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Siti Norziahidayu Amzee Zamri, Nor Haniza Sarmin, and Mustafa Anis El-Sanfaz

The Conjugate and Generalized Conjugacy Class Graphs for Metacyclic 3-Groups and Metacyclic 5-Groups

Siti Norziahidayu Amzee Zamri[1, a)], Nor Haniza Sarmin[1, b)] and Mustafa Anis El-Sanfaz[2, c)]

[1]*Department of Mathematical Sciences, Faculty of Science, Universiti Teknologi Malaysia, 81310 UTM Johor Bahru, Johor.*
[2]*Department of Mathematics, Faculty of Science, University of Benghazi, Libya.*

[a)]Corresponding author: norzisan@gmail.com
[b)]nhs@utm.my
[c)]kabeto_sanfaz@yahoo.com

Abstract. Let G be a metacyclic *p*-group where *p* is either 3 or 5, and let Ω be the set of all ordered pairs (x,y) in $G\times G$ such that $lcm(|x|,|y|)=p,\ xy=yx$ and $x\neq y$. In this paper, the conjugate graphs associated to metacyclic 3-groups and metacyclic 5-groups are found. Besides, the generalized conjugacy class graphs of these groups are also determined. We show that the conjugate graphs for both groups are a union complete graph and empty graph. Meanwhile, the generalized conjugacy class graphs for both groups are complete and null graph.

INTRODUCTION

A graph is a representation of a set of objects or vertices which are connected by links or edges. Graph theory can be applied in many fields such as atomic bonding, wireless networking and material physics. Besides, graph theory can also be related with group theory in the study of algebraic graph theory. The elements of the groups will be the vertices while the edges will be depending on the properties or behavior of the group elements.

First, we give some basic definitions on graph theory started with the definition of a graph. A graph, denoted as Γ, is a mathematical structure containing two sets, vertices and edges which are denoted by $V(\Gamma)$ and $E(\Gamma)$, respectively. We called a graph Γ as a connected graph if there exists a path between every pair of distinct vertices, and it is disconnected otherwise. Sub graphs, $\Gamma(V_1),\Gamma(V_2),...,\Gamma(V_n)$ are all components of Γ. The graph is connected if it has precisely one component. Moreover, a graph is said to be complete if each ordered pair of vertices are adjacent to each other, which is denoted by K_n, where *n* is the number of vertices. Also, the graph is empty if there is no edge between its vertices. In this paper, we denote empty graph of n nodes as $\overline{K}_n$. Additionally, a graph is called null if it has no vertices, which is denoted by K_0, [1,2]. Throughout this paper, the conjugate graph and generalized conjugacy class graph of metacyclic 3-groups and metacyclic 5-groups will be determined. We denote the generalized conjugacy class graph of a finite non-abelian group G as $\Gamma_G^{\Omega_C}$, with vertex set $K(\Omega)-|A|$, where A is $\{g\omega=\omega g:\omega\in\Omega\}$. Two vertices in $\Gamma_G^{\Omega_C}$ are connected by an edge if their cardinalities are not coprime [3]. The conjugate graph, denoted by Γ_G^C, is a graph of a finite non-abelian group G with vertex set $G\setminus Z(G)$ such that two distinct vertices join by an edge if they are conjugate [4].

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This paper is divided into three sections. The first section describes some basic concepts in graph theory. Next, the second section focuses on the metacyclic group and its presentation. Some previous research on the generalization of commutativity degree and earlier research on graphs related to groups are also provided. In the third section, our main results which are the conjugate and generalized conjugacy class graphs for metacyclic 3-groups and metacyclic 5-groups are presented.

PRELIMINARIES

In this section, we give some previous research on graphs related to groups, in general denoted as Γ_G since our concern is on metacyclic 3-group and metacyclic 5-group, we provide the definition of metacyclic group and its presentation.

Definition 1. [5] Let G be a group. G is called metacyclic if it has a normal cyclic subgroup N such that the factor group G/N is also cyclic.

An example of metacyclic group is dihedral group. Throughout this paper, we use the presentation of metacyclic p-groups, where p is an odd prime given by Basri [6] as in the following theorem:

Theorem 1. [6] Let G be a non abelian metacyclic p-group. Then G is one of the following:

(Type 1) $G \cong \left\langle a,b \middle| a^{p^{\alpha}} = b^{p^{\beta}} = 1, [b,a] = a^{p^{\alpha-\delta}} \right\rangle$ where p is an odd prime, $\alpha,\beta,\delta \in \mathrm{N}, \delta \le \alpha < 2\delta, \delta \le \beta, \delta \le \min\{\alpha-1,\beta\}$.

(Type 2) $G \cong \left\langle a,b \middle| a^{p^{\alpha}} = 1, b^{p^{\beta}} = a^{p^{\alpha-\varepsilon}}, [b,a] = a^{p^{\alpha-\delta}} \right\rangle$ where p is an odd prime, $\alpha,\beta,\delta,\varepsilon \in \mathrm{N}, \delta+\varepsilon \le \alpha < 2\delta, \delta \le \beta, \alpha < \beta+\varepsilon, \delta \le \min\{\alpha-1,\beta\}$.

Throughout the paper, we only consider the metacyclic p-groups of Type 1. Since our work is relating group theory to graph theory, we first provide some previous works on group theory, more precisely in the commutativity degree and the probability that an element of a group fixes a set. The probability that two random elements in a group commute is called commutativity degree [7]. In 2013, Omer *et al.* [8] extended this concept and introduced a new probability called the probability that an element of a group fixes a set, as given in the following theorems:

Theorem 2. [8] Let G be a finite group. Let S be the subset of $G \times G$ in the form of (x,y), in which $|x| = |y| = 2$, where $xy = yx$ and G acts on S by conjugation. Then the probability that an element of the group G fixes the set S is given by $P_G(S) = \dfrac{K}{|S|}$, where K is the number of conjugacy classes of S in G.

The probability that an element of a group fixes a set has been found for some finite non-abelian groups which included dihedral groups, metacyclic 2-groups, symmetric groups, quaternion groups and alternating groups [9-12].

Recently, Zamri *et al.* [13,14] has found the probability that a metacyclic 3-group and metacyclic 5-group fixes a set by conjugation action. The results are given in the following theorems:

Theorem 3. [13] Let G be a metacyclic 3-group such that $G \cong \left\langle a,b \middle| a^{5^{\alpha}} = b^{5^{\beta}} = 1, [b,a] = a^{5^{\alpha-\delta}} \right\rangle$ where $\alpha,\beta,\delta \in \mathbb{N}$, $\delta \le \alpha < 2\delta$, $\delta \le \beta$, $\delta \le \min\{\alpha-1,\beta\}$. Let Ω be the set of ordered pairs (x,y) in $G \times G$ such that $lcm(|x|,|y|) = 3$, $xy = yx$ and $x \ne y$. If G acts on Ω by conjugation, then $P_G(\Omega) = \begin{cases} \dfrac{7}{18}, & \text{when } \alpha > \beta = \delta, \\ 1, & \text{otherwise.} \end{cases}$

Theorem 4. [14] Let G be a metacyclic 5-group such that $G \cong \left\langle a,b \middle| a^{5^{\alpha}} = b^{5^{\beta}} = 1, [b,a] = a^{5^{\alpha-\delta}} \right\rangle$ where $\alpha,\beta,\delta \in \mathbb{N}$, $\delta \le \alpha < 2\delta$, $\delta \le \beta$, $\delta \le \min\{\alpha-1,\beta\}$. Let Ω be the set of ordered pairs (x,y) in $G \times G$ such that $lcm(|x|,|y|) = 5$, $xy = yx$ and $x \ne y$. If G acts on Ω by conjugation, then $P_G(\Omega) = \begin{cases} \frac{17}{75}, & \text{when } \alpha > \beta = \delta, \\ 1, & \text{otherwise.} \end{cases}$

In this paper, we used the probability given in Theorem 3 and Theorem 4 to determine the conjugate and generalized conjugacy class graphs of metacyclic 3-groups and metacyclic 5-groups.

Next, some previous works on graphs related to groups are given. In 1990, Bertram *et al*. [15] introduced a graph related to conjugacy class graph where the vertices of this graph are non-central conjugacy classes and two vertices are adjacent if their cardinalities are not coprime. Later on, Bianchi *et al.* [16] investigated the regularity of the graph related to conjugacy classes. In 2005, Moreto *et al.* [17] classified the groups in which the size of conjugacy classes are not coprimes for any five distinct classes. Later on, Moradipour *et al.* [18] used the graph related to conjugacy classes to find the graph properties on some finite metacyclic 2-groups. In the same year, Illangovan and Sarmin [19] found some graphs related to conjugacy classes for some 2-groups. In 2012, Erfanian and Tolue [4] introduced the conjugate graph in which two vertices of this graph are connected if they are conjugate. The conjugate graph is found for some finite non-abelian groups. In 2015, Omer *et al.* [3] introduced the generalized conjugacy class graph in which the vertices are orbits under the group action on set.

MAIN RESULTS

In this section, our results are presented, starting with the conjugate graph of metacylic 3-groups and metacyclic 5-groups, followed by the generalized conjugacy class graph of the same groups.

The conjugate graph of metacyclic 3-groups and metacyclic 5-groups are given in the following theorems:

Theorem 5. Let G be a metacyclic 3-group such that $G \cong \left\langle a,b \middle| a^{3^{\alpha}} = b^{3^{\beta}} = 1, [b,a] = a^{3^{\alpha-\delta}} \right\rangle$ where $\alpha,\beta,\delta \in \mathbb{N}$, $\delta \le \alpha < 2\delta$, $\delta \le \beta$, and $\delta \le \min\{\alpha-1,\beta\}$. Let Ω be the set of all ordered pairs (x,y) in $G \times G$ such that $lcm(|x|,|y|) = 3$, $xy = yx$ and $x \ne y$. If G acts on Ω by conjugation, then the conjugate graph of *G*,

$$\Gamma_G^C = \begin{cases} \bigcup_{i=1}^{11} K_3 \cup \overline{K}_3, & \text{when } \alpha > \beta = \delta \\ \overline{K}_{36}, & \text{otherwise.} \end{cases}$$

Proof: From the proof of Theorem 2, the set Ω consists of 36 elements. Thus by the definition of the conjugate graph, there are 36 vertices. From Theorem 2, we have 2 cases. Since the action is by conjugation, the orbit is considered as the conjugacy class. In Case 1, we have 3 orbits of size 1 and 11 orbits of size 3. By definition, the vertices are adjacent if they are conjugate. Thus, we have three isolated vertices or empty graph of three nodes and 11 complete graph of size three. In Case 2, all 36 vertices are isolated vertices since their orbits are of size 1. Therefore, the conjugate graph of metacyclic 3-groups, $\Gamma_G^C = \begin{cases} \bigcup_{i=1}^{11} K_3 \cup \overline{K}_3, & \text{when } \alpha > \beta = \delta \\ \overline{K}_{36}, & \text{otherwise.} \end{cases}$

Theorem 6. Let G be a metacyclic 5-group such that $G \cong \left\langle a,b \middle| a^{5^{\alpha}} = b^{5^{\beta}} = 1, [b,a] = a^{5^{\alpha-\delta}} \right\rangle$ where $\alpha,\beta,\delta \in \mathbb{N}$, $\delta \le \alpha < 2\delta$, $\delta \le \beta$, and $\delta \le \min\{\alpha-1,\beta\}$. Let Ω be the set of all ordered pairs (x,y) in $G \times G$ such that $lcm(|x|,|y|) = 5$, $xy = yx$ and $x \ne y$. If G acts on Ω by conjugation, then the conjugate graph of G,

$$\Gamma_G^C = \begin{cases} \bigcup_{i=1}^{58} K_5 \cup \overline{K}_{10}, & \text{when } \alpha > \beta = \delta \\ \overline{K}_{300}, & \text{otherwise.} \end{cases}$$

Proof: From the proof of Theorem 3, the set Ω consists of 300 elements. Thus by definition of the conjugate graph, there are 300 vertices. From Theorem 3, there are two cases. Since the action is by conjugation, the orbit is considered as the conjugacy class. In Case 1, we have 10 orbits of size 1 and 58 orbits of size 5. By definition, the vertices are adjacent if they are conjugate. Thus, we have 10 isolated vertices and 58 complete graphs of size five. In Case 2, all 300 vertices are isolated vertices since their orbits are of size 1. Therefore, the conjugate graph of metacyclic 5-groups, $\Gamma_G^C = \begin{cases} \bigcup_{i=1}^{58} K_5 \cup \overline{K}_{10}, & \text{when } \alpha > \beta = \delta \\ \overline{K}_{300}, & \text{otherwise.} \end{cases}$

Next, we determine the generalized conjugacy class graph for metacylic 3-groups and metacyclic 5-groups.

Theorem 7. Let G be a metacyclic 3-group such that $G \cong \left\langle a,b \middle| a^{3^{\alpha}} = b^{3^{\beta}} = 1, [b,a] = a^{3^{\alpha-\delta}} \right\rangle$ where $\alpha,\beta,\delta \in \mathbb{N}$, $\delta \le \alpha < 2\delta$, $\delta \le \beta$, and $\delta \le \min\{\alpha-1,\beta\}$. Let Ω be the set of all ordered pairs (x,y) in $G \times G$ such that $lcm(|x|,|y|) = 3$, $xy = yx$ and $x \ne y$. If G acts on Ω by conjugation, then the generalized conjugacy class graph of G, $\Gamma_G^{\Omega_C} = \begin{cases} K_{11}, & \text{when } \alpha > \beta = \delta \\ \text{null graph}, & \text{otherwise.} \end{cases}$

Proof: From the proof of Theorem 2, there are two cases. In Case 1, there are 14 orbits, three orbits of size one and 11 orbits of size three. By definition of generalized conjugacy class graph, the three orbits of size one are the elements of A, thus the number of vertices is 14-3=11. For the 11 orbits of size three, the greatest common divisor is not equal to one. Since the vertices are joined by an edge if their cardinalities are not coprime, thus the vertices are adjacent to each other. Therefore, we have K_{11}. In Case 2, we have 36 orbits of size one. By definition of generalized conjugacy class graph, the 36 orbits of size one are the elements of A, thus the number of vertices is 36-36=0. Thus, we have zero vertices, which is a null graph. Therefore, the generalized conjugacy class graph of metacyclic 3-groups, $\Gamma_G^{\Omega_C} = \begin{cases} K_{11}, & \text{when } \alpha > \beta = \delta \\ \text{null graph}, & \text{otherwise.} \end{cases}$

Theorem 8. Let G be a metacyclic 5-group such that $G \cong \left\langle a,b \middle| a^{5^{\alpha}} = b^{5^{\beta}} = 1, [b,a] = a^{5^{\alpha-\delta}} \right\rangle$ where $\alpha,\beta,\delta \in \mathbb{N}$, $\delta \le \alpha < 2\delta$, $\delta \le \beta$, $\delta \le \min\{\alpha-1,\beta\}$. Let Ω be the set of all ordered pairs (x,y) in $G \times G$ such that $lcm(|x|,|y|) = 5$, $xy = yx$ and $x \ne y$. If G acts on Ω by conjugation, then the generalized conjugacy class graph of G, $\Gamma_G^{\Omega_C} = \begin{cases} K_{58} & \text{when } \alpha > \beta = \delta \\ \text{null graph}, & \text{otherwise.} \end{cases}$

Proof: From the proof of Theorem 2, there are two cases. In Case 1, there are 68 orbits, 10 orbits of size one and 58 orbits of size five. By definition of generalized conjugacy class graph, the 10 orbits of size one are the elements of A, thus the number of vertices is 68-10=58. For the 58 orbits of size five, the greatest common divisor is not equal to one. Since the vertices are joined by an edge if their cardinalities are not coprime, thus the vertices are adjacent to each other. Therefore, we have K_{58}. In Case 2, we have 68 orbits of size one. By definition of generalized conjugacy class graph, the 68 orbits of size one are the elements of A, thus the number of vertices is 68-68=0. Thus, we have zero vertices, which is a null graph. Therefore, the generalized conjugacy class graph of metacyclic 5-groups, $\Gamma_G^{\Omega_C} = \begin{cases} K_{58}, \text{ when } \alpha > \beta = \delta \\ \text{null graph, otherwise.} \end{cases}$

CONCLUSION

In this paper, the conjugate graph and generalized conjugacy class graph for metacyclic 3-groups and metacyclic 5-groups of Type 1 are determined. The conjugate graphs for both groups are found to be a union of complete graphs and empty graphs for the first case, and empty graphs for the second case. Meanwhile, the generalized conjugacy class graphs for both groups are found to be a complete graph for the first case, and a null graph for the second case.

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