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THE TOPOLOGICAL INDICES OF NON-COMMUTING GRAPH OF A FINITE GROUP

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Abstract: Assume G is a non-abelian finite group. The non-commuting graph Γ_G of G is defined as a graph with vertex set $G - Z(G)$ in which $Z(G)$ is the center of G and two distinct vertices x and y are joined if and only if $xy \neq yx$. Various topological indices have been determined for simple and connected graphs. Since non-commuting graph is a simple and connected graph, topological indices could be defined for it. The main objective of this article is to calculate various topological indices including the Szeged index, Edge-Wiener index, the first Zagreb index and the second Zagreb index for the non-commuting graph of G.

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1. Introduction

In this paper, G is a non-abelian finite group. Various graphs could be attributed to G, one of which is the non-commuting graph, denoted by Γ_G. The set of vertices and edges of Γ_G are $V(\Gamma_G)$ and $E(\Gamma_G)$, respectively so that $V(\Gamma_G) = G - Z(G)$ in which $Z(G)$ is the center of G and for every $x, y \in V(\Gamma_G)$ we have $\{x, y\} \in E(\Gamma_G) \Leftrightarrow xy \neq yx$. The centralizer of x within G which is denoted by $C_G(x)$ is a subset of G which is defined as $\{g \in G : gx = xg\}$. According to [3], the non-commuting graph of a finite group G was first introduced by Paul Erdos.

Assume that $G = (V, E)$ is a graph in which V is the set of vertices and E is the set of edges. This graph is a finite graph whenever $|V|$ and $|E|$ are finite. The distance between two vertices x and y is denoted by $d(x, y)$, which the length of the shortest path between the two vertices x and y. The degree of the vertex x is denoted by $\deg(x)$, equal to the number of edges through x. The diameter of G is defined as follows:

$$\operatorname{diam}(G) = \max\{d(x, y) : x, y \in V(\Gamma_G)\}.$$

The Szeged index of the graph $G = (V, E)$ is defined as follows: This index is a recently introduced invariant of a graph which is based on the distances of the vertices of the graph [5] and [6]. Let $e = xy$ be an edge of G. We define the following sets:

$$N_x(e|G) = \{w \in V : d(w, x) < d(w, y)\},$$
$$N_y(e|G) = \{w \in V : d(w, y) < d(w, x)\}.$$

Hence $N_x(e|G)$ is the set of all vertices of G which are closer to x than y and $N_y(e|G)$ is the set of all vertices of G which are closer to y than x. The size of $N_x(e|G)$ are $N_y(e|G)$ are denoted by $n_x(e|G)$ and $n_y(e|G)$, respectively.

The Szeged index of the graph G is defined by

$$Sz(G) = \sum_{e=xy \in E(G)} n_x(e|G) \cdot n_y(e|G).$$

Let G be a connected graph. The Edge-Wiener index of G is defined as follows:

$$W_e(G) = \sum_{\{e,f\} \subseteq E(G)} d(e, f).$$

Where e, f are two edges in G and $d(e, f)$ is the distance between two vertices in the line-graph. In view of the above definition $W_e(G) = W(\overline{G})$ ($\overline{G}$ is the line-graph of G). For more details, refer to the [4]. The first Zagreb index of G is denoted by $Z_1(G)$ and is defined by:

$$Z_1(G) = \sum_{x \in V} (\deg(x))^2.$$

The second Zagreb index of the graph G is defined by:

$$Z_2(G) = \sum_{\{x,y\} \subseteq V} \deg(x) \cdot \deg(y).$$

The readers can refer to [7] for more details. Our main goal is to calculate the above mentioned indices for the non-commuting graph of G in terms of the order of G, $Z(G)$ and the number of conjugacy classes of G. The following lemmas will be used repeatedly.

Lemma 1. *[1]. Let G be a finite group. Then* $\mathrm{diam}(\Gamma_G) = 2$.

Lemma 2. *[1]. Let G be a finite group and $k(G)$ the number of conjugacy classes of G, then*

$$|E(\Gamma_G)| = \frac{1}{2}|G|(|G| - k(G)).$$

Lemma 3. *[1]. Let G be a finite group. If x be one of the vertices of Γ_G, then*

$$\deg(x) = |G| - |C_G(x)|.$$

2. The Szeged Index of a Non-Commuting Graph

In this section, we find the Szeged index for the non-commuting graph of a finite group.

Lemma 4. *Let G be a finite group. Then*

$$\sum_{x \notin Z(G)} |C_G(x)| = |G|(k(G) - |Z(G)|).$$

Proof. We know that G is the union of its conjugacy classes. Assume that $\{x_i\}_{i=1}^{k}$ are the representative of the conjugacy classes and $\mathrm{class}(x_i)$ denotes the conjugacy class of x_i and $G = \bigcup\limits_{i=1}^{k} \mathrm{class}(x_i)$.
Now, let $\{x_i\}_{i=1}^{t} \notin Z(G)$, thus we have $k(G) = t + |Z(G)|$. Every x which is not

placed within $Z(G)$ would be placed within one of class$(x_i)s$ in which $1 \leq i \leq t$. Therefore we have:

$$\sum_{x \notin Z(G)} |C_G(x)| = \sum_{i=1}^{t} |\operatorname{class}(x_i)||C_G(x_i)| = |G|t = |G|(k(G) - |Z(G)|).$$

$\square$

In the next theorem, we calculate the Szeged index of Γ_G.

Theorem 5. *Assume G is a finite group and Γ_G its non-commuting graph. Then the Szeged index of Γ_G is*

$$\begin{aligned} Sz(\Gamma_G) = \frac{1}{2}(&\sum_{i=1}^{n} \left(\sum_{x_j \notin C_G(x_i)} (|C_G(x_i) \cap C_G(x_j)|)^2 \right) \\ &+ \sum_{i=1}^{n} \deg(x_i) \left(\sum_{x_j \notin C_G(x_i)} +2|C_G(x_i) \cap C_G(x_j)| - |C_G(x_j)| \right) \\ &+ |G| \left(\sum_{i=1}^{n} \left(\sum_{x_j \notin C_G(x_i)} -2|C_G(x_i) \cap C_G(x_j)| + |C_G(x_j)| \right) \right)). \end{aligned}$$

Proof. Assume that x and y are two arbitrary vertices of the graph Γ_G that are joined together by e (where e is one of the edges of the non-commuting graph). Now we calculate $n_x(e|\Gamma_G)$ and $n_y(e|\Gamma_G)$:

$$N_x(e|\Gamma_G) = \{w \in V(\Gamma_G) : d(w,x) < d(w,y)\}.$$

According to Lemma 1, we have:
If $d(w,y) = 1$ then $d(w,x) = 0$ and $w = x$. If $d(w,y) = 2$ then $d(w,x) = 0$ or 1. So

$$\begin{aligned} n_x(e|\Gamma_G) &= (|C_G(y) - 1) - |C_G(x) \cap C_G(y)| + 1 \\ &= |C_G(y)| - |C_G(x) \cap C_G(y)| \end{aligned}$$

In order to

$$\begin{aligned} n_y(e|\Gamma_G) &= (|C_G(x)| - 1) - |C_G(x) \cap C_G(y)| + 1 \\ &= |C_G(x)| - |C_G(x) \cap C_G(y)|. \end{aligned}$$

$$\begin{aligned}
Sz(\Gamma_G) &= \sum_{e=xy\in E} n_x(e|\Gamma_G)\cdot n_y(e|\Gamma_G)\\
&= \sum_{e=xy\in E} (|C_G(y)| - |C_G(x)\cap C_G(y)|)(|C_G(x)| - |C_G(x)\cap C_G(y)|)\\
&= \sum_{e=xy\in E} |C_G(x)\cap C_G(y)|^2\\
&- \sum_{e=xy\in E} (|C_G(x)| + |C_G(y)|)|C_G(x)\cap C_G(y)| + \sum_{e=xy\in E} |C_G(x)||C_G(y)|
\end{aligned}$$

Now, we have to calculate the all of summations.
Letting $|G| - |Z(G)| = n$, we obtain $\sum\limits_{e=xy\in E} |C_G(x)\cap C_G(y)|^2$:

$$\sum_{e=xy\in E} |C_G(x)\cap C_G(y)|^2 = \frac{1}{2}\sum_{i=1}^{n}\left(\sum_{x_j\notin C_G(x_i)} (|C_G(x_i)\cap C_G(x_j)|)^2\right)$$

So we can gain $\sum\limits_{e=xy\in E} (|C_G(x)| + |C_G(y)|)|C_G(x)\cap C_G(y)|$,

$$\sum_{e=xy\in E} (|C_G(x)| + |C_G(y)|)|C_G(x)\cap C_G(y)| = \sum_{\substack{x\in G-Z(G)\\ y\notin C_G(x)}} |C_G(x)||C_G(x)\cap C_G(y)|$$

$$\begin{aligned}
&= \sum_{i=1}^{n} |C_G(x_i)|\left(\sum_{x_j\notin C_G(x_i)} |C_G(x_i)\cap C_G(x_j)|\right)\\
&= \sum_{i=1}^{n} (|G| - \deg(x_i))\left(\sum_{x_j\notin C_G(x_i)} |C_G(x_i)\cap C_G(x_j)|\right)\\
&= \sum_{i=1}^{n} -\deg(x_i)\left(\sum_{x_j\notin C_G(x_i)} |C_G(x_i)\cap C_G(x_j)|\right)\\
&+ |G|\sum_{i=1}^{n}\left(\sum_{x_j\notin C_G(x_i)} |C_G(x_i)\cap C_G(x_j)|\right).
\end{aligned}$$

Now, calculating $\sum\limits_{e=xy\in E} |C_G(x)||C_G(y)|$.

$$\sum_{e=xy\in E} |C_G(x)||C_G(y)| = \frac{1}{2}\sum_{i=1}^{n}\left(|C_G(x_i)|\sum_{x_j\notin C_G(x_i)} |C_G(x_j)|\right)$$

$$= \frac{1}{2}\sum_{i=1}^{n}\left((|G| - \deg(x_i)) \sum_{x_j \notin C_G(x_i)} |C_G(x_j)|\right)$$
$$= -\frac{1}{2}\sum_{i=1}^{n}\left(\deg(x_i) \sum_{x_j \notin C_G(x_i)} |C_G(x_j)|\right)$$
$$+ \frac{|G|}{2}\sum_{i=1}^{n}\left(\sum_{x_j \notin C_G(x_i)} |C_G(x_j)|\right).$$

Now, the Szeged index is equal to

$$Sz(\Gamma_G) = \sum_{e=xy\in E} |C_G(x) \cap C_G(y)|^2 - \sum_{e=xy\in E} (|C_G(x)| + |C_G(y)|)|C_G(x) \cap C_G(y)|$$
$$+ \sum_{e=xy\in E} |C_G(x)||C_G(y)|$$
$$= \frac{1}{2}\sum_{i=1}^{n}\left(\sum_{x_j \notin C_G(x_i)} (|C_G(x_i) \cap C_G(x_j)|)^2\right)$$
$$+ \sum_{i=1}^{n} \deg(x_i)\left(\sum_{x_j \notin C_G(x_i)} |C_G(x_i) \cap C_G(x_j)|\right)$$
$$- |G|\sum_{i=1}^{n}\left(\sum_{x_j \notin C_G(x_i)} |C_G(x_i) \cap C_G(x_j)|\right)$$
$$- \frac{1}{2}\sum_{i=1}^{n}\left(\deg(x_i) \sum_{x_j \notin C_G(x_i)} |C_G(x_j)|\right)$$
$$+ \frac{|G|}{2}\sum_{i=1}^{n}\left(\sum_{x_j \notin C_G(x_i)} |C_G(x_j)|\right)$$
$$= \frac{1}{2}(\sum_{i=1}^{n}\left(\sum_{x_j \notin C_G(x_i)} (|C_G(x_i) \cap C_G(x_j)|)^2\right)$$
$$+ \sum_{i=1}^{n} \deg(x_i)\left(\sum_{x_j \notin C_G(x_i)} +2|C_G(x_i) \cap C_G(x_j)| - |C_G(x_j)|\right)$$

$$+\,|G|\left(\sum_{i=1}^{n}\left(\sum_{x_j\notin C_G(x_i)} -2|C_G(x_i)\cap C_G(x_j)|+|C_G(x_j)|\right)\right)).$$

$\square$

3. The Edge-Wiener Index of a Non-Commuting Graph

In this section, we find the Edge-Wiener index of a non-commuting graph. We start with a couple of lemmas.

Lemma 6. *[2]. Assume G is a finite group and Γ_G its non-commuting graph. If $\overline{\Gamma_G}$ is a line-graph then,*

$$|V(\overline{\Gamma_G})|=|E(\Gamma_G)| \quad , \quad |E(\overline{\Gamma_G})|=\sum_{x\in V(\Gamma_G)}\binom{\deg(x)}{2}.$$

Lemma 7. *Assume G is a finite group and $\overline{\Gamma_G}$ a line-graph of Γ_G. Then $\overline{\Gamma_G}$ is a connected graph and* $\operatorname{diam}(\overline{\Gamma_G})=2$.

Proof. First, we prove that there is a path between two vertices of $\overline{\Gamma_G}$. Assume that two arbitrary vertices e and f belong to $\overline{\Gamma_G}$, thus e is an edge in Γ_G, so there are two vertices x and y of Γ_G that are joined together by e. Furthermore, there are two vertices x_1 and y_1 that are connected together by f. We know that $\operatorname{diam}(\Gamma_G)=2$, thus there is at least an edge between all mentioned vertices. It means: there is a path between two edges.
Now, we prove that $\operatorname{diam}(\overline{\Gamma_G})=2$. Suppose that $\operatorname{diam}(\overline{\Gamma_G})=1$, then $\overline{\Gamma_G}$ is a complete graph. Next

$$\exists\ x\in G\ \ni\ x\neq x^{-1}\Rightarrow \exists\ y\in G\ \ni\ x\xrightarrow{e} y\xrightarrow{f} x^{-1}$$

$$G\neq C_G(x)\cup C_G(y)\Rightarrow \exists\ z\in G-C_G(x)\cup C_G(y).$$

Therefore, we have $z\xrightarrow{h} x\xrightarrow{e} y\xrightarrow{g} z$, but $\overline{\Gamma_G}$ is a complete graph, so h and f are joined together, which is impossible. Since $z\neq x,y$ and $x\neq y,x^{-1}$. Thus $\operatorname{diam}(\overline{\Gamma_G})\neq 1$. Hence $\operatorname{diam}(\overline{\Gamma_G})=2$. $\square$

Theorem 8. *Let G be a finite group and $\overline{\Gamma_G}$ a line-graph of Γ_G. Then*

$$W_e(\overline{\Gamma_G})=|E(\Gamma_G)|^2+|G|^2\left(k(G)-\frac{1}{2}|Z(G)|-\frac{1}{2}|G|\right)-\frac{1}{2}\sum_{x\in G-Z(G)}|C_G(x)|^2.$$

Proof. By definition, $W_e(\overline{\Gamma_G}) = \sum_{\{e,f\}\subseteq E(\Gamma_G)} d(e,f) = \frac{1}{2}\sum_{e\in E(\Gamma_G)} d(e)$ where

$$d(e) = \sum_{f\in E(\Gamma_G)} d(e,f).$$

First we compute $d(e)$ for an arbitrary vertex of the graph $\overline{\Gamma_G}$. According to Lemma 7, $d(e) = \sum_{f\in E(\Gamma_G)} d(e,f) = 2$ (the number of vertices whose distance from e is 2)+1 (the number of vertices whose distance from e is 1). Let x and y be joined together by e. Then

$$\begin{aligned}
d(e) &= \sum_{f\in E(\Gamma_G)} d(e,f) \\
&= 1((\deg(x)-1)+(\deg(y)-1)) + 2(|E(\Gamma_G)| - \deg(x) - \deg(y) + 1) \\
&= 2|E(\Gamma_G)| - (\deg(x)+\deg(y)).
\end{aligned}$$

Using the above formula, we can calculate $W_e(\overline{\Gamma_G})$:

$$\begin{aligned}
W_e(\overline{\Gamma_G}) &= \frac{1}{2}\sum_{e\in E(\Gamma_G)} d(e) \\
&= \frac{1}{2}\sum_{e\in E(\Gamma_G)} 2|E(\Gamma_G)| - (\deg(x)+\deg(y)) \\
&= |E(\Gamma_G)|^2 - \frac{1}{2}\sum_{e\in E(\Gamma_G)} (\deg(x)+\deg(y)) \\
&= |E(\Gamma_G)|^2 - \frac{1}{2}\sum_{x\in G-Z(G)} (\deg(x))^2 \\
&= |E(\Gamma_G)|^2 - \frac{1}{2}\sum_{x\in G-Z(G)} (|G| - |C_G(x)|)^2 \\
&= |E(\Gamma_G)|^2 - \frac{1}{2}|G|^2(|G| - |Z(G)|) + |G|\sum_{x\in G-Z(G)} |C_G(x)| \\
&\quad - \frac{1}{2}\sum_{x\in G-Z(G)} |C_G(x)|^2 \\
&= |E(\Gamma_G)|^2 - \frac{1}{2}|G|^2(|G| - |Z(G)|) + |G|^2(k(G) - |Z(G)|) \\
&\quad - \frac{1}{2}\sum_{x\in G-Z(G)} |C_G(x)|^2
\end{aligned}$$

$$= |E(\Gamma_G)|^2 + |G|^2 \left(k(G) - \frac{1}{2}|Z(G)| - \frac{1}{2}|G| \right) - \frac{1}{2} \sum_{x \in G - Z(G)} |C_G(x)|^2.$$

□

4. The First Zagreb Index of a Non-Commuting Graph

In this section, the first Zagreb index of a non-commuting graph is computed.

Theorem 9. *Let G be a finite group and Γ_G its non-commuting graph. Then*

$$Z_1(\Gamma_G) = |G|^2(|G| + |Z(G)| - 2k(G)) + \sum_{x \in G - Z(G)} |C_G(x)|^2.$$

Proof. Using the definition of $Z_1(\Gamma_G)$, we have

$$\begin{aligned} Z_1(\Gamma_G) &= \sum_{x \in G - Z(G)} \deg(x)^2 \\ &= \sum_{x \in G - Z(G)} (|G| - |C_G(x)|)^2 \\ &= |G|^2(|G| - |Z(G)|) - 2|G| \sum_{x \in G - Z(G)} |C_G(x)| + \sum_{x \in G - Z(G)} |C_G(x)|^2 \\ &= |G|^2(|G| + |Z(G)| - 2k(G)) + \sum_{x \in G - Z(G)} |C_G(x)|^2. \end{aligned}$$

□

5. The Second Zagreb Index of a Non-Commuting Graph

In this section, we calculate the second Zagreb index of a non-commuting graph.

Theorem 10. *Let G be a finite group and Γ_G its non-commuting graph. Then*

$$Z_2(\Gamma_G) = \frac{1}{2} \left(|G|^2(|G| - k(G))^2 + |G|^2(k(G) - |Z(G)|) - \sum_{1 \le i \le n} |C_G(x_i)|^2 \right).$$

Proof. Assume that x is an arbitrary vertex of Γ_G and is fixed. Now be calculated $\sum_{x\neq y\in G-Z(G)} \deg(x)\cdot\deg(y)$:

$$\begin{aligned}
\sum_{x\neq y\in G-Z(G)} \deg(x)\cdot\deg(y) &= \sum_{x\neq y\in G-Z(G)} (|G|-|C_G(x)|)(|G|-|C_G(y)|) \\
&= \sum_{x\neq y\in G-Z(G)} (|G|-|C_G(x)|)|G| \\
&- \sum_{x\neq y\in G-Z(G)} (|G|-|C_G(x)|)|C_G(y)| \\
&= |G|(|G|-|Z(G)|-1)(|G|-|C_G(x)|) \\
&- (|G|-|C_G(x)|) \sum_{x\neq y\in G-Z(G)} |C_G(y)|.
\end{aligned}$$

We know that $\sum_{y\in G-Z(G)} |C_G(y)| = |G|(k(G)-|Z(G)|)$, thus $\sum_{x\neq y\in G-Z(G)} |C_G(y)|$. Can be found as follows:
$\sum_{x\neq y\in G-Z(G)} |C_G(y)| = |G|(k(G)-|Z(G)|) - |C_G(x)|$ where

$$\sum_{x\neq y\in G-Z(G)} \deg(x)\cdot\deg(y) = (|G|-|C_G(x)|)(|G|(|G|-k(G)) + |C_G(x)| - |G|).$$

Next, we calculate the second Zagreb index of the non-commuting graph. Let $G - Z(G) = \{x_1, x_2, \ldots, x_n\}$. Then

$$\begin{aligned}
Z_2(\Gamma_G) &= \sum_{\{x,y\}\subseteq V} \deg(x)\cdot\deg(y) \\
&= \frac{1}{2}\Big(\sum_{x_1\neq y\in G-Z(G)} \deg(x_1)\cdot\deg(y) \\
&+ \sum_{x_2\neq y\in G-Z(G)} \deg(x_2)\cdot\deg(y) \\
&+ \ldots + \sum_{x_n\neq y\in G-Z(G)} \deg(x_n)\cdot\deg(y)\Big) \\
&= \frac{1}{2}[(|G|-|C_G(x_1)|)(|G|(|G|-k(G)) + |C_G(x_1)| - |G|) \\
&+ (|G|-|C_G(x_2)|)(|G|(|G|-k(G)) + |C_G(x_2)| - |G|) \\
&+ \ldots + (|G|-|C_G(x_n)|)(|G|(|G|-k(G)) + |C_G(x_n)| - |G|)]
\end{aligned}$$

$$= \frac{1}{2}(|G|^2(|G| - k(G))(|G| - |Z(G)|) - |G|(|G| - k(G)) \sum_{1\leq i\leq n} |C_G(x_i)|$$
$$+ |G| \sum_{1\leq i\leq n} |C_G(x_i)| - \sum_{1\leq i\leq n} |C_G(x_i)|^2$$
$$- |G|^2(|G| - |Z(G)|) + |G| \sum_{1\leq i\leq n} |C_G(x_i)|)$$
$$= \frac{1}{2}\left(|G|^2(|G| - k(G))^2 + |G|^2(k(G) - |Z(G)|) - \sum_{1\leq i\leq n} |C_G(x_i)|^2\right).$$

□

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