee q_{\tilde{k}} \tilde{\mu}$.

If $x, y \in S$, such that $x_{\tilde{t}_1}, y_{\tilde{t}_2} \in \tilde{\mu}$, where $\tilde{t}_1, \tilde{t}_2 \in D(0, 1]$, then using ($c_{15}$), we have

$$\begin{aligned}\tilde{\mu}(xy) &\geq \mathrm{r\,min}\left\{\tilde{\mu}(x), \tilde{\mu}(y), \left[\frac{1-k^+}{2}, \frac{1-k^-}{2}\right]\right\}, \\ &\geq \mathrm{r\,min}\left\{\tilde{t}_1, \tilde{t}_2, \left[\frac{1-k^+}{2}, \frac{1-k^-}{2}\right]\right\}, \\ &= \begin{cases} \mathrm{r\,min}\{\tilde{t}_1, \tilde{t}_2\}, & \text{if } \mathrm{r\,min}\{\tilde{t}_1, \tilde{t}_2\} \leq \left[\frac{1-k^+}{2}, \frac{1-k^-}{2}\right], \\ \left[\frac{1-k^+}{2}, \frac{1-k^-}{2}\right], & \text{if } \mathrm{r\,min}\{\tilde{t}_1, \tilde{t}_2\} > \left[\frac{1-k^+}{2}, \frac{1-k^-}{2}\right], \end{cases}\end{aligned}$$

it follows that $(xy)_{\mathrm{r\,min}\{\tilde{t}_1,\tilde{t}_2\}} \in \vee q_{\tilde{k}} \tilde{\mu}$.

Finally, if $x, y \in S$, such that $y_{\tilde{t}} \in \tilde{\mu}$, where $\tilde{t} \in D(0, 1]$, then by (c_{15}),

$$\begin{aligned}\tilde{\mu}(xy) &\geq \mathrm{r\,min}\left\{\tilde{\mu}(y), \left[\frac{1-k^+}{2}, \frac{1-k^-}{2}\right]\right\}, \\ &\geq \mathrm{r\,min}\left\{\tilde{t}, \left[\frac{1-k^+}{2}, \frac{1-k^-}{2}\right]\right\}, \\ &= \begin{cases} \tilde{t}, & \text{if } \tilde{t} \leq \left[\frac{1-k^+}{2}, \frac{1-k^-}{2}\right], \\ \left[\frac{1-k^+}{2}, \frac{1-k^-}{2}\right], & \text{if } \tilde{t} > \left[\frac{1-k^+}{2}, \frac{1-k^-}{2}\right], \end{cases}\end{aligned}$$

follows that $(xy)_{\tilde{t}} \in \vee q_{\tilde{k}} \tilde{\mu}$. Hence, $\tilde{\mu}$ is an an interval-valued $(\in, \in \vee q_{\tilde{k}})$-fuzzy left ideal of S. Similarly, the result can be proved for the right case. Consequently, $\tilde{\mu}$ be an an interval-valued $(\in, \in \vee q_{\tilde{k}})$-fuzzy ideal of S. □

3.24 Proposition

Every interval-valued $(\in, \in \vee q_{\tilde{k}})$-fuzzy ideal of an ordered semigroup S is an interval-valued $(\in, \in \vee q_{\tilde{k}})$-fuzzy interior ideal of S.

Proof 10. Let $\tilde{\mu}$ be an interval-valued $(\in, \in \vee q_{\tilde{k}})$-fuzzy ideal of S. If $x, y \in S$, such that $x \leq y$, then

$$\tilde{\mu}(x) \geq \mathrm{r\,min}\left\{\tilde{\mu}(y), \left[\frac{1-k^+}{2}, \frac{1-k^-}{2}\right]\right\}.$$

If $x, a, y \in S$, then

$$\begin{aligned}\tilde{\mu}(xay) &= \tilde{\mu}(x(ay)), \\ &\geq \mathrm{r\,min}\left\{\tilde{\mu}(ay), \left[\frac{1-k^+}{2}, \frac{1-k^-}{2}\right]\right\}, \\ &\geq \mathrm{r\,min}\left\{\tilde{\mu}(a), \left[\frac{1-k^+}{2}, \frac{1-k^-}{2}\right]\right\}.\end{aligned}$$

If $x, y \in S$, then

$$\tilde{\mu}(xy) \geq \mathrm{r\,min}\left\{\tilde{\mu}(x), \tilde{\mu}(y), \left[\frac{1-k^+}{2}, \frac{1-k^-}{2}\right]\right\}.$$

Consequently, $\tilde{\mu}$ is an interval-valued $(\in, \in \vee q_{\tilde{k}})$-fuzzy interior ideal. □

3.25 Remark

In general, every interval-valued $(\in, \in\vee q_{\tilde{k}})$-fuzzy interior ideal of an ordered semigroup S is not an interval-valued $(\in, \in\vee q_{\tilde{k}})$-fuzzy ideal. However, in case of a regular ordered semigroup, every interval-valued $(\in, \in\vee q_{\tilde{k}})$-fuzzy interior ideal of an ordered semigroup S is an interval-valued $(\in, \in\vee q_{\tilde{k}})$-fuzzy ideal of S.

3.26 Proposition

Every interval-valued $(\in, \in\vee q_{\tilde{k}})$-fuzzy interior ideal of a regular ordered semigroup S is an interval-valued $(\in, \in\vee q_{\tilde{k}})$-fuzzy ideal of S.

Proof 11. Let $\tilde{\mu}$ is an interval-valued $(\in, \in\vee q_{\tilde{k}})$-fuzzy interior ideal of a regular ordered semigroup S. If $a, b\in S$, such that $a \le b$, then by (c_4),

$$\tilde{\mu}(a) \ge \mathrm{r\,min}\left\{\tilde{\mu}(b), \left[\frac{1-k^+}{2}, \frac{1-k^-}{2}\right]\right\}.$$

Also, for all $a, b\in S$, (c_5) implies that

$$\tilde{\mu}(ab) \ge \mathrm{r\,min}\left\{\tilde{\mu}(a), \tilde{\mu}(b), \left[\frac{1-k^+}{2}, \frac{1-k^-}{2}\right]\right\}.$$

Finally, if $a, b\in S$, then there exists $c\in S$, such that $a \le aca$, and therefore, $ab \le (aca)b = (ac)ab$. By (c_4),

$$\begin{aligned}\tilde{\mu}(ab) &\ge \mathrm{r\,min}\left\{\tilde{\mu}((ac)ab), \left[\frac{1-k^+}{2}, \frac{1-k^-}{2}\right]\right\},\\ &\ge \mathrm{r\,min}\left\{\tilde{\mu}(a), \left[\frac{1-k^+}{2}, \frac{1-k^-}{2}\right]\right\},\\ &(\tilde{\mu} \text{ is interval-valued}(\in, \in\vee q_{\tilde{k}})\text{-fuzzy interior ideal}).\end{aligned}$$

Hence, $\tilde{\mu}$ is an interval-valued $(\in, \in\vee q_{\tilde{k}})$-fuzzy right ideal of S. Similarly, we can prove that $\tilde{\mu}$ is an interval-valued $(\in, \in\vee q_{\tilde{k}})$-fuzzy left ideal. □

From Propositions (3.24) and (3.26), we have the following result.

3.27 Theorem

In regular ordered semigroups, the concepts of interval-valued $(\in, \in\vee q_{\tilde{k}})$-fuzzy interior ideal and interval-valued $(\in, \in\vee q_{\tilde{k}})$-fuzzy ideal coincide.

In the following, we define the Cartesian product of two interval-valued $(\in, \in\vee q_{\tilde{k}})$-fuzzy interior ideals.

3.28 Definition

The Cartesian product of two interval-valued $(\in, \in\vee q_{\tilde{k}})$-fuzzy interior ideals $\tilde{\mu}_1$ and $\tilde{\mu}_2$ of an ordered semigroup S is defined as

$$(\tilde{\mu}_1 \times \tilde{\mu}_2)(x, y) = \mathrm{r\,min}\left\{\tilde{\mu}_1(x), \tilde{\mu}_2(y), \left[\frac{1-k^+}{2}, \frac{1-k^-}{2}\right]\right\}$$

for all $x, y\in S$.

3.29 Theorem

The Cartesian product of two interval-valued $(\in, \in\vee q_{\tilde{k}})$-fuzzy interior ideals of S is an interval-valued $(\in, \in\vee q_{\tilde{k}})$ fuzzy interior ideal of $S\times S$.

Proof 12. Let $\tilde{\mu}_1$ and $\tilde{\mu}_2$ be interval-valued $(\in, \in\vee q_{\tilde{k}})$-fuzzy interior ideals of S. Let $(a, b), (c, d)\in S\times S$ with $(a, b) \leq (c, d)$ and consider

$$\begin{aligned}(\tilde{\mu}_1\times\tilde{\mu}_2)(a, b) &= \text{r min}\left\{\tilde{\mu}_1(a), \tilde{\mu}_2(b), \left[\frac{1-k^+}{2}, \frac{1-k^-}{2}\right]\right\}\\ &\geq \text{r min}\left\{\tilde{\mu}_1(c), \tilde{\mu}_2(d), \left[\frac{1-k^+}{2}, \frac{1-k^-}{2}\right]\right\}\\ &= (\tilde{\mu}_1\times\tilde{\mu}_2)(c, d).\end{aligned}$$

Next, we consider

$$\begin{aligned}(\tilde{\mu}_1\times\tilde{\mu}_2)((a, b).(c, d)) &= (\tilde{\mu}_1\times\tilde{\mu}_2)(ac, bd)\\ &= \text{r min}\left\{\tilde{\mu}_1(ac), \tilde{\mu}_2(bd), \left[\frac{1-k^+}{2}, \frac{1-k^-}{2}\right]\right\}\\ &\geq \text{r min}\left\{\begin{array}{c}\text{r min}\left\{\tilde{\mu}_1(a), \tilde{\mu}_1(c), \left[\frac{1-k^+}{2}, \frac{1-k^-}{2}\right]\right\},\\ \text{r min}\left\{\tilde{\mu}_2(b), \tilde{\mu}_2(d), \left[\frac{1-k^+}{2}, \frac{1-k^-}{2}\right]\right\},\\ \left[\frac{1-k^+}{2}, \frac{1-k^-}{2}\right]\end{array}\right\}\\ &= \text{r min}\left\{\begin{array}{c}\text{r min}\left\{\begin{array}{c}\tilde{\mu}_1(a), \tilde{\mu}_1(c), \tilde{\mu}_2(b),\\ \tilde{\mu}_2(d), \left[\frac{1-k^+}{2}, \frac{1-k^-}{2}\right]\end{array}\right\},\\ \left[\frac{1-k^+}{2}, \frac{1-k^-}{2}\right]\end{array}\right\}\\ &= \text{r min}\left\{\begin{array}{c}\text{r min}\left\{\tilde{\mu}_1(a), \tilde{\mu}_2(b), \left[\frac{1-k^+}{2}, \frac{1-k^-}{2}\right]\right\},\\ \text{r min}\left\{\tilde{\mu}_1(c), \tilde{\mu}_2(d), \left[\frac{1-k^+}{2}, \frac{1-k^-}{2}\right]\right\},\\ \left[\frac{1-k^+}{2}, \frac{1-k^-}{2}\right]\end{array}\right\}\\ &= \text{r min}\left\{\begin{array}{c}(\tilde{\mu}_1\times\tilde{\mu}_2)(a, b), (\tilde{\mu}_1\times\tilde{\mu}_2)(c, d),\\ \left[\frac{1-k^+}{2}, \frac{1-k^-}{2}\right]\end{array}\right\}.\end{aligned}$$

Finally, take $(x_1, y_1), (c, d), (x_2, y_2)\in S\times S$ and consider

$$
\begin{aligned}
(\tilde{\mu}_1 \times \tilde{\mu}_2)((x_1, y_1).(c, d).(x_2, y_2)) &= (\tilde{\mu}_1 \times \tilde{\mu}_2)(x_1 c x_2, y_1 d y_2) \\
&= \mathrm{r}\min\left\{ \begin{array}{c} \tilde{\mu}_1(x_1 c x_2), \tilde{\mu}_2(y_1 d y_2), \\ \left[\dfrac{1-k^+}{2}, \dfrac{1-k^-}{2}\right] \end{array} \right\} \\
&\geq \mathrm{r}\min\left\{ \begin{array}{c} \tilde{\mu}_1(c), \tilde{\mu}_2(d), \\ \left[\dfrac{1-k^+}{2}, \dfrac{1-k^-}{2}\right] \end{array} \right\} \\
&= \mathrm{r}\min\left\{ \begin{array}{c} (\tilde{\mu}_1 \times \tilde{\mu}_2)(c, d), \\ \left[\dfrac{1-k^+}{2}, \dfrac{1-k^-}{2}\right] \end{array} \right\}.
\end{aligned}
$$

Hence, $\tilde{\mu}_1 \times \tilde{\mu}_2$ is an interval-valued $(\in, \in \vee \mathrm{q}_{\tilde{k}})$ -fuzzy interior ideal of $S \times S$. □

4 Conclusion

In the world of contemporary mathematics, the use of algebraic structures in computer science, control theory, and fuzzy automata theory always gain the interest of researchers. Algebraic structures, particularly ordered semigroups, play a key role in such applied branches. Further, the fuzzification of several subsystems of ordered semigroups are used in various models involving uncertainties. In this article, we introduced new types of subsystems of ordered semigroup called interval-valued $(\in, \in \vee \mathrm{q}_{\tilde{k}})$ -fuzzy interior ideal in ordered semigroup. It is investigated that in case of regular ordered semigroups the interval-valued $(\in, \in \vee \mathrm{q}_{\tilde{k}})$ -fuzzy ideals and interval-valued $(\in, \in \vee \mathrm{q}_{\tilde{k}})$ -fuzzy interior ideals coincide. It is also shown that the intersection of non-empty class of interval-valued $(\in, \in \vee \mathrm{q}_{\tilde{k}})$ -fuzzy interior ideals of an ordered semigroup is also an interval-valued $(\in, \in \vee \mathrm{q}_{\tilde{k}})$ -fuzzy interior ideal. Finally, ordinary interior ideals and interval-valued $(\in, \in \vee \mathrm{q}_{\tilde{k}})$ -fuzzy interior ideals are connected by means of level subset.

Acknowledgments: H.U.K. acknowledges Universiti Teknologi Malaysia for providing partial financial support through International Doctoral Fellowship and the University of Malakand for granting study leave.

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