Spectrum of cayley graphs of dihedral groups and their energy

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Abstract: Let G be a finite group and S be a subset of G where S does not include the identity of G and is inverse closed. A Cayley graph of G with respect to S is a graph where its vertices are the elements of G and two vertices a and b are connected if ab^{-1} is in the subset S. The energy of a Cayley graph is the sum of all absolute values of the eigenvalues of its adjacency matrix. In this paper, we consider a specific subset $S = \{a^{n/2}, b\}$ of dihedral group of order $2n$, where $n \geq 4$ and n is even and find the Cayley graph with respect to the set. We also calculate their spectrum and compute the energy of the respected Cayley graphs. Finally, the generalization of the Cayley graphs and their energies are found.

Keywords: Energy of graph, eigenvalues, adjacency matrix, Cayley graph, Dihedral group.

1 Introduction

The study on the spectrum of Cayley graphs by using algebraic graph theory was first initiated by Babai [1] in 1979 and the method has been developed by many researchers. One of the many researchers, Cvetkovic [2] has also studied on the spectrum of graphs. One of his results which is on Cayley graphs of dihedral groups gives the following; if all eigenvalues of $\Gamma = Cay(D_{2n}, S)$ are symmetric with respect to the origin, then the Γ is bipartite. Although, there are also other approaches to find the spectrum of a graph where Diaconis and Shahshahani [3] have calculated the spectrum of Cayley graphs by using the character table of the groups.

Besides, the study on the energy of general simple graphs was first defined by Gutman in 1978 inspired from the Huckel Molecular Orbital (HMO) Theory proposed in 1930s. The Huckel Molecular Orbital Theory has been used by chemists in approximating the energies related with $\pi-$electron orbitals in conjugated hydrocarbon molecules [4]. In the early days, when computers were not widely available, the calculation of the HMO total $\pi-$electron energy was a serious problem. In order to overcome the difficulty, a variety of approaches have been offered to calculate the approximate calculation of the $\pi-$electron energy. Within the HMO approximation, the total energy of the $\pi-$electrons, denoted by ϵ is obtained by summing distinct electron energies. In conjugated hydrocarbons, the total number of $\pi-$electrons is equal to the number of vertices of the associated molecular graph. After some considerations, they arrived at the definition of the energy which is the sum of the absolute values of the eigenvalues of the graph [5].

Many researchers have studied on the topic of energy of graphs. In 2004, Bapat and Pati [6] have proved that the energy of a graph is never an odd integer. Meanwhile, the properties that the energy of a graph is never the square root of an odd integer has been proven by Pirzada and Gutman [7] in 2008. There are also a few other researchers who studied specifically on the energy of unitary Cayley graphs (see [8],

[9]).

Besides, there have also been many studies on the Cayley graphs for dihedral groups. In 2006, Wang and Xu [10] have considered the non-normal one-regular and 4−valent Cayley graphs of dihedral groups while Kwak and Oh [11] have classified the 4−valent and 6−valent one regular normal Cayley graphs of dihedral groups whose vertex stabilizers in $Aut(\Gamma)$ are cyclic. In addition, in 2008, Kwak *et al.*[12] have explored on the one-regular Cayley graphs on dihedral groups of any prescribed valency. Kim *et al.* [13] also have studied on the Cayley graphs of dihedral groups on the classification of $p-$valent regular Cayley graphs.

The objective of this study is to calculate the energy of the Cayley graphs associated to dihedral groups for the subset $S = \{a^{n/2}, b\}$ where $n \geq 3$ and n is even. The methodology includes the finding of elements, vertices and edges for the Cayley graphs of the dihedral groups, finding their isomorphism, building the adjacency matrix for the Cayley graph, finding the spectrum of the adjacency matrix of the graphs and lastly calculating the energy of the graphs.

2 Preliminaries

Definition 2.1 (14). **Dihedral Group**
If π_n is a regular polygon with n vertices, $v_1, v_2, \ldots, v_n$ and center O, then the symmetry group $\sum(\pi n)$ is called the dihedral group with $2n$ elements, and it is denoted by D_{2n}.

Definition 2.2 (15). **Cayley Graph of a Group**
Let G be a finite group with identity 1. Let S be a subset of G satisfying $1 \notin S$ and $S = S^{-1}$; that is, $s \in S$ if and only if $s^{-1} \in S$. The Cayley graph $Cay(G; S)$ on G with connection set S is defined as follows:

- the vertices are the elements of G
- there is an edge joining g and h if and only if $h = sg$ for some $s \in S$.

The set of all Cayley graphs on G is denoted by $Cay(G)$.

Definition 2.3 (16). **Adjacency Matrix**
Let Γ be a graph with $V(\Gamma) = \{1, \ldots, n\}$ and $E(\Gamma) = \{e_1, \ldots, e_m\}$. The adjacency matrix of Γ denoted by $A(\Gamma)$ is the $n \times n$ matrix defined as follows. The rows and the columns of $A(\Gamma)$ are indexed by $V(\Gamma)$. If $i \neq j$ then the $(i, j)-$entry of $A(\Gamma)$ is 0 for vertices i and j nonadjacent, and the $(i, j)-$entry is 1 for i and j adjacent. The $(i, i)-$entry of $A(\Gamma)$ is 0 for $i = 1, \ldots, n$. $A(\Gamma)$ is often simply denoted by A.

Proposition 2.1 (17). *Consider the undirected n-cycle C_n . The spectrum consists of the numbers* $\{2cos(\frac{2\pi j}{n})\}; j = \{0, 1, \ldots, n-1\}$

Definition 2.4 (16). **Energy of Graph**
For any graph Γ, the energy of the graph is defined as $\varepsilon(\Gamma) = \sum_{i=1}^{n} |\lambda_i|$ where $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of the adjacency matrix of Γ.

3 Main results

In this section, the main results are specified in term of proposition, lemma and theorem. Proposition 2 presents the generalization of the Cayley graphs of D_{2n} with respect to the set $S = \{a^{n/2}, b\}$ while Lemma 1 states the spectrum of the Cayley graphs in Proposition 2. Then, the results on the energy of the Cayley graphs with respect to the set $S = \{a^{n/2}, b\}$ are presented in Theorem 2. The findings in Proposition 2 and Lemma 1 will be used in proving Theorem 2. One example is also presented for better understandings.

Proposition 3.1. *Let D_{2n} be a dihedral group of order $2n$ where $n \geq 3$ and n even. Let $S = \{a^{n/2}, b\}$ be a subset of D_{2n}. The Cayley graphs of D_{2n} with respect to the set S, $Cay(D_{2n}, \{a^{n/2}, b\})$, are the cycle graphs $\bigcup_i^{\frac{n}{2}} C_4$.*

Proof Consider the dihedral groups D_{2n} of order $2n$ and the Cayley graphs of D_{2n} with respect to the subset $S = \{a^{n/2}, b\}$, denoted as $Cay(D_{2n}, \{a^{n/2}, b\})$. Note that $|S| = 2$. By the definition of Cayley graph, the vertices of $Cay(D_{2n}, \{a^{n/2}, b\})$ are the elements ofD_{2n} and there is an edge joining vertices h and g if and only if $hg^{-1} \in S$. Since the order of S is 2, this forms $\frac{n}{2}$ cycle graphs of order 4 where all vertices are from the group D_{2n}, $\bigcup_i^{\frac{n}{2}} C_4$. ■

Lemma 3.1. *Let D_{2n} be a dihedral group of order $2n$ where $n \geq 3$ and n even. Let $S = \{a^{n/2}, b\}$ be a subset of D_{2n}. Thus, the spectrum of the $Cay(D_{2n}, \{a^{n/2}, b\})$, denoted as $Spec(Cay(D_{2n}, \{a^{n/2}, b\}))$ are 0 with multiplicity n and ± 2 with multiplicity to $n/2$.*

Proof Consider the dihedral groups D_{2n} of order $2n$. By Proposition 2, The Cayley graphs of D_{2n} with respect to the subset S , $Cay(D_{2n}, \{a^{n/2}, b\})$, are the cycle graph $\bigcup_i^{\frac{n}{2}} C_4$. Since the adjacency spectrum of a cycle graph C_n is $\{2cos(\frac{2\pi j}{n})\}; j = \{0, 1, \ldots, n-1\}$, then $Spec(\bigcup_i^{\frac{n}{2}} C_4) = \frac{n}{2}\{2cos(\frac{2\pi j}{4})\}; j = \{0, 1, 2, 3\}$ which also can be written as $\lambda = 0$ with multiplicity n and $\lambda = \pm 2$ with multiplicity to $n/2$. ■

Theorem 3.1. *Let D_{2n} be a dihedral group of order $2n$ where $n \geq 3$ and $S = \{a^{n/2}, b\}$ is a subset of D_{2n}. The energy of the Cayley graphs of D_{2n} with respect to the set S, $E(Cay(D_{2n}, \{a^{n/2}, b\})) = 2n$.*

Proof Consider the dihedral groups D_{2n} of order $2n$. By Proposition 2 and Lemma 1, The Cayley graphs of D_{2n} with respect to the subset S , $Cay(D_{2n}, \{a^{n/2}, b\})$, are the cycle graph $\bigcup_i^{\frac{n}{2}} C_4$ with the spectrum $\lambda = 0$ with multiplicity n and $\lambda = \pm 2$ with multiplicity to $n/2$. Therefore, the energy of the Cayley graphs of D_{2n} with respect to the subset S,

$$E(Cay(D_{2n}, \{a^{n/2}, b\})) = n|0| + \tfrac{n}{2}|2| + \tfrac{n}{2}|-2| = 2n.$$

■

The computation of the energy of Cayley graph with respect to the set $S = \{a^2, b\}$ of dihedral groups of order 8, D_8 is as shown in the following example.

Example 3.1. Let D_8 be a dihedral group of order 8 where $D_8 = < a, b | a^4 = b^2 = 1, bab = a^{-1} >$ and $S = \{a^2, b\}$ be a subset of D_8. The Cayley graph of D_8 with respect to the subset S, $Cay(D_8, \{a^2, b\})$ are the cycle graph $\bigcup_i^2 C_4$. The spectrum of $Cay(D_8, \{a^2, b\})$ are 0 with multiplicity 4 and ± 2 with multiplicity 2. Thus, the energy of the Cayley graphs of D_8 with respect to the subset S, $E(Cay(D_8, \{a^2, b\}))$ is 8.

Proof Consider a dihedral group of order $D_8 =< a, b|a^4 = b^2 = 1, bab = a^{-1} >$ and $S = \{a^2, b\}$ be a subset of D_8. Then, by Definition 2, the vertex x is connected to sx where $s \in S$.

$1 - a^2$ since $a^2.1 = a^2$,
$1 - b$ since $b.1 = b$,
$a - a^3$ since $a^2.a = a^3$,
$a - a^3b$ since $b.a = a^3b$,
$a^2 - 1$ since $a^2.a^2 = 1$,
$a^2 - a^2b$ since $b.a^2 = a^2b$,
$a^3 - a$ since $a^2.a^3 = a$,
$a^3 - ab$ since $b.a^3 = ab$,
$b - a^2b$ since $a^2.b = a^2b$,
$b - 1$ since $b.b = 1$,
$ab - a^3b$ since $a^2.ab = a^3b$,
$ab - a^3$ since $b.ab = a^3$,
$a^2b - b$ since $a^2.a^2b = b$,
$a^2b - a^2$ since $b.a^2b = a^2$,
$a^3b - ab$ since $a^2.a^3b = ab$,
$a^3b - a$ since $b.a^3b = a$.

The connected elements form the $Cay(D_8, \{a^2, b\}) = \bigcup_i^2 C_4$ as illustrated in Figure 1.

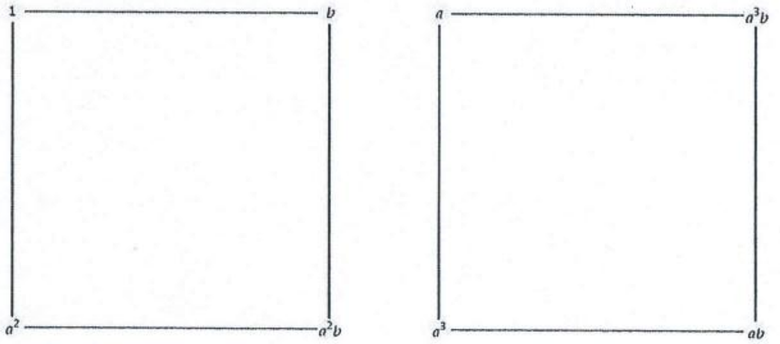

Figure 1: $Cay(D_8, \{a^2, b\})$

By the definition of adjacency matrix,

$$A(Cay(D_8, \{a^2, b\})) = \begin{bmatrix} 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}.$$

From the adjacency matrix, the characteristic polynomials found is $f(\lambda) = \lambda^8 - 8\lambda^6 + 16\lambda^4$ which gives the spectrum, $\lambda = 0$ with multiplicity 4 and $\lambda = \pm 2$ with multiplicity 2. By using the generalization of spectrum of a cycle graph, the spectrum can also be found as $Spec(\bigcup_i^2 C_4) = 2\{2cos(\frac{0}{4}), 2cos(\frac{2\pi}{4}), 2cos(\frac{4\pi}{4}), 2cos(\frac{6\pi}{4})\} = 2\{2, 0, -2, 0\}$ which can be written as $\lambda = 0$ with multiplicity 4 and $\lambda = \pm 2$ with multiplicity 2. Therefore,

$$E(Cay(D_8, \{a^2, b\})) = 4|0| + 2|2| + 2|-2| = 8.$$

■

4 Conclusion

As a conclusion, it has been found that the energy of the Cayley graphs of the dihedral groups of order $2n$ with respect to the subset $S = \{a^{n/2}, b\}$ where $n \geq 4$ and n is even are $2n$.

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Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors have contributed to all parts of the article. All authors read and approved the final manuscript.

References

[1] Babai, L. (1979). Spectra of Cayley graphs. Journal of Combinatorial Theory, Series B, 27(2), 180-189.

[2] Cvetkovic, D. M., Rowlinson, P., and Simic, S. (2010). An introduction to the theory of graph spectra. Cambridge, UK: Cambridge University Press.

[3] Diaconis, P., and Shahshahani, M. (1981). Generating a random permutation with random transpositions. Zeitschrift für Wahrscheinlichkeitstheorie und verwandte Gebiete, 57(2), 159-179.

[4] Woods, C. (2013). My Favorite Application Using Graph Eigenvalues: Graph Energy.

[5] Li, X., Shi., and Gutman, I. (2012). Graph Energy. Springer Science and Business Media.

[6] Bapat, R. B., and Pati, S. (2004). Energy of a graph is never an odd integer. Bull. Kerala Math. Assoc, 1(2), 129-132.

[7] Pirzada, S., and Gutman, I. (2008). Energy of a graph is never the square root of an odd integer. Applicable Analysis and Discrete Mathematics, 2(1), 118-121.

[8] Foster-Greenwood, B. and Kriloff, C. Spectra of Cayley graphs of complex reflection groups. Journal of Algebraic Combinatorics 44(1) (2016), 33-57.

[9] Liu, X. and Li, B. Distance powers of unitary Cayley graphs. Applied Mathematics and Computation 289 (2016), 272-280.

[10] Wang, C. and Xu, M., Non-normal one-regular and 4-valent Cayley graphs of dihedral groups D_{2n}. European Journal of Combinatorics, 27(5), pp.750-766, 2006.

[11] Kwak, J.H. and Oh, J.M., OneŨregular Normal Cayley Graphs on Dihedral Groups of Valency 4 or 6 with Cyclic Vertex Stabilizer. Acta Mathematica Sinica, 22(5), pp.1305-1320, 2006.

[12] Kwak, J.H., Kwon, Y.S. and Oh, J.M., Infinitely many one-regular Cayley graphs on dihedral groups of any prescribed valency. Journal of Combinatorial Theory, Series B, 98(3), pp.585-598, 2008.

[13] Kim, D. S., Kwon, Y. S., and Lee, J. U. (2010). A classification of prime-valent regular Cayley maps on abelian, dihedral and dicyclic groups. Bulletin of the Korean Mathematical Society, 47(1), 17-27.

[14] Rotman, J. J., Advanced Modern Algebra, USA: Pearson Education, 2002.

[15] Beineke, L. W. and Wilson, R. J., Topics in algebraic graph theory, 102, USA : Cambridge University Press, 2004.

[16] Bapat, R. B., Graphs and matrices, New York (NY): Springer, 2010.

[17] Brouwer, A.E. and Haemers, W.H., Spectra of graphs. Springer Science & Business Media. (2011).