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On the Energy of Non-Commuting Graph of Dihedral Groups

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Abstract. In mathematics, the energy of a graph is the sum of the values of the eigenvalues of the adjacency matrix of the graph. This quantity is studied in the context of spectral graph theory. In this paper the concepts of non-commuting graph of dihedral groups are presented and the general formula for the energy of this associated graph is found.

INTRODUCTION

The study of algebraic structure using the properties of graphs has become an exciting research topic. There are many researches on assigning a graph to a group and investigation of algebraic properties of these graph in relation to the groups are well studied. For instance, Abdollahi *et al*. [1] studied on the non-commuting graph of finite group while Talibi [2] did the same study for the dihedral groups. In this paper, all graphs considered are assumed to be finite, simple and undirected.

Let □be a graph with vertex-set $V(\Gamma)=\{1,\ldots,n\}$ $E(\Gamma)=\{e_1,\ldots,e_n\}$. The adjacency matrix of Γ, denoted by $A(\Gamma)$, is a square matrix with rows and columns labeled by graph vertices, with a 1 or 0 in position (v_i,v_j) representing according whether v_i and v_j are adjacent or not. If the vertices are adjacent, then they are represented by the value 1, otherwise they are represented by 0. For a simple graph, the adjacency matrix must have $0's$ on the diagonal. For an undirected graph, the adjacency matrix is symmetric [3].

Let Γ be a simple graph, A be its adjacency matrix and $\lambda_1,\lambda_2,\ldots,\lambda_n$ be the eigenvalues of the graph Γ. By eigenvalues of the graph Γ we mean the eigenvalues of its adjacency matrix. The energy of Γ is defined as the sum of the absolute values of its eigenvalues [4]. The energy of graph definition was pioneered by Gutman [4] in 1978. It is applied in chemistry to approximate the total π- electron energy of molecules. The carbon atoms are represented by vertices of the graph while the single edge between each pair of distinct vertices represented the hydrogen bonds between the carbon atoms [5].

This paper consists of three sections. The first section is the introduction of the energy of the graph which is constructed by a group. The second section includes some concepts related to non-commuting graph which are used in this paper. Our main results are presented in the third section, in which we compute the eigenvalues and the energy of the non-commuting graphs of dihedral groups using the adjacency matrices of non-commuting graph of dihedral groups. Finally, the general formulas for the energy of non-commuting graph of dihedral groups of order $2n$ are found.

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PRELIMINARIES

In this section, some basic concepts and previous results on the adjacency matrix and the eigenvalues of the graph as well as the definitions of non-commuting graph of finite group and dihedral group are provided. The references can be found in [6,7,8].

First, the dihedral group D_{2n}, is the group of all symmetries of a regular polygon. The group is of order $2n$ where n is an integer, and it has a presentation as in the following:

$$D_{2n} \cong \langle a,b : a^n = b^2 = 1, bab = a^{-1} \rangle.$$

The dihedral group D_{2n} has $2n$ elements which are listed as below:

$$D_{2n} = \{1, a, a^2, \ldots, a^{n-1}, b, ab, a^2b, \ldots, a^{n-1}b\}.$$

Let *G* be a finite non-abelian group with the center denoted by $Z(G)$. A non-commuting graph is a graph whose vertices are non-central elements of *G* (i.e. $G - Z(G)$). Two vertices v_1 and v_2 are adjacent whenever $v_1v_2 \neq v_2v_1$. Let *A* be a $n \times n$ matrix, then the determinant of $\lambda I - A$, $\det(\lambda I - A)$, is a polynomial with the variable λ of degree *n* and is called the characteristic polynomial of *A*. The equation $\det(\lambda I - A) = 0$ is called the characteristic equation of *A*. By the fundamental theorem of algebra, the equation has *n* roots and these roots are called the eigenvalues of *A*.

A square matrix *A* is called symmetric if $A = A^T$. The eigenvalues of symmetric matrix are real and the rank of the matrix is equal to the number of nonzero eigenvalues with counting multiplicities.

MAIN RESULTS

In this section we obtain the eigenvalues of the non-commuting graph of dihedral groups and found the general formulas for the energy of non-commuting graph of dihedral groups D_{2n}. We break into two cases, namely when *n* is odd and when *n* is even.

Proposition 1 *Let G be a dihedral group of order 2n where n is an odd integer and let* $\Gamma_{D_{2n}}$ *be its non-commuting graph. Then the eigenvalues of* $\Gamma_{D_{2n}}$ *are as follows:*

$$\lambda = 0 \text{ with multiplicity } (n-2), \quad \lambda = -1 \text{ with multiplicity } (n-1), \text{ and } \lambda = \frac{n-1}{2} \pm \sqrt{\frac{5n^2 - 6n + 1}{4}}.$$

Proof: Consider the dihedral group $D_{2n} \cong \langle a,b : a^n = b^2 = 1, bab = a^{-1} \rangle = \{1, a, a^2, \ldots, a^{n-1}, b, ab, a^2b, \ldots, a^{n-1}b\}$. For *n* odd, $Z(D_{2n}) = \{1\}$, hence by the definition of the non-commuting graph of a finite group, the vertex-set of the graph $V(\Gamma_{D_{2n}}) = \{a, a^2, \ldots, a^{n-1}, b, ab, a^2b, \ldots, a^{n-1}b\}$. The elements $a, a^2, \ldots, a^{n-1}$ are pairwise commute but a^i do not commute with a^ib, for $i = 0,1,2,\ldots,n-1$, thus all vertices of the form $a^i, i = 1,2,\ldots,n-1$ and $a^ib, i = 0,1,2,\ldots,n-1$ are adjacent. Hence, the adjacency matrix of this graph is a $(2n-1)\times(2n-1)$ matrix given in the following:

$$A(\Gamma_{D_{2n}}) = \begin{bmatrix} O_{(n-1)\times(n-1)} & J_{(n-1)\times n} \\ J^T_{(n-1)\times n} & B_{n\times n} \end{bmatrix},$$

where $O_{(n-1)\times(n-1)}$ is the $(n-1)\times(n-1)$ zero matrix, $J_{(n-1)\times n}$ is an $(n-1)\times n$ matrix of all 1s and $B_{n\times n}$ is an $n \times n$ matrix, in which all elements in the diagonal are zeros and the remaining elements are 1's. Hence, the degree of the characteristic polynomial of this matrix is $(2n-1)$, and the rank of the adjacency matrix of non-commuting graph of dihedral group D_{2n} when *n* is an odd integer is equal to *n*+1. Therefore, we have *n*+1 nonzero eigenvalues of graph $\Gamma_{D_{2n}}$, when *n* is an odd integer. Thus we have $(2n-1)-(n+1)$ zero eigenvalues and this gives $\lambda = 0$ with multiplicity $(n-2)$. The other eigenvalues are $\lambda = -1$ with multiplicity $(n-1)$, and the characteristic equation of the adjacency matrix of the graph $\Gamma_{D_{2n}}$ when n is an odd integer is in the form $\lambda^{n-2}(\lambda+1)^{n-1}(\lambda^2-(n-1)\lambda-(n^2-n))=0$. Using the quadratic formula, we get that the last eigenvalues to be $\lambda=\frac{n-1}{2}\pm\sqrt{\frac{5n^2-6n+1}{4}}$.

Proposition 2 *Let G be a dihedral group of order 2n where n is an even integer and* $n>4$, *and let* $\Gamma_{D_{2n}}$ *be its non-commuting graph. Then the eigenvalues of* $\Gamma_{D_{2n}}$ *are as follows:*

$\lambda=0$ *(with multiplicity* $\frac{3n-6}{2}$ *),* $\lambda=-2$ *with multiplicity* $(\frac{n}{2}-1)$, *and* $\lambda=(\frac{n}{2}-1)\pm\sqrt{\frac{5n^2-12n+4}{4}}$.

Proof: Consider the dihedral group $D_{2n}\cong\langle a,b:a^n=b^2=1,bab=a^{-1}\rangle=\{1,a,a^2,...,a^{n-1},b,ab,a^2b,...,a^{n-1}b\}$. For n is even integer, $Z(D_{2n})=\left\{1,a^{\frac{n}{2}}\right\}$, hence by the definition of the non-commuting graph of a finite group, the vertex-set $V(\Gamma_{D_{2n}})$ has $2n$-2 elements and the adjacency matrix of the graph $\Gamma_{D_{2n}}$ when n is even integer and $n>4$ is a $(2n-2)\times(2n-2)$ matrix given in the following:

$$A\left(\Gamma_{D_{2n}}\right)=\begin{bmatrix} O_{(n-2)\times(n-2)} & J_{(n-2)\times n} \\ J^T_{(n-2)\times n} & B_{n\times n}\end{bmatrix},$$

where $O_{(n-2)\times(n-2)}$ is an $(n-2)\times(n-2)$ zero matrix, $J_{(n-2)\times n}$ is an $(n-2)\times n$ matrix of all 1's and $B_{n\times n}$ is an $n\times n$ matrix, all elements in the diagonal are zeros and the remaining elements are equal 0 or 1 according to the adjacent of the elements. Hence, the degree of the characteristic polynomial of this matrix is $(2n-2)$, and the rank of the adjacency matrix of non-commuting graph of dihedral group D_{2n} when n is even integer and $n>4$ is equal to n-2. Therefore, we have n-2 nonzero eigenvalues of graph $\Gamma_{D_{2n}}$, when n is even integer. Thus we have $(2n-2)-(n-2)$ zero eigenvalues and this gives $\lambda=0$ with multiplicity (n). The other eigenvalues are $\lambda=-2$ with multiplicity $\left(\frac{n}{2}-1\right)$, and the characteristic equation of the adjacency matrix of the graph $\Gamma_{D_{2n}}$ when is even and $n>4$ is in the form $\lambda^n(\lambda+2)^{\frac{n}{2}-1}(\lambda^2-(n-2)\lambda-(n^2-2n))=0$ and by using the quadratic formula we get that rest of the eigenvalues are $\lambda=(\frac{n}{2}-1)\pm\sqrt{\frac{5n^2-12n+4}{4}}$.

Next, we present our results on the energy of the non-commuting graph of dihedral groups, which are given in proposition and two theorems according to the cases. First is the case when $n=4$.

Proposition 3 *Let G be a dihedral group of order 8, i.e.* $G=D_8\cong\langle a,b:a^4=b^2=1,bab=a^3\rangle$ *and let* Γ_{D_8} *be its non-commuting graph. Then the energy of the graph* Γ_{D_8} *is* $\varepsilon\left(\Gamma_{D_8}\right)=8$.

Proof: Let $D_8=\{1,a,a^2,a^3,b,ab,a^2b,a^3b\}$, the center of $D_8, Z(D_8)=\{1,a^2\}$, hence by the definition of the non-commuting graph of finite group, the vertex-set of the graph Γ_{D_8} is $V\left(\Gamma_{D_8}\right)=\{a,a^3,b,ab,a^2b,a^3b\}$, where the elements a and a^3 commute, b and a^2b commute and ab and a^3b also commute. Hence the adjacency matrix of the graph Γ_{D_8} is

$$A\left(\Gamma_{D_8}\right)=\begin{bmatrix} 0&0&1&1&1&1\\ 0&0&1&1&1&1\\ 1&1&0&1&0&1\\ 1&1&1&0&1&0\\ 1&1&0&1&0&1\\ 1&1&1&0&1&0 \end{bmatrix}$$

and the characteristic equation of this matrix is $\lambda^6-12\lambda^4-16\lambda^3=0$. Thus the eigenvalues of the graph Γ_{D_8} are $\lambda=0$ with the multiplicity 3, $\lambda=-2$ with multiplicity 2 and $\lambda=4$. By the definition of the energy of the graph $\varepsilon\left(\Gamma_{D_8}\right)=3|0|+2|-2|+|4|=8$.

Theorem 1 *Let G be a dihedral group of order 2n, where n is an odd integer, i.e.* $G=D_{2n}\cong\left\langle a,b:a^n=b^2=1,bab=a^{-1}\right\rangle$ *and let* $\Gamma_{D_{2n}}$ *be its non-commuting graph. Then the energy of the graph* $\Gamma_{D_{2n}}$ *is* $\varepsilon\left(\Gamma_{D_{2n}}\right)=(n-1)+\sqrt{5n^2-6n+1}$.

Proof: From Proposition 1, the eigenvalues of the non-commuting graph of dihedral group D_{2n}, when n is odd integer, are as follows: $\lambda=0$ with multiplicity $(n-2)$, $\lambda=-1$ with multiplicity $(n-1)$ and $\lambda=\frac{n-1}{2}\pm\sqrt{\frac{5n^2-6n+1}{4}}$, hence by the definition of the energy of the graph we found that the energy of the non-commuting graph of dihedral group D_{2n} when n is odd integer is:

$$\varepsilon(\Gamma_{D_{2n}})=|0|+(n-1)|-1|+\left|\lambda=\frac{n-1}{2}\pm\sqrt{\frac{5n^2-6n+1}{4}}\right|=(n-1)+\sqrt{5n^2-6n+1}.$$

We illustrate the above theorem by the following example.

Example 1 Let G be a dihedral group of order 10, $G=D_{10}\cong\left\langle a,b:a^5=b^2=1,bab=a^{-1}\right\rangle$ and let $\Gamma_{D_{10}}$ be its non-commuting graph. The adjacency matrix of the graph $\Gamma_{D_{10}}$ is a 9×9 matrix $A(\Gamma_{D_{10}})$ and $\mathrm{rank}(A(\Gamma_{D_{10}}))=6$. Thus the characteristic equation of this matrix is: $\lambda^9-30\lambda^7-100\lambda^6-135\lambda^5-84\lambda^4-20\lambda^3=0$ and the eigenvalues of the graph $\Gamma_{D_{10}}$ are: $\lambda=0$ with multiplicity 3, $\lambda=-1$ with multiplicity 4 and $\lambda=2\pm 2\sqrt{6}$. Thus the energy of the non-commuting graph $\Gamma_{D_{10}}$ is $\varepsilon(\Gamma_{D_{10}})=4+4\sqrt{6}$.

Theorem 2 *Let G be a dihedral group of order 2n, where n is an even integer and* $n>4$*, i.e.* $G=D_{2n}\cong\left\langle a,b:a^n=b^2=1,bab=a^{-1}\right\rangle$ *and let* $\Gamma_{D_{2n}}$ *be its non-commuting graph. Then the energy of the graph* $\Gamma_{D_{2n}}$ *is* $\varepsilon\left(\Gamma_{D_{2n}}\right)=(n-2)+\sqrt{5n^2-12n+4}$.

Proof: From Proposition 1, the eigenvalues of the non-commuting graph of dihedral group D_{2n}, when n is even integer and $n>4$, are as follows: $\lambda=0$ with multiplicity (n), $\lambda=-2$ with multiplicity $(\frac{n}{2}-1)$ and $\lambda=\left(\frac{n}{2}-1\right)\pm\sqrt{\frac{5n^2-12n+4}{4}}$, hence by the definition of the energy of the graph we found that the energy of the non-commuting graph of dihedral group D_{2n} when n is even integer and $n>4$ is:

$$\varepsilon(\Gamma_{D_{2n}}) = |0| + \left(\frac{n}{2}-1\right)|-2| + \left|\lambda = \left(\frac{n}{2}-1\right) \pm \sqrt{\frac{5n^2-12n+4}{4}}\right| = (n-2) + \sqrt{5n^2-12n+4}.$$

Theorem 2 is illustrated in the following example.

Example 2 Let G be a dihedral group of order 12, $G = D_{12} \cong \langle a,b : a^6 = b^2 = 1, bab = a^{-1} \rangle$ and $\Gamma_{D_{12}}$ be its non-commuting graph. The adjacency matrix of the graph $\Gamma_{D_{12}}$ is a 10×10 matrix $A(\Gamma_{D_{12}})$ and $\mathrm{rank}(A(\Gamma_{D_{12}})) = 4$. Thus the characteristic equation of this matrix is: $\lambda^{10} - 36\lambda^8 - 112\lambda^7 - 96\lambda^6 = 0$ and the eigenvalues of the graph $\Gamma_{D_{12}}$ are: $\lambda = 0$ with multiplicity 6, $\lambda = -2$ with multiplicity 2 and $\lambda = 2 \pm 2\sqrt{7}$. Hence the energy of the non-commuting graph $\Gamma_{D_{12}}$ is $\varepsilon(\Gamma_{D_{12}}) = 4 + 4\sqrt{7}$.

CONCLUSION

In this paper, the general formulas for the energy of non-commuting graph of all dihedral groups are found. For n an odd integer, the energy of the non-commuting graph, $\varepsilon\left(\Gamma_{D_{2n}}\right) = (n-1) + \sqrt{5n^2-6n+1}$ while for n even integer and $n > 4$, $\varepsilon\left(\Gamma_{D_{2n}}\right) = (n-2) + \sqrt{5n^2-12n+4}$ and when $n = 4$, $\varepsilon\left(\Gamma_{D_8}\right) = 8$.

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