

RECENT ADVANCES IN SOME FINITE GROUP of NILPOTENCY CLASS TWO

Over the years, many authors have delved into research on finite groups of nilpotency class two, while applications of groups in general have also seen increasing interest. This book contains updates and some recent research findings by the academic staff, researchers and graduate students associated with Applied Algebra and Analysis Group (AAAG) in Universiti Teknologi Malaysia on some properties of finite groups of nilpotency class two. These include the classification of two-generator p -groups of nilpotency class two and their automorphisms. Others include a graph of the groups, which are related to conjugacy classes and their properties. The final part of the book includes the commutativity degree of these groups based on the determination of the number of their conjugacy classes.



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Editors
NOR HANIZA SARMIN • YUSOF YAACOB

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Editors

NOR HANIZA SARMIN
YUSOF YAACOB





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Nor Haniza Sarmin and Yusof Yaacob

2016

Contents

<i>List of Contributors</i>	vii
<i>Preface</i>	ix
Chapter 1 Two-Generator p-Groups of Nilpotency Class Two: A Review	1
<i>Nor Haniza Sarmin and Yasamin Barakat</i>	
Chapter 2 On the Automorphism of Two-Generator p-Groups of Nilpotency Class Two	7
<i>Nor Haniza Sarmin and Yasamin Barakat</i>	
Chapter 3 Some 2-Groups of Nilpotency Class Two with Graphs Related to Conjugacy Classes	17
<i>Nor Muhainiah Mohd Ali, Sheila Ilangovan, and Kayvan Moradipour</i>	
Chapter 4 On the Commutativity Degree of Two-Generator p-Groups of Nilpotency Class Two	29
<i>Nor Muhainiah Mohd Ali and Fadila Normahia Abd Manaf</i>	
Index	51

List of Contributors

Nor Haniza Sarmin

Department of Mathematical
Sciences, Faculty of Science,
Universiti Teknologi Malaysia,
Johor Bahru, Johor, Malaysia.

Nor Muhainiah Mohd Ali

Department of Mathematical
Sciences, Faculty of Science,
Universiti Teknologi Malaysia,
Johor Bahru, Johor, Malaysia.

Kayvan Moradipour

Department of Mathematics,
Khorramabad Technical and
Vocational College, Technical
and Vocational University, Iran.

Sheila a/p Ilangovan

Foundation of Science,
Faculty of Science,
University of Nottingham Malaysia
Campus, Selangor, Malaysia.

Fadila Normahia Abd Manaf

Department of Mathematical
Sciences, Faculty of Science,
Universiti Teknologi Malaysia,
Johor Bahru, Johor, Malaysia.

Yasamin Barakat

Department of Mathematics,
Faculty of Science,
Islamic Azad University,
Ahvaz Branch, Iran.

Preface

Applied Algebra and Analysis Group (AAAG) is one of the research groups under the Frontier Material Research Alliance, Universiti Teknologi Malaysia. The research interests of AAAG are Algebra, Group Theory and Formal Language Theory and Splicing Systems.

This book chapter consists of four chapters that focus on different types of application for two-generator p -groups of nilpotency class two which are automorphism, graph and commutativity degree. New updates and some research findings by the academic staff, researchers and graduate students associated to AAAG are presented in this book chapter.

Chapter 1 presents an overview on the properties and classifications of finite two generated p -groups of nilpotency class two. Consequently, some classifications of these groups are given in this chapter. These classifications are selected based on their accuracy to be applied in the next chapters. Additionally, nonabelian groups of order p^3 are discussed as an example of two generated groups of nilpotency class two, and various classifications of them are stated.

Chapter 2 presents some recent findings on automorphism group of G , where G is a finite two generated p -group of nilpotency class two. In fact, two methods are provided which give the necessary and sufficient conditions to obtain an automorphism on G from a map defined on the generating set of G . The application of these methods is elaborated by examples. Moreover, a practical way to characterize automorphism group of G is discussed by introducing a specific normal subgroup of G and its quotient group.

Chapter 3 focuses on graph theory, more precisely on the graph related to conjugacy classes. The graph of conjugacy classes

is found for some finite two-generator two-groups of class two. Besides, the numbers of connected components, regularity, diameter and edges are found. Consequently, some graph properties such as the chromatic number and the clique number are obtained.

Chapter 4 discusses on the determination of commutativity degree of two-generator p -groups of nilpotency class two for any prime p . In 1973, the commutativity degree was found by using the fact that it is equal to the number of conjugacy class of a group divides the order of the group. In this chapter, a general formula for finding the conjugacy classes of nilpotent p -groups of nilpotency class two is presented. In addition, some concepts and basic results on the commutativity degree are given.

Nor Haniza Sarmin

Yusof Yaacob

*Department of Mathematical Sciences, Faculty of Science,
Universiti Teknologi Malaysia*

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1

Two-Generator p -Groups of Nilpotency Class Two: A Review

Nor Haniza Sarmin and Yasamin Barakat

1.1 INTRODUCTION

Let g and h be two elements in an arbitrary group G . Then, the commutator of g and h is defined to be $[g, h] = g^{-1}h^{-1}gh$. The group generated by all commutators is called the derived subgroup and is denoted by G' . Indeed, $G' = \langle [g, h] : g, h \in G \rangle$. The group G is of nilpotency class two if and only if $G' \leq Z(G)$, where $Z(G)$ is the centre of G . Moreover, a group G is referred to be a p -group, in case the order of every element in G is p^k for some nonnegative integer k . Hence, every finite subgroup of a p -group is of order p^n for a nonnegative integer n [1].

Now, let G be a finite 2-generated p -group of nilpotency class two. If G is generated by a and b , then its derived subgroup, G' is generated by $[a, b]$ [2]. Moreover, if G' is of order p^m , then $Z(G) \cap \langle a \rangle = \langle a^{p^m} \rangle$ and $Z(G) \cap \langle b \rangle = \langle b^{p^m} \rangle$ [2]. Consequently, we find $Z(G) = \langle a^{p^m}, b^{p^m}, [a, b] \rangle$. Usually, 2-generated groups of exactly class two are considered nonabelian, or in other words $[a, b] \neq 1$.

There are several classifications of finite 2-generated p -groups of nilpotency class two. Some of these classifications are stated in the next section. These classifications will be applied in the next chapters to present some recent findings concerning these kind of groups.

1.2 CLASSIFICATIONS OF 2-GENERATED p -GROUPS OF CLASS TWO

In 1993, Bacon and Kappe [3] published their findings and gave a classification for finite 2-generated p -groups of nilpotency exactly two, where p is an odd prime. This classification is stated in Theorem 1.1.

Theorem 1.1 [3] *Let G be a finite 2-generated p -group of class two, where p is an odd prime. Then G is isomorphic to exactly one group of the following three types:*

- (a) $G \cong (\langle c \rangle \times \langle a \rangle) \rtimes \langle b \rangle$, where $[a, b] = c, [a, c] = [b, c] = 1, |a| = p^\alpha, |b| = p^\beta, |c| = p^\gamma$; α, β, γ are integers, and $\alpha \geq \beta \geq \gamma$;
- (b) $G \cong \langle a \rangle \rtimes \langle b \rangle$, where $[a, b] = a^{p^{\alpha-\gamma}}, |a| = p^\alpha, |b| = p^\beta, |[a, b]| = p^\gamma$; α, β, γ are integers, $\alpha \geq 2\gamma$ and $\beta \geq \gamma$;
- (c) $G \cong (\langle c \rangle \times \langle a \rangle) \rtimes \langle b \rangle$, where $[a, b] = a^{p^{\alpha-\gamma}}c, [c, b] = a^{-p^{2\alpha-\gamma}}c^{-p^{\alpha-\gamma}}, |a| = p^\alpha, |b| = p^\beta, |c| = p^\sigma, |[a, b]| = p^\gamma$; α, β, γ are integers, $\beta \geq \gamma \geq \sigma \geq 1$ and $\alpha + \sigma \geq 2\gamma$.

In 1999, Kappe *et al.* [2] extended the previous classification to include the case $p = 2$. Their result is given in Theorem 1.2 as follows:

Theorem 1.2 [2] *Let G be a finite 2-generated 2-group of class two. Then G is isomorphic to exactly one group of the following four types:*

- (a) $G \cong (\langle c \rangle \times \langle a \rangle) \rtimes \langle b \rangle$, where $[a, b] = c, [a, c] = [b, c] = 1, |a| = 2^\alpha, |b| = 2^\beta, |c| = 2^\gamma, \alpha, \beta, \gamma \in \mathbb{N}, \alpha \geq \beta \geq \gamma$;
- (b) $G \cong \langle a \rangle \rtimes \langle b \rangle$, where $[a, b] = a^{2^{\alpha-\gamma}}, |a| = 2^\alpha, |b| = 2^\beta, |[a, b]| = 2^\gamma, \alpha, \beta, \gamma \in \mathbb{N}, \alpha \geq 2\gamma, \beta \geq \gamma, \alpha + \beta > 3$;
- (c) $G \cong (\langle c \rangle \times \langle a \rangle) \rtimes \langle b \rangle$, where $[a, b] = a^{2^{\alpha-\gamma}}c, [c, b] = a^{-2^{2\alpha-\gamma}}c^{-2^{\alpha-\gamma}}, |a| = 2^\alpha, |b| = 2^\beta, |c| = 2^\sigma, |[a, b]| = 2^\gamma, \alpha, \beta, \gamma \in \mathbb{N}, \beta \geq \gamma > \sigma, \alpha + \sigma \geq 2\gamma$;
- (d) $G \cong (\langle c \rangle \times \langle a \rangle) \rtimes \langle b \rangle$, where $[a, b] = a^2c, [c, b] = a^{-4}c^{-2}, |a| = |b| = 2^{\gamma+1}, |c| = 2^{\gamma-1}, |[a, b]| = 2^\gamma$,

$$a^{2^\gamma} = b^{2^\gamma}, \gamma \in \mathbb{N}.$$

Clearly, Theorem 1.2 is applicable for finite 2-generated 2-groups of class two and hence covers the case $p = 2$. Several years later, Magidin [4] gives the classification in Theorem 1.2 using presentation notations. This version is stated in Theorem 1.3.

Theorem 1.3[4] *Let G be a finite 2-generated 2-group of class two. Then G is isomorphic to exactly one group of the following three types:*

- (a) $G \cong \langle a, b \mid a^{2^\alpha} = b^{2^\beta} = [a, b]^{2^\gamma} = [a, b, a] = [a, b, b] = 1 \rangle$, where α, β and γ are positive integers satisfying $\alpha \geq \beta \geq \gamma$;
- (b) $G \cong \langle a, b \mid a^{2^\alpha} = b^{2^\beta} = [a, b, a] = [a, b, b] = 1, a^{2^{\alpha+\sigma-\gamma}} = [a, b]^{2^\sigma} \rangle$, with $\alpha, \beta, \gamma, \sigma$ integers satisfying $\beta \geq \gamma > \sigma \geq 0, \alpha + \sigma \geq 2\gamma$ and $\alpha + \beta + \sigma > 3$;
- (c) $G \cong \langle a, b \mid a^{2^{\gamma+1}} = b^{2^{\gamma+1}} = [a, b]^{2^\gamma} = [a, b, a] = [a, b, b] = 1, a^{2^\gamma} = b^{2^\gamma} = [a, b]^{2^{\gamma-1}} \rangle$, where $\gamma \in \mathbb{N}$.

Finally, Ahmad *et al.* [5] modified this version to include all finite 2-generated p -groups of class two. Their result, which is presented in Theorem 1.4 is the most updated version that is published recently in 2012.

Theorem 1.4[5] *Let p be a prime and $n > 2$ an integer. Every 2-generated p -group of class exactly two and order p^n , corresponds to an ordered 5-tuple of integers, $(\alpha, \beta, \gamma; \rho, \sigma)$ such that:*

- (a) $\alpha \geq \beta \geq \gamma \geq 1$;
- (b) $\alpha + \beta + \gamma = n$;
- (c) $0 \leq \rho \leq \gamma$ and $0 \leq \sigma \leq \gamma$;

where $(\alpha, \beta, \gamma; \rho, \sigma)$ corresponds to the group presented by

$$G = \langle a, b \mid [a, b]^{p^\gamma} = [a, b, a] = [a, b, b] = 1, \\ a^{p^\alpha} = [a, b]^{p^\rho}, b^{p^\beta} = [a, b]^{p^\sigma} \rangle.$$

Moreover,

4 Recent Advances in Some Finite Groups of Nilpotency Class Two

- (a) if $\alpha > \beta$, then G is isomorphic to:
- (i) $(\alpha, \beta, \gamma; \rho, \gamma)$ when $\rho \leq \sigma$;
 - (ii) $(\alpha, \beta, \gamma; \gamma, \sigma)$ when $0 \leq \sigma \leq \sigma + \alpha - \beta \leq \rho$ or $\sigma < \rho = \gamma$;
 - (iii) $(\alpha, \beta, \gamma; \rho, \sigma)$ when $0 \leq \sigma \leq \rho < \min(\gamma, \sigma + \alpha - \beta)$;
- (b) if $\alpha = \beta > \gamma$, or $\alpha = \beta = \gamma$ and $p > 2$, then G is isomorphic to $(\beta, \beta, \gamma; \min(\rho, \sigma), \gamma)$;
- (c) if $\alpha = \beta = \gamma$ and $p = 2$, then G is isomorphic to:
- (i) $(\gamma, \gamma, \gamma; \min(\rho, \sigma), \gamma)$ when $0 \leq \min(\rho, \sigma) < \gamma - 1$;
 - (ii) $(\gamma, \gamma, \gamma; \gamma - 1, \gamma - 1)$ when $\rho = \sigma = \gamma - 1$;
 - (iii) $(\gamma, \gamma, \gamma; \gamma, \gamma)$ when $\min(\rho, \sigma) \geq \gamma - 1$ and $\max(\rho, \sigma) = \gamma$.

The groups listed in 1(a)-3(c) are pairwise non-isomorphic.

According to Ahmad *et al.* [5], those groups classified in part 1(c) of Theorem 1.4 were missing in the previous classifications.

1.3 AN EXAMPLE

Let p be a prime positive integer. The class of nonabelian groups of order p^3 is an example of finite nonabelian 2-generated p -groups of nilpotency class two. It is known that $G' = Z(G)$ if G is a nonabelian group of order p^3 . Hence, G is of nilpotency class two. Moreover, G is 2-generated, since it is isomorphic to one of the following groups, in case $p = 2$:

$$D_4 = \langle a, b \mid a^4 = b^2 = 1, a^b = a^{-1} \rangle, \quad (1.1)$$

or,

$$Q_8 = \langle a, b \mid a^4 = 1, a^2 = b^2, a^b = a^{-1} \rangle; \quad (1.2)$$

where D_4 is the dihedral group of order eight, and Q_8 is the quaternion group. Moreover, in case $p > 2$, the group G is isomorphic to:

$$\langle a, b \mid a^p = 1 = b^p, [a, b]^a = [a, b] = [a, b]^b \rangle, \quad (1.3)$$

when the exponent of G is p , and it is isomorphic to:

$$\langle a, b \mid a^{p^2} = 1 = b^p, a^b = a^{p+1} \rangle, \quad (1.4)$$

if the exponent of G is p^2 . Recall that the exponent of a group H is the least positive integer m such that $h^m = 1$ for any $h \in H$. Clearly, the exponent of every p -group is a power of p .

A presentation of G in the form given in Theorem 1.4 was provided in [6], where G is a nonabelian group of order p^3 . For instance, it was shown that the groups given in (1.3) and (1.4) are corresponded to $(1, 1, 1; 1, 1)$ and $(1, 1, 1; 0, 1)$, respectively. Additionally, applying Theorem 1.1 implies that groups of the form (1.3) are isomorphic to $(\mathbb{Z}_p \times \mathbb{Z}_p) \rtimes \mathbb{Z}_p$, and groups of the form (1.4) are isomorphic to $\mathbb{Z}_{p^2} \rtimes \mathbb{Z}_p$.

1.4 CONCLUSION

In this chapter some properties of finite 2-generated p -groups of nilpotency class two have been provided. For instance, the structure of the derived subgroup and also the centre. Moreover, some classifications for this kind of groups are presented, which are applied in the next chapters.

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6 *Recent Advances in Some Finite Groups of Nilpotency Class Two*

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2

On the Automorphism of Two-Generator p -Groups of Nilpotency Class Two

Nor Haniza Sarmin and Yasamin Barakat

2.1 INTRODUCTION

An automorphism on a group G is a homomorphism of G , which is one to one and onto. Recall that a homomorphism of G is a function f from G into itself that preserves the operation on G , that is $f(gh) = f(g)f(h)$ for every $g, h \in G$. The set of all automorphisms on G together with composition forms a group that is called the automorphism group of G , and denoted by $Aut(G)$.

Now, let G be a finite nonabelian 2-generated p -group of class two. In Chapter 1, some classifications of G have been stated. In this chapter, the latest version of the classifications given in Theorem 1.4 is applied to give two techniques to find the automorphisms on G . Some examples are provided to show how these methods work. Additionally, some properties of $Aut(G)$ are provided to help characterize it.

2.2 METHODS TO RECOGNIZE AUTOMORPHISMS ON FINITE 2-GENERATED p -GROUPS OF CLASS TWO

The group of exactly class two is usually considered nonabelian, more precisely, the derived subgroup is assumed nontrivial. If G is a

finite nonabelian 2-generated p -group of class two, then according to Theorem 1.4, G is presented as follows:

$$\langle a, b \mid [a, b]^{p^\gamma} = [a, b, a] = [a, b, b] = 1, \\ a^{p^\alpha} = [a, b]^{p^\rho}, b^{p^\beta} = [a, b]^{p^\sigma} \rangle. \quad (2.1)$$

This presentation can be used to find the automorphisms on G . Recall that an automorphism is onto, so its image should span G . This fact is used to give the first technique for finding the automorphisms on G . This technique is stated in Theorem 2.1. To elaborate our results, the following proposition is applied which shows the multiplication of commutators in nilpotent groups of class two.

Proposition 2.1 [1] *Let H be a group of nilpotency class two. For any $x, y, z \in H$ and $n \in \mathbb{Z}$, the following equations hold:*

- (a) $[x, yz] = [x, y][x, z]$;
- (b) $[xy, z] = [x, z][y, z]$;
- (c) $[x^n, y] = [x, y]^n = [x, y^n]$;
- (d) $(xy)^n = x^n y^n [y, x]^{(n(n-1))/2}$.

Theorem 2.1 [2] *First Method to Find Automorphisms on G*

Let G be a finite 2-generated p -group of class two and f be a map of G to itself. Then f extends to an automorphism on G if and only if it satisfies in the following conditions:

- (a) $G = \langle f(a), f(b) \rangle$;
- (b) $[f(a), f(b)]^{p^\gamma} = [f(a), f(b), f(a)] \\ = [f(a), f(b), f(b)] = 1$;
- (c) $[f(a)]^{p^\alpha} = [f(a), f(b)]^{p^\rho}$;
- (d) $[f(b)]^{p^\beta} = [f(a), f(b)]^{p^\sigma}$.

Sketch of Proof. If f is an automorphism on G , then it is onto and its image group is G . In other words, $f(a)$ and $f(b)$ are generators of G that satisfy in (2.1), considering the fact that (2.1) is a presentation of G . In converse, let f be a mapping on G into

itself that satisfies in conditions (a) - (d). Since these conditions are written based on properties of G that are given in (2.1), it can be concluded that f extends to an automorphism on G .

The next example gives an application of the first method.

Example 2.1 Let G be a nonabelian group of order p^3 . Then the following map

$$f : \begin{cases} a \mapsto a^{p+1}, \\ b \mapsto b \end{cases} \quad (2.2)$$

extends to an automorphism on G .

Solution According to the discussion given in Section 1.3, if $p = 2$, then $G \cong D_4$ or Q_8 . However for both groups, $[a, b]^2 = 1$ and $a^2 = [a, b]$. Hence,

$$a^4 = 1, \quad [a, b]^{-1} = [a, b], \quad f(a) = a^{p+1} = a^3 = a^{-1}.$$

Thus,

$$\langle [f(a)]^{-1}, f(b) \rangle = \langle a, b \rangle = G.$$

Moreover,

$$[f(a), f(b)] = [a^{-1}, b] = [a, b]^{-1} = [a, b] \in G' = Z(G).$$

Therefore,

$$\begin{aligned} |[f(a), f(b)]| &= |[a, b]|, \\ [f(a), f(b), f(a)] &= [f(a), f(b), f(b)] = 1. \end{aligned}$$

Finally,

$$\begin{aligned} |[f(b)]| &= |b|, \\ [f(a)]^2 &= (a^3)^2 = a^4 a^2 = a^2 = [a, b] = [f(a), f(b)]. \end{aligned}$$

Hence, f satisfies conditions (a) – (d) of Theorem 2.1.

10 Recent Advances in Some Finite Groups of Nilpotency Class Two

Now, let $p > 2$. If G is of the form (1.3), then

$$a^p = b^p = [a, b]^p = 1.$$

Hence, $f(a) = a^{p+1} = a$ or f is the identity map, which obviously extends to identity automorphism on G . On the other hand, if G is presented by (1.4), then $(\alpha, \beta, \gamma; \rho, \sigma) = (1, 1, 1; 0, 1)$, $a^{p^2} = b^p = [a, b]^p = 1$ and $a^p = [a, b]$. Hence, we find

$$a = a^{(p^3+1)} = [a^{p+1}]^{(p^2-p+1)} = [f(a)]^{(p^2-p+1)}.$$

This leads to $G = \langle [f(a)]^{(p^2-p+1)}, f(b) \rangle$. In addition, we have

$$[f(a), f(b)] = [a^{p+1}, b] = [a, b]^{p+1} = [a, b] \in G' \leq Z(G).$$

This implies that f satisfies all relations that are given in condition (b) of Theorem 2.1. The next statement completes the solution:

$$\begin{aligned} [f(a)]^{p^\alpha} &= [f(a)]^p = [a^{p+1}]^p \\ &= a^{p^2+p} = a^p = [a, b] \\ &= [f(a), f(b)] = [f(a), f(b)]^{p^\rho}. \end{aligned}$$

Next, the Frattini subgroup and some of its properties are applied to find the second method, as provided in the following:

Definition 2.1 [3] Let H be an arbitrary group. A non-generator element of H is an element that could be removed from any generating set. The set of all non-generators of H forms a normal subgroup that is called Frattini subgroup, and denoted by $\Phi(H)$. Indeed,

$$H = \langle \Phi(H), x_1, x_2, \dots, x_n \rangle$$

if and only if

$$H = \langle x_1, x_2, \dots, x_n \rangle.$$

Proposition 2.2 [3] Let H be a p -group. Then

On the Automorphism of Two-Generator p -Groups of Nilpotency Class Two 11

- (a) $H' \leq \Phi(H)$;
- (b) $h^p \in \Phi(H)$ for all $h \in H$.

In our study, G is considered a finite 2-generated p -group of nilpotency class two. Let $\{a, b\}$ spans G . Then by Proposition 2.2, $a^p, b^p \in \Phi(G)$. Hence, $|a\Phi(G)| = |b\Phi(G)| = p$, and so then

$$G/\Phi(G) = \langle a\Phi(G), b\Phi(G) \rangle \cong \mathbb{Z}_p \times \mathbb{Z}_p$$

is of order p^2 . The automorphism group of $\mathbb{Z}_p \times \mathbb{Z}_p$ is isomorphic to $GL(2, p)$ the general linear group of degree two. Moreover, $|GL(2, p)| = (p^2 - 1)(p^2 - p)$ [1]. It is known that $GL(2, p)$ consists of all nonsingular 2×2 matrices. i.e.

$$GL(2, p) \cong \left\{ \begin{pmatrix} k & l \\ m & n \end{pmatrix} : k, l, m, n \in \mathbb{Z}_p, kn - lm \not\equiv 0 \pmod{p} \right\}.$$

This fact and the following proposition are used in the proof of method 2, which is given in [4].

Proposition 2.3[4] *Let $G = \langle a, b \rangle$ be a finite 2-generated group of class two. Then every element $g \in G$ is of the form $g = a^{x_1}b^{x_2}[a, b]^{x_3}$ where x_i 's are positive integers such that $x_1 \leq |a|$, $x_2 \leq |b|$ and $x_3 < |[a, b]|$. Moreover, we have $ba^k = a^k b[a, b]^{-k}$, for any integer k .*

Theorem 2.2[4] *Second Method to Find Automorphisms on G*
Let G be a 2-generated group of class two and order p^n that corresponded to $(\alpha, \beta, \gamma; \rho, \sigma)$. Let x_i and y_i be nonnegative integers for $i = 1, 2, 3$ such that $0 \leq x_1, y_1 < |a|$, $0 \leq x_2, y_2 < |b|$ and $0 \leq x_3, y_3 < |[a, b]|$. Then the map defined as:

$$f : \begin{cases} a \mapsto a^{x_1}b^{x_2}[a, b]^{x_3}, \\ b \mapsto a^{y_1}b^{y_2}[a, b]^{y_3} \end{cases}$$

can be extended to a unique automorphism on G if and only if the following conditions hold:

- (a) $d(f) := \det \begin{pmatrix} x_1 & x_2 \\ y_1 & y_2 \end{pmatrix} \not\equiv 0 \pmod{p}$;
- (b) $[f(a), f(b)]^{p^\gamma} = [f(a), f(b), f(a)]$
 $= [f(a), f(b), f(b)] = 1$;
- (c) $[f(a)]^{p^\alpha} = [f(a), f(b)]^{p^\rho}$;
- (d) $[f(b)]^{p^\beta} = [f(a), f(b)]^{p^\sigma}$.

An example of the usage of the second method can be found in [5], where all automorphisms on D_4 were found by applying Method 2. As another example of the application of Method 2, one can consider the map f given in (2.2). Clearly, conditions (b) - (d) in both methods are similar. Thus, it is enough to show that the map f given in (2.2) satisfies in condition (a) of Method 2. However,

$$d(f) = \det \begin{pmatrix} p+1 & 0 \\ 0 & 1 \end{pmatrix} = p+1 \equiv 1 \not\equiv 0 \pmod{p}.$$

The second method enables us to find an upper bound for the order of $\text{Aut}(G)$ in our case. To achieve this goal, the following proposition that reveals the orders of a and b in G is used.

Proposition 2.4 [4] *Let $G = \langle a, b \rangle$ be a nilpotent group of class two and order p^n that is corresponded to $(\alpha, \beta, \gamma; \rho, \sigma)$. Then*

$$|a| = p^{\alpha+\gamma-\rho}, \quad |b| = p^{\beta+\gamma-\sigma}, \quad |[a, b]| = p^\gamma.$$

Lemma 2.1 *Let $G = \langle a, b \rangle$ be a nilpotent group of class two and order p^n that is corresponded to $(\alpha, \beta, \gamma; \rho, \sigma)$. Then*

$$|\text{Aut}(G)| \leq p^{2[(n+(\gamma-\rho)+(\gamma-\sigma))]}.$$

Proof According to Theorem 2.2, we have

$$|\text{Aut}(G)| \leq |a|^2 |b|^2 |[a, b]|^2.$$

Proposition 2.4 and the fact that $\alpha + \beta + \gamma = n$, which is excerpted from Theorem 1.4, imply that:

$$\begin{aligned} |\text{Aut}(G)| &\leq p^{2(\alpha+\gamma-\rho)} p^{2(\beta+\gamma-\sigma)} p^{2\gamma} \\ &= p^{[2(\alpha+\beta+\gamma)+2(2\gamma-\rho-\sigma)]} \\ &= p^{2[(n+(\gamma-\rho)+(\gamma-\sigma))]} \end{aligned}$$

In the next section, an idea on the characterization of $\text{Aut}(G)$, where G is a finite 2-generated p -group of nilpotency class two is discussed.

2.3 CHARACTERIZATION OF AUTOMORPHISMS ON FINITE TWO-GENERATED p -GROUPS OF CLASS TWO

To proceed our discussion, another concept is needed. In [4], $A_\Phi(G)$ is introduced as the following:

Definition 2.2[4] Let $G = \langle a, b \rangle$ be a nilpotent p -group of class two. Then $A_\Phi(G)$ is defined to be the set consisting of all those elements f in $\text{Aut}(G)$ that induce the identity automorphism on $G/\Phi(G)$. i.e. $\bar{f} \in A_\Phi(G)$ if and only if for each $g\Phi(G) \in G/\Phi(G)$, we have:

$$\bar{f}(g\Phi(G)) = f(g)\Phi(G) = \Phi(G) = \mathbf{i}_{G/\Phi(G)}.$$

Proposition 2.5[4] Let G be a finite 2-generated p -group of nilpotency class two. Then $A_\Phi(G)$ is a normal subgroup of $\text{Aut}(G)$.

The following lemma explains the reason of our interest in $A_\Phi(G)$. In fact, it shows that how $A_\Phi(G)$ can be used to study and characterize $\text{Aut}(G)$.

Theorem 2.3 Let G be a finite 2-generated p -group of nilpotency class two. Then $\text{Aut}(G)/A_\Phi(G)$ is isomorphic to a subgroup of $GL(2, p)$.

Proof Consider the following map:

$$\begin{cases} \tau : \text{Aut}(G) \rightarrow \text{Aut}(G/\Phi(G)), \\ \tau(f) = \bar{f}, \end{cases}$$

where, for every $\bar{f} \in \text{Aut}(G/\Phi(G))$ we have $\bar{f}(g\Phi(G)) = f(g)\Phi(G)$. We need to prove that τ is a homomorphism and $A_\Phi(G)$ is its kernel. In other words, $A_\Phi(G) = \tau^{-1}(\{\mathbf{i}_{G/\Phi(G)}\})$.

To show that τ is well-defined, consider $f_1 = f_2$ in $\text{Aut}(G)$. Then, $f_1(g) = f_2(g)$ for every $g \in G$. Hence, $f_1(g)\Phi(G) = f_2(g)\Phi(G)$, or $\tau(f_1) = \bar{f}_1 = \bar{f}_2 = \tau(f_2)$, which prove that τ is well-defined. Furthermore, τ is a homomorphism since the following relations hold:

$$\begin{aligned} \tau(f_1 f_2)(g\Phi(G)) &= \overline{f_1 f_2}(g\Phi(G)) \\ &= [f_1 f_2(g)]\Phi(G) \\ &= [f_1(f_2(g))]\Phi(G) \\ &= \bar{f}_1(f_2(g)\Phi(G)) \\ &= \bar{f}_1(\bar{f}_2(g\Phi(G))) \\ &= [\bar{f}_1 \bar{f}_2](g\Phi(G)) \\ &= \tau(f_1)\tau(f_2)(g\Phi(G)). \end{aligned}$$

Therefore, according to the first theorem of isomorphism $\text{Aut}(G)/\text{Ker}(\tau) \cong \text{Im}(\tau)$. Recall that

$$\text{Im}(\tau) \leq \text{Aut}(G/\Phi(G)) \cong GL(2, p).$$

In other words, $\text{Im}(\tau)$ is isomorphic to a subgroup of $GL(2, p)$. Hence, $\text{Im}(\tau)$ is almost known, and it remains to characterize $\text{Ker}(\tau)$. The following relations show that $\text{Ker}(\tau) = A_\Phi(G)$.

$$\begin{aligned} \text{Ker}(\tau) &= \{f \in \text{Aut}(G) : \tau(f) = \mathbf{i}_{G/\Phi(G)}\} \\ &= \{f \in \text{Aut}(G) : \bar{f} = \mathbf{i}_{G/\Phi(G)}\} \\ &= A_\Phi(G). \end{aligned}$$

This shows the importance of $A_\Phi(G)$. □

2.4 CONCLUSION

Let G be a 2-generated group of nilpotency class two and order p^n . In this chapter, firstly two methods with examples have been stated

On the Automorphism of Two-Generator p -Groups of Nilpotency Class Two 15

to recognize the automorphisms on G among the maps that can be defined from G to itself. Next, a technique that can be applied to characterize and study $\text{Aut}(G)$ is provided. This technique is based on introducing a specific normal subgroup of $\text{Aut}(G)$ called $A_{\Phi}(G)$ and its quotient group.

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3

Some 2-Groups of Nilpotency Class Two with Graphs Related to Conjugacy Classes

Nor Muhainiah Mohd Ali, Sheila Ilangovan,
and Kayvan Moradipour

3.1 INTRODUCTION

The graph related to conjugacy classes, Γ_G , where G denotes a finite group or an infinite FC-group has always sparked many interests among researchers in group theory over the past decades. These include the works in [1–6]. In 1990, Bertram *et al.* [3] attached a simple graph Γ_G to a group G in the following way. The vertices are the non-central conjugacy class sizes of G for which two non-central class sizes are adjacent if and only if they are not coprime. More precisely, the vertices A and B are joined by an edge if $|A|$ and $|B|$ have a common prime divisor. For ease of understanding, we refer the reader to an example.

Example 3.1 Let G be the Dihedral group of order 8, D_4 . The elements of D_4 are $\{e, a, a^2, a^3, b, ab, a^2b, a^3b\}$. There are five conjugacy classes in D_4 , where C_1 to C_5 represent the conjugacy classes. The elements of the respective conjugacy classes are given as follows:

- (a) $C_1 : 1$;
- (b) $C_2 : a^2$;

- (c) $C_3 : a, a^3;$
- (d) $C_4 : b, a^2b;$
- (e) $C_5 : ab, a^3b.$

Since the vertices are the non-central conjugacy class sizes of G , there are only three vertices in D_4 and the order of each vertices are 2, respectively.

In this section we revoke some definitions and known facts that are related to this chapter. The vertex set and the edge set of Γ_G are denoted by $V(\Gamma_G)$ and $E(\Gamma_G)$, respectively. Most of the notations are standard and can be referred to [7, 8]. Furthermore, some notations and definitions will be introduced as needed.

Let $n(G)$ denotes the number of connected components of Γ_G . In 1990, Bertram *et al.* [3] proved that if G is a finite or FC-group, then $n(G) \leq 2$.

Suppose that Γ_G is connected and if x and y are vertices in Γ_G , then $d(x, y)$ denotes the length of the shortest path between x and y . The largest distance between all pairs of the vertices of Γ_G is called the *diameter* of Γ_G , and is denoted by $d(G)$. The diameter is the number of edges between the furthest vertices of Γ_G . Consequently, it is known as *longest shortest path*. In 1993, Chillag *et al.* [5] improved the lower bound for the diameter of Γ_G in [3]. They proved that the diameter of the graph is at most 3, and it is the best possible bound. In the case of 2-generator 2-group of class two, we have the following result.

It is worthwhile to notice the recent research by Bianchi *et al.* in [2]. We recall that a graph is called *k-regular* if every vertex is adjacent to exactly k vertices, where k is a positive integer. They proved that Γ_G is a 2-regular graph if and only if it is a complete graph with three vertices, and Γ_G is a 3-regular graph if and only if it is a complete graph with four vertices where G is a finite group. The following question was also posed by Bianchi *et al.* in [2].

Question: *Is it true that, given a positive integer k , the graph Γ_G is k -regular if and only if it is a complete graph with $(k + 1)$ vertices?*

Indeed, they proved that it certainly holds for $1 \leq k \leq 4$. Inspired by this, we study the regularity of a graph related to conjugacy classes of 2-generator 2-groups of class two.

The *chromatic number* of a graph Γ_G , denoted by $\chi(\Gamma_G)$, is the smallest number of colors, which Γ_G needed to color the vertices of Γ_G so that no two adjacent vertices share the same color. The maximum size of a subset A of the vertices of Γ_G is called the *clique number* of Γ_G and is denoted by $\omega(\Gamma_G)$.

This chapter is structured as follows: The first section is the Introduction. In the second section, the notations have been fixed and a few propositions are stated to enact some preliminary results which are useful for the proofs of our results. The third section is devoted to the proofs of the above-mentioned results and some corollaries are also presented.

3.2 PRELIMINARIES

Note that G is abelian if and only if Γ_G is without vertices. Bertram *et al.* [3] showed that the symmetric group of order 6, S_3 is the only group that has vertices but no edges. Likewise, they stated without proof that non-abelian p -groups are clearly complete. For the non-abelian simple group, the following proposition is given which uses the classification of finite simple groups.

Proposition 3.1 [3, 9] *Let G be a non-abelian finite simple group. Then Γ_G is a complete graph, i.e., there is an edge between any two vertices in Γ_G .*

In this chapter, we focus on the completeness of Γ_G , the number of connected components, $n(G)$, diameter, $d(G)$, the number of edges, $E(\Gamma_G)$, and the regularity of the graph for some 2-groups concerning all types. It may be worth recalling several results concerning the number of the conjugacy classes and conjugacy class sizes which have a counterpart in the context of this research. Consequently, Kappe's classification [10] is used by Ilangovan and

Sarmin to formulate the number of conjugacy classes and conjugacy class sizes for 2-generator 2-groups of class two in [11] and [12], respectively. Kappe's classification is stated in the following theorem.

Theorem 3.1 [10] *Let G be a finite non-abelian 2-generator 2-group of nilpotency class two, then G is isomorphic to exactly one group of the following types:*

- (a) $G \cong (\langle c \rangle \times \langle a \rangle) \rtimes \langle b \rangle$, where $[a, b] = c, [a, c] = [b, c] = 1, |a| = 2^\alpha, |b| = 2^\beta, |c| = 2^\gamma, \alpha, \beta, \gamma \in \mathbb{N}, \alpha \geq \beta \geq \gamma$;
- (b) $G \cong \langle a \rangle \rtimes \langle b \rangle$, where $[a, b] = a^{2^{\alpha-\gamma}}, |a| = 2^\alpha, |b| = 2^\beta, |[a, b]| = 2^\gamma, \alpha, \beta, \gamma \in \mathbb{N}, \alpha \geq 2\gamma, \beta \geq \gamma, \alpha + \beta > 3$;
- (c) $G \cong (\langle c \rangle \times \langle a \rangle) \rtimes \langle b \rangle$, where $[a, b] = a^{2^{\alpha-\gamma}}c, [c, b] = a^{-2(\alpha-\gamma)}c^{-2^{\alpha-\gamma}}, |a| = 2^\alpha, |b| = 2^\beta, |c| = 2^\sigma, |[a, b]| = 2^\gamma, \alpha, \beta, \gamma, \sigma \in \mathbb{N}, \beta \geq \gamma > \sigma, \alpha + \sigma \geq 2\gamma$;
- (d) $G \cong (\langle c \rangle \times \langle a \rangle) \langle b \rangle$, where $|a| = |b| = 2^{\gamma+1}, |[a, b]| = 2^\gamma, |c| = 2^{\gamma-1}, [a, b] = a^2c, [c, b] = a^{-4}c^{-2}, a^{2^\gamma} = b^{2^\gamma}, \gamma \in \mathbb{N}$.

The classification is then modified and classified into three type. From now on each type in the classification is called as Type 1, 2 and 3. To study the number of edges of Γ_G , the following results from [11, 12] are needed.

Proposition 3.2 [12] *Let G be a 2-generator 2-group of class two. If $|G'| = 2^\gamma$ and $\gamma \in \mathbb{N}$, then the conjugacy classes sizes, $Sz(G) = \{2^\rho \mid 0 \leq \rho \leq \gamma\}$.*

Proposition 3.3 [11] *Let G be a 2-generator 2-group of class two of Type 1 denoted by $G = G_1(\alpha, \beta, \gamma)$. Then $|G| = 2^{\alpha+\beta+\gamma}$, and $|Z(G)| = 2^{\alpha+\beta-\gamma}$.*

Proposition 3.4 [11] *Let G be a 2-generator 2-group of class two of Type 2 denoted by $G = G_2(\alpha, \beta, \gamma, \sigma)$. Then $|G| = 2^{\alpha+\beta+\sigma}$, and $|Z(G)| = 2^{\alpha+\beta-2\gamma+\sigma}$.*

Proposition 3.5 [11] *Let G be a 2-generator 2-group of class two of Type 3 denoted by $G = G_3(\gamma)$. Then $|G| = 2^{3\gamma}$, and $|Z(G)| = 2^\gamma$.*

Proposition 3.6 [3] *If G is a 2-generator 2-group of class two, then Γ_G is a complete graph.*

The following results from [2] are needed.

Theorem 3.2 [2] *Let G be a finite group. The graph Γ_G is 2-regular if and only if it is a triangle.*

Consequently our problem can be modified as follows. Using this results, Γ_G will be shown as a k -regular graph with $(k + 1)$ vertices. First, the following proposition is proven.

Proposition 3.7 *If G is a 2-generator 2-group of class two, then Γ_G contains at least one triangle.*

Proof Let G be a 2-generator 2-group of class two. In view of Theorem 1.1, the smallest group of G is from Type 1 and Type 2 which is of order 8, namely the Dihedral group and Quaternion group of order 8. A straightforward calculation shows that $|Z(G)| = 2$. Certainly, the graph Γ_G has at least 3 vertices since the number of conjugacy classes is 5. Using Proposition 3.6, Γ_G is a complete graph. Thus, Γ_G contains at least one triangle since the smallest group G has 3 vertices. The proof is now complete. \square

The following theorem from the book Bondy gives the diameter of Γ_G in terms of edge set.

Theorem 3.3 (Theorem 1.1, [14])

$$d(\Gamma_G) = \sum_{i=1}^{|V(G)|} d(g_i) = 2|E(\Gamma_G)|.$$

In [13], Ahmad defined the base group as a group which is not an extension such that the image is of the same type. Therefore, this viewpoint is used in [11] to obtain the conjugacy classes for the base group for 2-generator 2-groups of class two concerning all types. In addition, the authors in [11] also formulated the conjugacy classes for the non-base group using the central extension method. The following theorem provides such an analysis and will be used in proving Theorem 3.1 in the next section.

Theorem 3.4[11] *Let G be a 2-generator 2-group of class two of Type 1 denoted by $G = G_1(\alpha, \beta, \gamma)$. If $G = G_1(\alpha, \beta, \gamma = 1)$ is the base group, then $cl_G = 2^{\alpha+\beta} + |Z(G)| - 2^{\alpha+\beta-2\gamma}$. Otherwise, if $G = G_1(\alpha, \beta, \gamma)$, then G can be reduced to $H = (\alpha, \beta, \gamma - 1)$ down to the base group such that $cl_G = cl_H + \frac{|Z(G)|}{2}$.*

Theorem 3.5[11] *Let G be a 2-generator 2-group of class two of Type 2 denoted by $G = G_2(\alpha, \beta, \gamma, \sigma)$. If $G = G_2(\alpha, \beta, \gamma, \sigma = 0)$ is a base group, then $cl_G = 2^{\alpha+\beta-\gamma} + 2^{\alpha+\beta-2\gamma} - 2^{\alpha+\beta-3\gamma}$. Otherwise, if $G = G_2(\alpha, \beta, \gamma, \sigma > 0)$, then G can be reduced down to the base group $K = (\alpha - 1, \beta, \gamma - 1, \sigma - 1)$ such that $cl_G = 2cl_K + \frac{|Z(G)|}{2}$.*

Theorem 3.6[11] *Let G be a 2-generator 2-group of class two of Type 3 denoted by $G = G_3(\gamma)$. If $G = G_3(\gamma = 1)$ is the base group, then $cl_G = 2^{2\gamma} + |Z(G)| - 1$. Otherwise, if $G = G_3(\gamma)$, then G can be reduced to $H = (\gamma - 1)$ down to the base group such that $cl_G = 4cl_H + \frac{|Z(G)|}{2}$.*

3.3 ON THE GRAPH Γ_G FOR SOME FINITE 2-GROUPS

In this section, the results mentioned in the Introduction are stated and proven.

Theorem 3.7 *If G is a 2-generator 2-group of class 2, then $n(G) = 1$.*

Proof Using the definition of Γ_G , the conjugacy classes are of coprime cardinalities greater than 1, and each and every vertex has at least a path connecting to the other vertex. Clearly, the complement graph of a complete graph is an empty graph. Hence, there is only one connected component, $n(G) = 1$. \square

Theorem 3.8 *If G is a 2-generator 2-group of class 2, then $d(G) = 1$.*

Proof Let G be a 2-generator 2-group of class two. Applying Proposition 3.6, Γ_G is a complete graph. Consequently, every pair of vertices is adjacent. Since the diameter of a graph is the number of edges between the furthest nodes, $d(G) = 1$, as claimed. \square

Theorem 3.9 *If G is a 2-generator 2-group of class two, then Γ_G is k -regular with $(k + 1)$ vertices, where $k \geq 2$.*

Proof Using definition of k -regular graph, if all the vertices of Γ_G have the same degree k , then Γ_G is k -regular. In view of Proposition 3.6 and Proposition 3.7, we can see that Γ_G is a complete graph which has at least three vertices. Certainly if it is a complete graph with exactly three vertices, then it is 2-regular. Assume that there are $k + 1$ vertices. Since Γ_G is a complete graph, all the $k + 1$ vertices are pairwise adjacent. This yields that Γ_G is a k -regular graph. Thus, Γ_G is k -regular with $(k + 1)$ vertices, where $k \geq 2$. \square

The following theorem yields the number of vertices and edges of graph Γ_G in terms of cl_G , conjugacy classes and $|Z(G)|$, the order of the center of G .

Theorem 3.10 *If G is a 2-generator 2-group of class 2, then $|E(\Gamma_G)| = \frac{1}{2}(cl_G - |Z(G)|)(cl_G - |Z(G)| - 1)$.*

Proof If G is a 2-generator 2-group of nilpotency class two, we proved that the associated graph Γ_G is a complete graph. Hence,

$d(g) = |V(\Gamma_G)| - 1$ for every $g \in V(\Gamma_G)$. We recall from Theorem 3.3. This yields the following equality:

$$\sum_{i=1}^{|V(G)|} (|V(\Gamma_G)| - 1) = 2|E(\Gamma_G)|.$$

It follows that

$$|E(\Gamma_G)| = \frac{1}{2} |V(\Gamma_G)| (|V(\Gamma_G)| - 1). \quad (3.1)$$

The vertices of graph Γ_G are the non-central conjugacy class sizes of G . Therefore,

$$|V(\Gamma_G)| = cl_G - |Z(G)|. \quad (3.2)$$

Replacing (3.2) in (3.1) yields:

$$|E(\Gamma_G)| = \frac{1}{2} (cl_G - |Z(G)|) (cl_G - |Z(G)| - 1). \quad \square$$

As an immediate consequence of Theorem 3.10, we have the following corollaries.

Corollary 3.1 *Let G be a 2-generator 2-group of class two of Type 1 denoted by $G = G_1(\alpha, \beta, \gamma)$. If $G = G_1(\alpha, \beta, \gamma = 1)$ is the base group, then*

$$|E(\Gamma_G)| = \left(2^{\alpha+\beta-1} - 2^{\alpha+\beta-2\gamma-1}\right) \left(2^{\alpha+\beta} - 2^{\alpha+\beta-2\gamma} - 1\right).$$

Proof By putting Theorems 3.4, 3.10 and Equation (3.2) together, we have

$$\begin{aligned} |V(\Gamma_G)| &= 2^{\alpha+\beta} + |Z(G)| - 2^{\alpha+\beta-2\gamma} - |Z(G)| \\ &= 2^{\alpha+\beta} - 2^{\alpha+\beta-2\gamma}. \end{aligned}$$

Thus, $|E(\Gamma_G)| = \left(2^{\alpha+\beta-1} - 2^{\alpha+\beta-2\gamma-1}\right) \left(2^{\alpha+\beta} - 2^{\alpha+\beta-2\gamma} - 1\right).$ \square

Corollary 3.2 *Let G be a 2-generator 2-group of class two of Type 2 denoted by $G = G_2(\alpha, \beta, \gamma, \sigma)$. If $G = G_2(\alpha, \beta, \gamma, \sigma = 1)$ is the base group, then*

$$|E(\Gamma_G)| = \frac{3|G|}{2^{2\gamma+2}}(2^\gamma - 1) \left(\frac{3|G|}{2^{2\gamma+1}}(2^\gamma - 1) - 1 \right).$$

Proof In this case, we have $|V(\Gamma_G)| = 2^{\alpha+\beta-\gamma+1} + 2^{\alpha+\beta-\gamma} - 2^{\alpha+\beta-2\gamma} - 2^{\alpha+\beta-2\gamma+1}$. In view of Proposition 3.4, $|G| = 2^{\alpha+\beta+1}$. Applying Theorems 3.5, 3.10 and Equation (3.2) together, we have

$$|V(\Gamma_G)| = 2^{\alpha+\beta+1} \left(\frac{1}{2^\gamma} + \frac{1}{2^{\gamma+1}} - \frac{1}{2^{2\gamma+1}} - \frac{1}{2^{2\gamma}} \right).$$

From which it follows that

$$|V(\Gamma_G)| = \frac{3|G|}{2^{2\gamma+1}}(2^\gamma - 1).$$

Hence, the number of edges,

$$|E(\Gamma_G)| = \frac{3|G|}{2^{2\gamma+2}}(2^\gamma - 1) \left(\frac{3|G|}{2^{2\gamma+1}}(2^\gamma - 1) - 1 \right). \quad \square$$

We turn now to a specific case of Type 2. If $\sigma = 0$, this type is a metacyclic 2-group of class two. More recently, the second and third authors computed the exact number of conjugacy classes and the commutativity degree of G where G is a finite non-abelian metacyclic p -group where p is any prime in [15].

Corollary 3.3 *Let $G \cong \langle a, b \mid a^{2^\alpha} = b^{2^\beta} = e, [a, b] = a^{2^{\alpha-\gamma}} \rangle$, $\alpha, \beta, \gamma \in \mathbb{N}$, $\beta \geq \gamma > 0$, $\alpha \geq 2\gamma$, $\alpha + \beta > 3$ which is a metacyclic 2-group of class two. Then*

$$|E(\Gamma_G)| = \frac{9}{2}|G|^2 V_\gamma \left(V_\gamma - \frac{1}{3|G|} \right),$$

where $V_\gamma = \frac{2^\gamma - 1}{2^{2\gamma+1}}$.

Proof Referring to Theorem 4.1 in [15], the exact number of conjugacy classes of G is $cl_G = |G| \left(\frac{1}{2^\gamma} + \frac{1}{2^{\gamma+1}} - \frac{1}{2^{2\gamma+1}} \right) = |G| \left(\frac{2^{\gamma+1} + 2^\gamma - 1}{2^{2\gamma+1}} \right)$. Hence, $|V(\Gamma_G)| = |G| \left(\frac{3 \cdot 2^\gamma - 1}{2^{2\gamma+1}} \right) - \frac{|G|}{2^{2\gamma}}$ since $|Z(G)| = 2^{\alpha+\beta-2\gamma}$. Therefore, we have

$$|V(\Gamma_G)| = |G| \left(\frac{3 \cdot 2^\gamma - 1}{2^{2\gamma+1}} - \frac{1}{2^{2\gamma}} \right).$$

A straightforward calculation shows that $|V(\Gamma_G)| = 3|G| \left(\frac{2^\gamma - 1}{2^{2\gamma+1}} \right)$. We shorten $\frac{2^\gamma - 1}{2^{2\gamma+1}}$ to V_γ for convenience. Then we have $|E(\Gamma_G)| = \frac{3}{2}|G| \cdot V_\gamma (3|G|V_\gamma - 1)$. Thus the number of edges of Γ_G is $|E(\Gamma_G)| = \frac{9}{2}|G|^2 V_\gamma \left(V_\gamma - \frac{1}{3|G|} \right)$. \square

The following corollary gives a formula for the number of edges of Γ_G in terms of γ .

Corollary 3.4 *Let G be a 2-generator 2-group of class two of Type 3 denoted by $G = G_3(\gamma)$. If $G = G_3(1)$ is the base group, then*

$$|E(\Gamma_G)| = (2^{2\gamma} - 1)(2^{2\gamma-1} - 1).$$

Proof By Theorem 3.6, we are aware that the number of conjugacy classes of a group G , $cl_G = 2^{2\gamma} + |Z(G)| - 1$. Using similar counting principle as before, there are $|V(\Gamma_G)| = 2^{2\gamma} + |Z(G)| - 1 - |Z(G)|$. Clearly, we have $|V(\Gamma_G)| = 2^{2\gamma} - 1$. Therefore, we obtain the following equality:

$$2|E(\Gamma_G)| = (2^{2\gamma} - 1)(2^{2\gamma} - 2).$$

It is easy to show that $|E(\Gamma_G)| = (2^{2\gamma} - 1)(2^{2\gamma-1} - 1)$. \square

The following proposition shows that the clique number and the chromatic number of all groups of Type 1, 2 and 3 are identical.

Proposition 3.8 *Let G be a 2-generator 2-group of class 2. Then $\omega(\Gamma_G) = \chi(\Gamma_G)$.*

Proof If A is a clique of Γ_G , then the maximum size of A is $|V(\Gamma_G)| = k(G) - |Z(G)|$, since Γ_G is complete. It follows that $\omega(\Gamma_G) = k(G) - |Z(G)|$. On the other hand, $|V(\Gamma_G)|$ is the smallest number of colors needed to color the vertices of graph Γ_G so that no two adjacent vertices have the same color. Thus we deduce that $\chi(\Gamma_G) = k(G) - |Z(G)| = \omega(\Gamma_G)$. \square

3.4 CONCLUSION

This chapter has been devoted to the general formulas for the number of connected components, diameter, regularity and the number of edges of Γ_G . Some new structural results are generalized and used to derive directly the formulas for the number of edges of Γ_G .

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28 *Recent Advances in Some Finite Groups of Nilpotency Class Two*

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4

On the Commutativity Degree of Two-Generator p -Groups of Nilpotency Class Two

Nor Muhainiah Mohd Ali and Fadila Normahia Abd Manaf

4.1 INTRODUCTION

The commutativity degree of a group G , which is denoted by $P(G)$, is the probability that two elements of the group G , chosen randomly with replacement, commute. It can be written as,

$$P(G) = \frac{|\{(x, y) \in G \times G | xy = yx\}|}{|G|^2}.$$

Obviously, for an abelian group, this probability is clearly equal to one. Such probability can be associated with every finite group and this number can be considered as a measurement on how abelian a nonabelian group is. In other words, the commutativity degree is a kind of measure for the abelianness of a group.

This notion was first introduced by Miller [1] in 1944 and has been determined for many groups by other authors. The well-known investigation of this research was written in a paper by Erdos and Turan [2] which was published in 1968 where some problems of statistical group theory and commutativity degree in nonabelian groups have been considered. In 1973 and 1974, Gustafson [3] and MacHale [4] showed that the commutativity degree of a group G is at most $5/8$ if and only if G is a finite nonabelian group. The

commutativity degree also has been determined for other groups such as 2-Engel groups [5], dihedral groups [6] and alternating groups [7]. Thus, in this chapter, the commutativity degree of two-generator p -groups of nilpotency class two is determined and divided into two cases, namely for p are odd and $p = 2$.

4.2 PRELIMINARY RESULTS

In this section, some definitions and theorems that will be used in the subsequent sections are presented.

Definition 4.1 [1] Let G be a finite group. Then the commutativity degree of G can be defined as the following ratio:

$$P(G) = \frac{|\{(x, y) \in G \times G | xy = yx\}|}{|G|^2}.$$

Theorem 4.1 [3] Let G be a finite group and let $k(G)$ denote the number of conjugacy classes of G . Then

$$P(G) = \frac{k(G)}{|G|}.$$

Theorem 4.2 [8] If G is a nonabelian group and p is the least prime number which divides $|G|$, then

$$P(G) \leq \frac{p^2 + p - 1}{p^3}.$$

Moreover, this equality holds if and only if $G/Z(G)$ has order p^2 .

Earlier research on the classification of finite two-generator p -groups of nilpotency class two was introduced by Bacon and Kappe [9] in 1993 for the case p an odd prime and Kappe *et al.* [10] for the case $p = 2$. However, Magidin [11] gives the classification for both cases in terms of generators and relations in 2006. Finally, Ahmad *et al.* [12] modified this version to include all finite 2-generated p -groups of class two. Their result, which is presented in Theorem 4.3 is the most updated version that is published recently in 2012.

Theorem 4.3[12] *Let p be a prime and $n > 2$ an integer. Every 2-generated p -group of class exactly two and order p^n , corresponds to an ordered 5-tuple of integers, $(\alpha, \beta, \gamma; \rho, \sigma)$ such that:*

- (a) $\alpha \geq \beta \geq \gamma \geq 1$;
- (b) $\alpha + \beta + \gamma = n$;
- (c) $0 \leq \rho \leq \gamma$ and $0 \leq \sigma \leq \gamma$;

where $(\alpha, \beta, \gamma; \rho, \sigma)$ corresponds to the group presented by:

$$G = \langle a, b \mid [a, b]^{p^\gamma} = [a, b, a] = [a, b, b] = 1, \\ a^{p^\alpha} = [a, b]^{p^\rho}, b^{p^\beta} = [a, b]^{p^\sigma} \rangle.$$

Moreover,

- (a) if $\alpha > \beta$, then G is isomorphic to:
 - (i) $(\alpha, \beta, \gamma; \rho, \gamma)$ when $\rho \leq \sigma$;
 - (ii) $(\alpha, \beta, \gamma; \gamma, \sigma)$ when $0 \leq \sigma \leq \sigma + \alpha - \beta \leq \rho$ or $\sigma < \rho = \gamma$;
 - (iii) $(\alpha, \beta, \gamma; \rho, \sigma)$ when $0 \leq \sigma \leq \rho < \min(\gamma, \sigma + \alpha - \beta)$;
- (b) if $\alpha = \beta > \gamma$, or $\alpha = \beta = \gamma$ and $p > 2$, then G is isomorphic to $(\beta, \beta, \gamma; \min(\rho, \sigma), \gamma)$;
- (c) if $\alpha = \beta = \gamma$ and $p = 2$, then G is isomorphic to:
 - (i) $(\gamma, \gamma, \gamma; \min(\rho, \sigma), \gamma)$ when $0 \leq \min(\rho, \sigma) < \gamma - 1$.
 - (ii) $(\gamma, \gamma, \gamma; \gamma - 1, \gamma - 1)$ when $\rho = \sigma = \gamma - 1$.
 - (iii) $(\gamma, \gamma, \gamma; \gamma, \gamma)$ when $\min(\rho, \sigma) \geq \gamma - 1$ and $\max(\rho, \sigma) = \gamma$.

The groups listed in a(i)–c(iii) are pairwise non-isomorphic.

According to Ahmad *et al.*, those groups classified in part a(iii) of Theorem 4.3 were missing in the previous classifications.

4.3 THE COMMUTATIVITY DEGREE OF TWO-GENERATOR p -GROUPS OF NILPOTENCY CLASS TWO (p AN ODD PRIME)

In this section, the number of conjugacy classes of two-generator p -groups of nilpotency class two where p is an odd prime are

determined in order to compute the commutativity degree of these groups. Ahmad [13] in 2008 found the orders of two-generator p -groups of nilpotency class two where p is an odd prime and their centers. All these facts are used to compute the number of conjugacy classes of these groups. The orders of the groups and their centers for each type of these groups are given in the following propositions.

Proposition 4.1 [13] *Let G be a two-generator p -group of nilpotency class two (p an odd prime).*

- (a) *If $G \cong \langle a, b : a^{p^\alpha} = b^{p^\beta} = [a, b]^{p^\gamma} = [a, b, a] = [a, b, b] = 1 \rangle$, then $|G| = p^{\alpha+\beta+\gamma}$.*
- (b) *If $G \cong \langle a, b : a^{p^\alpha} = b^{p^\beta} = [a, b, a] = [a, b, b] = 1, a^{p^{\alpha-\gamma}} = [a, b] \rangle$, then $|G| = p^{\alpha+\beta}$.*
- (c) *If $G \cong \langle a, b : a^{p^\alpha} = b^{p^\beta} = [a, b, a] = [a, b, b] = 1, a^{p^{\alpha+\sigma-\gamma}} = [a, b]^{p^\sigma} \rangle$, then $|G| = p^{\alpha+\beta+\sigma}$.*

Proposition 4.2 [13] *Let G be a two-generator p -group of nilpotency class two (p an odd prime).*

- (a) *If $G \cong \langle a, b : a^{p^\alpha} = b^{p^\beta} = [a, b]^{p^\gamma} = [a, b, a] = [a, b, b] = 1 \rangle$, then $|Z(G)| = p^{\alpha+\beta-\gamma}$.*
- (b) *If $G \cong \langle a, b : a^{p^\alpha} = b^{p^\beta} = [a, b, a] = [a, b, b] = 1, a^{p^{\alpha-\gamma}} = [a, b] \rangle$, then $|Z(G)| = p^{\alpha+\beta-2\gamma}$.*
- (c) *If $G \cong \langle a, b : a^{p^\alpha} = b^{p^\beta} = [a, b, a] = [a, b, b] = 1, a^{p^{\alpha+\sigma-\gamma}} = [a, b]^{p^\sigma} \rangle$, then $|Z(G)| = p^{\alpha+\beta-2\gamma+\sigma}$.*

Initially, Ahmad [13] computed the number of conjugacy classes of the base groups of each type of finite non-abelian two-generator p -groups of nilpotency class two. The definition of a base group is given in the following:

Definition 4.2 [13] A base group G is a group which is not an extension such that the image is of the same type.

In other words, the group is a base group in the case $\gamma = 1$ in the classification. The number of conjugacy classes of the base groups of each type of finite nonabelian two-generator p -groups of nilpotency class two is given as follows.

Theorem 4.4[13] *Let G be a finite two-generator p -group of nilpotency class two (p an odd prime).*

- (a) *If G is isomorphic to the group mentioned in part (a) of Theorem 4.3 and G is a base group, then*

$$k(G) = p^{\alpha+\beta} + p^{\alpha+\beta-1} - p^{\alpha+\beta-2}.$$

- (b) *If G is isomorphic to the group mentioned in part (b) of Theorem 4.3 and G is a base group, then*

$$k(G) = p^{\alpha+\beta-1} + p^{\alpha+\beta-2} - p^{\alpha+\beta-3}.$$

- (c) *If G is isomorphic to the group mentioned in part (c) of Theorem 4.3 and G is a base group, then*

$$k(G) = p^{\alpha+\beta-\gamma+1} + p^{\alpha+\beta-\gamma} - p^{\alpha+\beta-2\gamma}.$$

The above results are used in [13] to compute the number of conjugacy classes for each class of groups.

Theorem 4.5[13] *Let G be a finite two-generator p -group of nilpotency class two (p an odd prime) and $k(H)$ be the number of conjugacy classes of the base group.*

- (a) *If G is isomorphic to the group mentioned in part (a) or (b) of Theorem 4.3, then*

$$k(G) = k(H) + |Z(G)| - \frac{|Z(G)|}{p}.$$

- (b) *If G is isomorphic to the group mentioned in part (c) of Theorem 4.3, then*

$$k(G) = pk(H) + |Z(G)| - \frac{|Z(G)|}{p}.$$

However, the results of Theorem 4.5 are simplified to obtain the following theorem which is the general formula for the number of conjugacy classes of all finite nonabelian two-generator p -groups of nilpotency class two.

Theorem 4.6 *Let G be a finite nonabelian two-generator p -group of nilpotency class two. Then the general formula for the number of conjugacy classes of the group G is given as follows:*

- (a) *If G is isomorphic to the group mentioned in part (a) of Theorem 4.3, then the number of conjugacy classes of G is*

$$p^{\alpha+\beta-\gamma-1}(p^{\gamma+1} + p^{\gamma} - 1).$$

- (b) *If G is isomorphic to the group mentioned in part (b) of Theorem 4.3, then the number of conjugacy classes of G is*

$$p^{\alpha+\beta-2\gamma-1}(p^{\gamma+1} + p^{\gamma} - 1).$$

- (c) *If G is isomorphic to the group mentioned in part (c) of Theorem 4.3, then the number of conjugacy classes of G is*

$$p^{\alpha+\beta-2\gamma+\sigma-1}(p^{\gamma+1} + p^{\gamma} - 1).$$

Proof

- (a) Recall that the number of conjugacy classes of G is denoted as $k(G)$. Let G_n be a finite two-generator p -group of nilpotency class two (p is an odd prime) which is isomorphic to the group mentioned in part (i) of Theorem 4.3 and $\gamma = n$. By Theorem 4.4(a), the number of conjugacy classes of the base group is $k(G_1) = p^{\alpha+\beta} + p^{\alpha+\beta-1} - p^{\alpha+\beta-2}$. Suppose that G is an extension of H and $G = G_n$, then $H = G_{n-1}$. By Theorem 4.5(a),

$$k(G_n) = k(G_{n-1}) + \left(1 - \frac{1}{p}\right)p^{\alpha+\beta-n}$$

and

$$k(G_{n-1}) = k(G_{n-2}) + \left(1 - \frac{1}{p}\right)p^{\alpha+\beta-n+1}.$$

Therefore,

$$\begin{aligned} k(G_n) &= k(G_{n-2}) + \left(1 - \frac{1}{p}\right)(p^{\alpha+\beta-n+1} + p^{\alpha+\beta-n}) \\ &= k(G_{n-2}) + (p^2 - 1)p^{\alpha+\beta-n-1}. \end{aligned}$$

The above equation can be concluded as follows:

$$k(G_n) = k(G_{n-i}) + (p^i - 1)p^{\alpha+\beta-n-1} \quad (4.1)$$

for $i = 1, \dots, n-1$. Equation (4.1) can be proven by induction on i . If $i = 1$, then

$$k(G_n) = k(G_{n-1}) + (p - 1)p^{\alpha+\beta-n-1}.$$

Next, assume that it is true for $i - 1$, that is

$$k(G_n) = k(G_{n-(i-1)}) + (p^{i-1} - 1)p^{\alpha+\beta-n-1}.$$

We now prove equation (4.1) is true for i . By taking the right hand side,

$$\begin{aligned} & k(G_{n-i}) + (p^i - 1)p^{\alpha+\beta-n-1} \\ &= k(G_{n-i+1}) - (p - 1)p^{\alpha+\beta-n+i-2} + (p^i - 1)p^{\alpha+\beta-n-1} \\ &= k(G_n) - (p^{i-1} - 1)p^{\alpha+\beta-n-1} - (p - 1)p^{\alpha+\beta-n+i-2} \\ &\quad + (p^i - 1)p^{\alpha+\beta-n-1} \\ &= k(G_n) - p^{\alpha+\beta-n+i-2} + p^{\alpha+\beta-n-1} - p^{\alpha+\beta-n+i-1} \\ &\quad + p^{\alpha+\beta-n+i-2} + p^{\alpha+\beta-n+i-1} - p^{\alpha+\beta-n-1} \\ &= k(G_n). \end{aligned}$$

Equation (4.1) is true for $i = 1$. In addition, by assuming that the equation is true for $i - 1$, we have proven that it is true for i . Thus, the equation is true for all $i \in \mathbb{N}$.

Now, let $i = n - 1$, then

$$\begin{aligned} k(G_n) &= k(G_{n-(n-1)}) + (p^{n-1} - 1)p^{\alpha+\beta-n-1} \\ &= k(G_1) + (p^{n-1} - 1)p^{\alpha+\beta-n-1} \\ &= p^{\alpha+\beta} + p^{\alpha+\beta-1} - p^{\alpha+\beta-2} + (p^{n-1} - 1)p^{\alpha+\beta-n-1} \\ &= p^{\alpha+\beta-n-1}(p^{n+1} + p^n - 1). \end{aligned}$$

Since $\gamma = n$, then $k(G) = p^{\alpha+\beta-\gamma-1}(p^{\gamma+1} + p^\gamma - 1)$ as needed.

- (b) Let $G_{m,n}$ be a finite two-generator p -group of nilpotency class two (p is an odd prime) which is isomorphic to the group mentioned in part (b) of Theorem 4.3 and $\alpha = m, \gamma = n$. By Theorem 4.4(b), the number of conjugacy classes of the base group is $k(G_{m,1}) = p^{m+\beta-1} + p^{m+\beta-2} - p^{m+\beta-3}$. Suppose that G is an extension of H and $G = G_{m,n}$, then $H = G_{m-1,n-1}$. By Theorem 4.5(a),

$$k(G_{m,n}) = k(G_{m-1,n-1}) + \left(1 - \frac{1}{p}\right)p^{m+\beta-2n}$$

and

$$k(G_{m-1,n-1}) = k(G_{m-2,n-2}) + \left(1 - \frac{1}{p}\right)p^{m-1+\beta-2n+2}.$$

Therefore,

$$\begin{aligned} k(G_{m,n}) &= k(G_{m-2,n-2}) \\ &\quad + \left(1 - \frac{1}{p}\right)(p^{m+\beta-2n+1} + p^{m+\beta-2n}) \\ &= k(G_{m-2,n-2}) + (p^2 - 1)p^{m+\beta-2n-1}. \end{aligned}$$

From the preceding equation, it can be concluded that

$$k(G_{m,n}) = k(G_{m-i,n-i}) + (p^i - 1)p^{m+\beta-2n-1} \quad (4.2)$$

for $i = 1, \dots, n-1$. Equation (4.2) can be proved by induction on i . If $i = 1$, then

$$k(G_{m,n}) = k(G_{m-1,n-1}) + (p - 1)p^{m+\beta-2n-1}.$$

Next, assume that it is true for $i - 1$, namely

$$k(G_{m,n}) = k(G_{m-(i-1),n-(i-1)}) + (p^{i-1} - 1)p^{m+\beta-2n-1}.$$

We now prove Equation (4.2) is true for i . By taking the right hand side,

$$\begin{aligned}
 & k(G_{m-i,n-i}) + (p^i - 1)p^{m+\beta-2n-1} \\
 &= k(G_{m-i+1,n-i+1}) - (p - 1)p^{m+\beta-2n+i-2} \\
 &\quad + (p^i - 1)p^{m+\beta-2n-1} \\
 &= k(G_{m,n}) - (p^{i-1} - 1)p^{m+\beta-2n-1} - (p - 1)p^{m+\beta-2n+i-2} \\
 &\quad + (p^i - 1)p^{m+\beta-2n-1} \\
 &= k(G_{m,n}) - p^{m+\beta-2n+i-2} + p^{m+\beta-2n-1} - p^{m+\beta-2n+i-1} \\
 &\quad + p^{m+\beta-2n+i-2} + p^{m+\beta-2n+i-1} - p^{m+\beta-2n-1} \\
 &= k(G_{m,n}).
 \end{aligned}$$

Equation (4.2) is true for $i = 1$. In addition, by assuming that the equation is true for $i - 1$, we have proven that it is true for i . Thus, the equation is true for all $i \in \mathbb{N}$.

Now, let $i = n - 1$, then

$$\begin{aligned}
 k(G_{m,n}) &= k(G_{m-(n-1),n-(n-1)}) + (p^{n-1} - 1)p^{m+\beta-2n-1} \\
 &= k(G_{m-n+1,1}) + (p^{n-1} - 1)p^{m+\beta-2n-1} \\
 &= p^{m-n+1+\beta-1} + p^{m-n+1+\beta-2} - p^{m-n+1+\beta-3} \\
 &\quad + (p^{n-1} - 1)p^{m+\beta-2n-1} \\
 &= p^{m+\beta-2n-1}(p^{n+1} + p^n - 1).
 \end{aligned}$$

Since $\alpha = m$ and $\gamma = n$, $k(G) = p^{\alpha+\beta-2\gamma-1}(p^{\gamma+1} + p^\gamma - 1)$ as needed.

- (c) Let $G_{m,n,q}$ be a finite two-generator p -group of nilpotency class two (p is an odd prime) which is isomorphic to the group mentioned in part (c) of Theorem 4.3 and $\alpha = m$, $\gamma = n$, $\sigma = q$. By Theorem 4.4(c), the number of conjugacy classes of the base group is

$$k(G_{m,n,1}) = p^{m+\beta-n+1} + p^{m+\beta-n} - p^{m+\beta-2n}.$$

38 *Recent Advances in Some Finite Groups of Nilpotency Class Two*

Assume that G is an extension of H and $G = G_{m,n,q}$, then $H = G_{m-1,n-1,q-1}$. By Theorem 4.5(b),

$$k(G_{m,n,q}) = pk(G_{m-1,n-1,q-1}) + \left(1 - \frac{1}{p}\right)p^{m+\beta-2n+q}$$

and

$$k(G_{m-1,n-1,q-1}) = pk(G_{m-2,n-2,q-2}) + \left(1 - \frac{1}{p}\right)p^{m+\beta-2n+q}.$$

Therefore,

$$\begin{aligned} k(G_{m,n,q}) &= p^2k(G_{m-2,n-2,q-2}) + \left(1 - \frac{1}{p}\right)(p^{m+\beta-2n+q+1} \\ &\quad + p^{m+\beta-2n+q}) \\ &= p^2k(G_{m-2,n-2,q-2}) + (p^2 - 1)p^{m+\beta-2n+q-1}. \end{aligned}$$

It can be concluded from the above equation that,

$$\begin{aligned} k(G_{m,n,q}) &= p^i.k(G_{m-i,n-i,q-i}) \\ &\quad + (p^i - 1)p^{m+\beta-2n+q-1} \end{aligned} \quad (4.3)$$

for $i = 1, \dots, q-1$. Equation (4.3) can be proved by induction on i . If $i = 1$, then

$$k(G_{m,n,q}) = pk(G_{m-1,n-1,q-1}) + (p - 1)p^{m+\beta-2n+q-1}.$$

Next, assume that it is true for $i - 1$, namely

$$\begin{aligned} k(G_{m,n,q}) &= p^{i-1}k(G_{m-(i-1),n-(i-1),q-(i-1)}) \\ &\quad + (p^{i-1} - 1)p^{m+\beta-2n+q-1}. \end{aligned}$$

We now prove Equation (4.3) is true for i . By taking the right hand side,

$$\begin{aligned}
 & p^i k(G_{m-i, n-i, q-i}) + (p^i - 1)p^{m+\beta-2n+q-1} \\
 &= p^i k(G_{m-i+1, n-i+1, q-i+1}) - p^{i-1}(p-1)p^{m+\beta-2n+q-1} \\
 &\quad + (p^i - 1)p^{m+\beta-2n+q-1} \\
 &= k(G_{m, n, q}) - p^{m+\beta-2n+q+i-2} + p^{m+\beta-2n+q-1} \\
 &\quad - p^{m+\beta-2n+q+i-1} + p^{m+\beta-2n+q+i-2} + p^{m+\beta-2n+q+i-1} \\
 &\quad - p^{m+\beta-2n+q-1} \\
 &= k(G_{m, n, q}).
 \end{aligned}$$

Equation (4.3) is true for $i = 1$. In addition, by assuming that the equation is true for $i - 1$, we have proven that it is true for i . Hence, the equation is true for all $i \in \mathbb{N}$.

Now, let $i = q - 1$, then

$$\begin{aligned}
 k(G_{m, n, q}) &= p^{q-1} k(G_{m-(q-1), n-(q-1), q-(q-1)}) \\
 &\quad + (p^{q-1} - 1)p^{m+\beta-2n+q-1} \\
 &= p^{q-1} k(G_{m-q+1, n-q+1, 1}) \\
 &\quad + (p^{q-1} - 1)p^{m+\beta-2n+q-1} \\
 &= p^{q-1}(p^{m+\beta-n+1} + p^{m+\beta-n} - p^{m+\beta-2n+q-1}) \\
 &\quad + (p^{q-1} - 1)p^{m+\beta-2n+q-1} \\
 &= p^{m+\beta-2n+q-1}(p^{n+1} + p^n - 1).
 \end{aligned}$$

Since $\alpha = m$, $\gamma = n$ and $\sigma = q$, then $k(G) = p^{\alpha+\beta-2\gamma+\sigma-1}(p^{\gamma+1} + p^\gamma - 1)$ as needed. \square

Next, by applying the formula of commutativity degree which is the number of conjugacy classes divided by the order of the group, the result in the following theorem holds.

Theorem 4.7 *Let G be a finite two-generator p -group of nilpotency*

class two, where p is an odd prime and $\gamma \in \mathbb{N}$. Then

$$P(G) = \frac{p^{\gamma+1} + p^\gamma - 1}{p^{2\gamma+1}}.$$

Proof If G is one of the groups in Theorem 4.3, then the possible number of conjugacy classes is $p^{\alpha+\beta-\gamma-1}(p^{\gamma+1} + p^\gamma - 1)$, $p^{\alpha+\beta-2\gamma-1}(p^{\gamma+1} + p^\gamma - 1)$ or $p^{\alpha+\beta-2\gamma+\sigma-1}(p^{\gamma+1} + p^\gamma - 1)$. Since the commutativity degree of a non-abelian group is equal to the number of conjugacy classes divided by the order of the group [3], hence by using the order of the group in Proposition 4.1, $P(G) = (p^{\gamma+1} + p^\gamma - 1)/(p^{2\gamma+1})$ for all cases. \square

4.4 THE COMMUTATIVITY DEGREE OF TWO-GENERATOR p -GROUPS OF NILPOTENCY CLASS TWO ($p=2$)

In this section, the exact formula for the commutativity degree of two-generator p -groups of nilpotency class two whenever $p=2$ is given. By using similar methods, in 2011, Ilangoan and Sarmin obtained the order of two-generator two-groups of nilpotency class two and their centers according to the types. Their results in [14–17] are combined and stated in the following proposition.

Proposition 4.3 [14–17] *Let G be a two-generator two-group of nilpotency class two.*

- (a) *If $G \cong \langle a, b : a^{2^\alpha} = b^{2^\beta} = [a, b]^{2^\gamma} = [a, b, a] = [a, b, b] = 1 \rangle$, then $|G| = 2^{\alpha+\beta+\gamma}$.*
- (b) *If $G \cong \langle a, b : a^{2^\alpha} = b^{2^\beta} = [a, b, a] = [a, b, b] = 1, a^{2^{\alpha+\sigma-\gamma}} = [a, b]^{2^\sigma} \rangle$, then $|G| = 2^{\alpha+\beta+\gamma}$.*
- (c) *If $G \cong \langle a, b : a^{2^{\gamma+1}} = b^{2^{\gamma+1}} = [a, b]^{2^\gamma} = [a, b, a] = [a, b, b] = 1, a^{2^\gamma} = b^{2^\gamma} = [a, b]^{2^{\gamma-1}} \rangle$, then $|G| = 2^{3\gamma}$.*

The following propositions give the orders of the centre of each types of the groups.

Proposition 4.4 [14–17] *Let G be a two-generator two-group of nilpotency class two.*

- (a) *If $G \cong \langle a, b : a^{2^\alpha} = b^{2^\beta} = [a, b]^{2^\gamma} = [a, b, a] = [a, b, b] = 1 \rangle$, then $|Z(G)| = 2^{\alpha+\beta-\gamma}$.*
- (b) *If $G \cong \langle a, b : a^{2^\alpha} = b^{2^\beta} = [a, b, a] = [a, b, b] = 1, a^{2^{\alpha+\sigma-\gamma}} = [a, b]^{2^\sigma} \rangle$, then $|Z(G)| = 2^{\alpha+\beta-2\gamma+\sigma}$.*
- (c) *If $G \cong \langle a, b : a^{2^{\gamma+1}} = b^{2^{\gamma+1}} = [a, b]^{2^\gamma} = [a, b, a] = [a, b, b] = 1, a^{2^\gamma} = b^{2^\gamma} = [a, b]^{2^{\gamma-1}} \rangle$, then $|Z(G)| = 2^\gamma$.*

Thus it remains to deal with the finite two-generator two-groups of nilpotency class two. The following results, which can be found in Ilangoan and Sarmin [14–17] are needed.

Theorem 4.8 [14–17] *Let G be a finite two-generator two-group of nilpotency class two.*

- (a) *If G is isomorphic to the group mentioned in part (a) of Theorem 4.3 and G is a base group, then $k(G) = 2^{\alpha+\beta} + 2^{\alpha+\beta-1} - 2^{\alpha+\beta-2}$.*
- (b) *If G is isomorphic to the group mentioned in part (b) of Theorem 4.3 and G is a base group, then $k(G) = 2^{\alpha+\beta-1-\gamma} + 2^{\alpha+\beta-\gamma} - 2^{\alpha+\beta-2\gamma}$.*
- (c) *If G is isomorphic to the group mentioned in part (iii) of Theorem 4.3 and G is a base group, then $k(G) = 5$.*

The above results are used to compute the number of conjugacy classes for each type of groups.

Theorem 4.9 [14–17] *Let G be a finite two-generator two-group of nilpotency class two and $k(H)$ be the number of conjugacy classes of the base group.*

- (a) *If G is the group of part (i) in Theorem 4.3, then*

$$k(G) = k(H) + \frac{|Z(G)|}{2}.$$

- (b) If G is isomorphic to the group mentioned in part (b) of Theorem 4.3, then

$$k(G) = 2k(H) + \frac{|Z(G)|}{2}.$$

- (c) If G is isomorphic to the group mentioned in part (c) of Theorem 4.3, then

$$k(G) = 4k(H) + \frac{|Z(G)|}{2}.$$

Similar to Theorem 4.6, the following result for a finite non-abelian two-generator two-group of nilpotency class two is obtained.

Theorem 4.10 *Let G be a finite non-abelian two-generator two-group of nilpotency class two. Then the general formula for the number of conjugacy classes of the group G is given as follows:*

- (a) If $G \cong \langle a, b : a^{2^\alpha} = b^{2^\beta} = [a, b]^{2^\gamma} = [a, b, a] = [a, b, b] = 1 \rangle$, then the number of conjugacy classes of G is $2^{\alpha+\beta-\gamma-1}(2^{\gamma+1} + 2^\gamma - 1)$.
- (b) If $G \cong \langle a, b : a^{2^\alpha} = b^{2^\beta} = [a, b, a] = [a, b, b] = 1, a^{2^{\alpha+\sigma-\gamma}} = [a, b]^{2^\sigma} \rangle$, then the number of conjugacy classes of G is $2^{\alpha+\beta-2\gamma+\sigma-1}(2^{\gamma+1} + 2^\gamma - 1)$.
- (c) If $G \cong \langle a, b : a^{2^{\gamma+1}} = b^{2^{\gamma+1}} = [a, b]^{2^\gamma} = [a, b, a] = [a, b, b] = 1, a^{2^\gamma} = b^{2^\gamma} = [a, b]^{2^{\gamma-1}} \rangle$, then the number of conjugacy classes of G is $2^{\gamma-1}(2^{\gamma+1} + 2^\gamma - 1)$.

Proof

- (a) Let G_n be a finite two-generator two-group of nilpotency class two which is isomorphic to the group mentioned in part (a) of Theorem 4.3 and $\gamma = n$. By Theorem 4.8(a), the number of conjugacy classes of the base group is

$$k(G_1) = 2^{\alpha+\beta} + 2^{\alpha+\beta-1} - 2^{\alpha+\beta-2}.$$

Suppose that G is an extension of H and $G = G_n$, then $H = G_{n-1}$. By Theorem 4.9(i),

$$k(G_n) = k(G_{n-1}) + \frac{2^{\alpha+\beta-n}}{2}$$

and

$$k(G_{n-1}) = k(G_{n-2}) + \frac{2^{\alpha+\beta-n+1}}{2}.$$

Therefore,

$$\begin{aligned} k(G_n) &= k(G_{n-2}) + \frac{1}{2}(2^{\alpha+\beta-n+1} + 2^{\alpha+\beta-n}) \\ &= k(G_{n-2}) + (2^2 - 1)2^{\alpha+\beta-n-1}. \end{aligned}$$

It can be concluded from the above equation that

$$k(G_n) = k(G_{n-i}) + (2^i - 1)2^{\alpha+\beta-n-1} \quad (4.4)$$

for $i = 1, \dots, n-1$. Equation (4.4) can be proved by induction on i . If $i = 1$, then

$$k(G_n) = k(G_{n-1}) + (2 - 1)2^{\alpha+\beta-n-1}.$$

Next, assume that it is true for $i - 1$, namely

$$k(G_n) = k(G_{n-(i-1)}) + (2^{i-1} - 1)2^{\alpha+\beta-n-1}.$$

We now prove that Equation (4.4) is true for i . By taking the right hand side,

$$\begin{aligned} &k(G_{n-i}) + (2^i - 1)2^{\alpha+\beta-n-1} \\ &= k(G_{n-i+1}) - 2^{\alpha+\beta-n+i-2} + (2^i - 1)2^{\alpha+\beta-n-1} \\ &= k(G_n) - (2^{i-1} - 1)2^{\alpha+\beta-n-1} - 2^{\alpha+\beta-n+i-2} \\ &\quad + (2^i - 1)2^{\alpha+\beta-n-1} \\ &= k(G_n) - 2^{\alpha+\beta-n+i-2} + 2^{\alpha+\beta-n-1} - 2^{\alpha+\beta-n+i-1} \\ &\quad + 2^{\alpha+\beta-n+i-2} + 2^{\alpha+\beta-n+i-1} - 2^{\alpha+\beta-n-1} \\ &= k(G_n). \end{aligned}$$

Equation (4.4) is true for $i = 1$. In addition, by assuming that the equation is true for $i - 1$, we have proven that it is true for i . Thus, the equation is true for all $i \in \mathbb{N}$.

Now, let $i = n - 1$, then

$$\begin{aligned}
 k(G_n) &= k(G_{n-(n-1)}) + (2^{n-1} - 1)2^{\alpha+\beta-n-1} \\
 &= k(G_1) + (2^{n-1} - 1)2^{\alpha+\beta-n-1} \\
 &= 2^{\alpha+\beta} + 2^{\alpha+\beta-1} - 2^{\alpha+\beta-2} + (2^{n-1} - 1)2^{\alpha+\beta-n-1} \\
 &= 2^{\alpha+\beta-n-1}(2^{n+1} + 2^n - 1).
 \end{aligned}$$

Since $\gamma = n$, $k(G) = 2^{\alpha+\beta-\gamma-1}(2^{\gamma+1} + 2^\gamma - 1)$ as needed.

- (b) Let $G_{m,n,q}$ be a finite two-generator two-group of nilpotency class two which is isomorphic to the group mentioned in part (b) of Theorem 4.3 and $\alpha = m, \gamma = n, \sigma = q$. By Theorem 4.8(b), the number of conjugacy classes of the base group is $k(G_{m,n,1}) = 2^{m+\beta-n+1} + 2^{m+\beta-n} - 2^{m+\beta-2n}$. Assume that G is an extension of H and $G = G_{m,n,q}$, then $H = G_{m-1,n-1,q-1}$. By Theorem 4.9(b),

$$k(G_{m,n,q}) = 2k(G_{m-1,n-1,q-1}) + \frac{2^{m+\beta-2n+q}}{2}$$

$$\text{and } k(G_{m-1,n-1,q-1}) = 2k(G_{m-2,n-2,q-2}) + \frac{2^{m+\beta-2n+q}}{2}.$$

Therefore,

$$\begin{aligned}
 k(G_{m,n,q}) &= 2^2 k(G_{m-2,n-2,q-2}) \\
 &\quad + \frac{1}{2}(2^{m+\beta-2n+q-1} + 2^{m+\beta-2n+q}) \\
 &= 2^2 k(G_{m-2,n-2,q-2}) + (2^2 - 1)2^{m+\beta-2n+q-1}.
 \end{aligned}$$

The above equation can be concluded as follows:

$$\begin{aligned}
 k(G_{m,n,q}) &= 2^i k(G_{m-i,n-i,q-i}) \\
 &\quad + (2^i - 1)2^{m+\beta-2n+q-1} \quad (4.5)
 \end{aligned}$$

for $i = 1, \dots, q-1$. Equation (4.5) can be proved by induction on i . If $i = 1$, then

$$k(G_{m,n,q}) = 2k(G_{m-1,n-1,q-1}) + (2 - 1)2^{m+\beta-2n+q-1}.$$

Next, assume that it is true for $i - 1$, that is

$$k(G_{m,n,q}) = 2^{i-1}k(G_{m-(i-1),n-(i-1),q-(i-1)}) \\ + (2^{i-1} - 1)2^{m+\beta-2n+q-1}.$$

We now prove Equation (4.5) is true for i . By taking the right hand side,

$$\begin{aligned} & 2^i k(G_{m-i,n-i,q-i}) + (2^i - 1)2^{m+\beta-2n+q-1} \\ &= 2^{i-1}k(G_{m-i+1,n-i+1,q-i+1}) - 2^{m+\beta-2n+q+i-2} \\ & \quad + 2^{m+\beta-2n+q+i-1} - 2^{m+\beta-2n+q-1} \\ &= k(G_{m,n,q}) - (2^{i-1} - 1)2^{m+\beta-2n+q-1} - 2^{m+\beta-2n+q+i-1} \\ & \quad + 2^{m+\beta-2n+q+i-2} + 2^{m+\beta-2n+q+i-1} - 2^{m+\beta-2n+q-1} \\ &= k(G_{m,n,q}) - 2^{m+\beta-2n+q+i-2} + 2^{m+\beta-2n+q-1} \\ & \quad - 2^{m+\beta-2n+q+i-1} + 2^{m+\beta-2n+q+i-2} + 2^{m+\beta-2n+q+i-1} \\ & \quad - 2^{m+\beta-2n+q-1} \\ &= k(G_{m,n,q}). \end{aligned}$$

Equation (4.5) is true for $i = 1$. In addition, by assuming that the equation is true for $i - 1$, we have proven that it is true for i . Thus, the equation is true for all $i \in \mathbb{N}$.

Now, let $i = q - 1$, then

$$\begin{aligned} k(G_{m,n,q}) &= 2^{q-1}k(G_{m-(q-1),n-(q-1),q-(q-1)}) \\ & \quad + (2^{q-1} - 1)2^{m+\beta-2n+q-1} \\ &= 2^{q-1}k(G_{m-q+1,n-q+1,1}) \\ & \quad + (2^{q-1} - 1)2^{m+\beta-2n+q-1} \\ &= 2^{q-1}(2^{m+\beta-n+1} + 2^{m+\beta-n} - 2^{m+\beta-2n+q-1}) \\ & \quad + (2^{q-1} - 1)2^{m+\beta-2n+q-1} \\ &= 2^{m+\beta-2n+q-1}(2^{n+1} + p^n - 1). \end{aligned}$$

Since $\alpha = m$, $\gamma = n$ and

$$\sigma = q, k(G) = 2^{\alpha+\beta-2\gamma+\sigma-1}(2^{\gamma+1} + 2^\gamma - 1)$$

as needed.

- (c) Let G_n be a finite two-generator two-group of nilpotency class two which is isomorphic to the group mentioned in part (c) of Theorem 4.3 and $\gamma = n$. By Theorem 4.8(c), the number of conjugacy classes of the base group is $k(G_1) = 5$. Suppose that G is an extension of H and $G = G_n$, then $H = G_{n-1}$. By Theorem 4.9(c),

$$k(G_n) = 4k(G_{n-1}) + \frac{2^n}{2}$$

and

$$k(G_{n-1}) = 4k(G_{n-2}) + \frac{2^{n-1}}{2}.$$

Therefore,

$$\begin{aligned} k(G_n) &= 4^2k(G_{n-2}) + 2(2^{n-1}) + \frac{2^n}{2} \\ &= 4^2k(G_{n-2}) + (2^2 - 1)2^{n-1}. \end{aligned}$$

It can be concluded from the above equation that

$$k(G_n) = 4^i k(G_{n-i}) + (2^i - 1)2^{n-1} \quad (4.6)$$

for $i = 1, \dots, n-1$. Equation (4.6) can be proved by induction on i . If $i = 1$, then

$$k(G_n) = 4k(G_{n-1}) + (2 - 1)2^{n-1}.$$

Next, assume that it is true for $i - 1$, that is

$$k(G_n) = 4^{i-1} k(G_{n-(i-1)}) + (2^{i-1} - 1)2^{n-1}.$$

We now prove Equation (4.6) is true for i . By taking the right hand side,

$$\begin{aligned}
 & 4^i k(G_{n-i}) + (2^i - 1)2^{n-1} \\
 &= 4^{i-1} k(G_{n-i+1}) - 2^{n+i-2} + 2^{n+i-1} - 2^{n-1} \\
 &= k(G_n) - 2^{n+i-2} + 2^{n-1} - 2^{n+i-1} + 2^{n+i-2} + 2^{n+i-1} \\
 &\quad - 2^{n-1} \\
 &= k(G_n).
 \end{aligned}$$

Equation (4.6) is true for $i = 1$. In addition, by assuming that the equation is true for $i - 1$, we have proven that it is true for i . Thus, the equation is true for all $i \in \mathbb{N}$.

Now, let $i = n - 1$, then

$$\begin{aligned}
 k(G_n) &= 4^{n-1} k(G_{n-(n-1)}) + (2^{n-1} - 1)2^{n-1} \\
 &= 4^{n-1} k(G_1) + (2^{n-1} - 1)2^{n-1} \\
 &= 4^{n-1} (5) + (2^{n-1} - 1)2^{n-1} \\
 &= 2^{n-1} (2^{n+1} + 2^n - 1).
 \end{aligned}$$

Since $\gamma = n$, therefore $k(G) = 2^{\gamma-1}(2^{\gamma+1} + 2^\gamma - 1)$ as needed. \square

Using similar method in Theorem 4.7, the commutativity degree for finite two-generator two-groups of nilpotency class two is presented in the following theorem.

Theorem 4.11 *Let G be a finite two-generator two-group of nilpotency class two and $\gamma \in \mathbb{N}$. Then*

$$P(G) = \frac{2^{\gamma+1} + 2^\gamma - 1}{2^{2\gamma+1}}.$$

Proof If G is one of the groups in Theorem 4.3, then the possible number of conjugacy classes is

$$2^{\alpha+\beta-\gamma-1}(2^{\gamma+1} + 2^\gamma - 1), 2^{\alpha+\beta-2\gamma+\sigma-1}(2^{\gamma+1} + 2^\gamma - 1),$$

48 *Recent Advances in Some Finite Groups of Nilpotency Class Two*

or

$$2^{\gamma+1}(2^{\gamma+1} + 2^{\gamma} - 1).$$

Note that the commutativity degree of a non-abelian group is equal to the number of conjugacy classes divided by the order of the group [3]. Hence, by using the order of the group in Proposition 4.3,

$$P(G) = \frac{2^{\gamma+1} + 2^{\gamma} - 1}{2^{2\gamma+1}}$$

for all cases. □

4.5 CONCLUSION

The general formulas for the exact number of conjugacy classes for all two-generator p -groups of nilpotency class two are found according to the classification of these groups. The commutativity degree of two-generator p -groups of nilpotency class two for any prime p are computed.

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50 *Recent Advances in Some Finite Groups of Nilpotency Class Two*

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Index

A

abelian groups, 29
automorphism, 7

B

base group, 22, 32

C

central extension, 22
chromatic number, 19, 26
classification, 2, 20
clique number, 19, 26
commutativity degree, 29
conjugacy classes, 31
connected components, 18

D

diameter, 18

E

edge, 18
Engel
 2-Engel, 30

G

generated
 2-generated, 1
graph

complete, 18

conjugacy classes, 17
simple, 17

group

p -group, 1, 30

H

homomorphism, 7

N

nilpotency class two, 1
nonabelian groups of order p^3 , 4

P

probability, 29

R

regular
 k -regular, 18

T

type, 20
 type 1, 20
 type 2, 21
 type 3, 21

V

vertex, 17, 18