## On the orbit graph of some 3-generator 2-groups

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# On the Orbit Graph of Some 3-generator 2-groups 

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#### Abstract

In this research, $G$ is a finite 3-generator 2-group of order 16, where $G$ can be a graph which is called the orbit graph. The orbit graph is a graph whose vertices are non-central orbits under the group action of $G$ on a set. The two vertices are adjacent when they are conjugated to each other. Based on the definition of the orbit graph and group action, the orbit graph is determined for $G$ in the case that $G$ acts on itself by conjugation. Throughout this research, the graph that will be considered is a simple undirected graph.


Keywords: Conjugation action, Orbit graph, 3-generator 2-groups

## INTRODUCTION

A graph in graph theory consists of vertices or points which are connected by edges or lines. Two vertices $u$ and $v$ are said to be connected if there is a $(u, v)$-path in $G[1]$. The graph is called a complete graph, when all of the vertices in a graph are adjacent to each other. The symbol $K_{n}$ is used to denote a complete graph with $n$ vertices [2]. Graphs can be divided into two types, directed graph and undirected graph. Directed graph is a graph in which the set of vertices are connected by edges where the edges have a direction associated with them. Meanwhile, undirected graph is a graph with no distinction between two vertices where the edges have no orientation.

In general, the application of graph theory can be found in various fields including chemistry, biology and also mathematics. A natural model for a molecule in chemistry can be built by using a graph. The vertices will be represented by atoms and the edges are made from bonds. In mathematics, graph theory is useful in geometry whereas algebraic graph theory has close links with the group theory.

In this section, some basic concepts in group theory and graph theory are discussed.

## Definition 1. [3] Group action

Let $G$ be a finite group and $S$ a set. $G$ acts on $S$ if there is a function $G \times S \rightarrow S$ such that

1. $(g h) s=g(h s), \forall g, h \in G, s \in S$
2. $1_{G} \times S=S, \forall s \in S$.

## Definition 2. [4] Orbit of a Group

Let $G$ act on a set $S$ and $x \in S$. The orbit of $x$, denoted by $O(x)$, is the subset

$$
O(x)=\{g x: g \in G\} \subseteq S
$$

If a group $G$ acts on itself by conjugation, the orbit $O(x)$ is:

$$
y \in G, y=a x a^{-1} \text { for some } a \in G
$$

The orbit is also called the conjugacy class of $x$ in $G . K(G)$ is used to represent the number of conjugacy classes.
In this paper, the classifications of 3-generator $p$-groups of order $p^{4}$ by Kim [5] are considered:-

$$
\begin{aligned}
H_{1} & =\left\langle x, y, z \mid x^{p}=y^{p}=z^{p^{2}}=1,[x, z]=[y, z]=1,[x, y]=z^{p}\right\rangle, \\
H_{2} & =\left\langle x, y, z \mid x^{p^{2}}=y^{p}=z^{p}=1,[x, z]=[y, z]=1, x^{y}=x^{p+1}\right\rangle, \\
H_{3} & =\left\langle x, y, z \mid x^{p^{2}}=y^{p}=z^{p}=1,[x, y]=z,[x, z]=[y, z]=1\right\rangle .
\end{aligned}
$$

Since the scope of this paper is 3-generator 2 -groups of order 16 , therefore in this case, we let $p=2$. Thus, the classifications of the groups, named as Type 1, Type 2 and Type 3 are given as follows:-

Type 1: $H_{1}=\left\langle x, y, z \mid x^{2}=y^{2}=z^{4}=1,[x, z]=[y, z]=1,[x, y]=z^{2}\right\rangle$,
Type 2: $\quad H_{2}=\left\langle x, y, z \mid x^{4}=y^{2}=z^{2}=1,[x, z]=[y, z]=1, x^{y}=x^{3}\right\rangle$,
Type 3: $H_{3}=\left\langle x, y, z \mid x^{4}=y^{2}=z^{2}=1,[x, y]=z,[x, z]=[y, z]=1\right\rangle$.

This paper is divided into three sections. The first section focuses on some fundamental concepts in graph theory and group theory. Previous results on orbit graph are discussed in the second section meanwhile our main results are presented in the last section.

## PRELIMINARIES

This section discusses the previous research on orbit graph for different types of group.
Omer et al. [6] has done a research on the orbit graph of finite $p$-groups and groups of order $p q$ in 2014. Three types of group action are discussed including conjugation, transitive and regular. The orbit graph of a finite $p$-group in case of conjugation action is shown as follows:

## Theorem 1. [6]

Let $G$ be a finite $p$-group and acts on itself by conjugation. Then, $\Gamma_{G}^{\Omega}=\bigcup_{x \in G}$
The orbit graph of a group of order $p q$ is also calculated and stated in Theorem 2.

## Theorem 2. [6]

Let $G$ be a finite group of order $p q$, where $p$ and $q$ are distinct primes. Let $\Omega$ be the set of all subsets of commuting elements in the form $(a, b)$ of size two in $G$. If $G$ acts on $\Omega$ by conjugation, then, $\Gamma_{G}^{\Omega}=K_{|\Omega|}$.

In addition, El-sanfaz et al. [7] worked on orbit graph of metacylclic 2-groups of negative type. Zamri et al. [8] computed the probability that an element of a metacyclic 5-group fixes a set and applied the result into orbit graph. In this case, $\Omega$ is the set of all ordered pairs $(x, y)$ in $G \times G$ such that $\operatorname{lcm}(|x|,|y|)=5, x y=y x$ and $x \neq y$.

In this paper, the orbit graph of 3 -generator $p$-groups of order $p^{4}$ when $p=2$ is considered. Therefore, the definition of the orbit graph is given as follows:

## Definition 3. [9] Orbit graph

Let $G$ be a finite group and $\Omega$ be a set of elements of $G$. Let $A$ be the set of commuting elements in $\Omega$, i.e $A=\{v \in \Omega: v g=g v, g \in G\}$. The orbit graph $\Gamma_{G}^{\Omega}$ consists of two sets, namely vertices and edges, denoted by $V\left(\Gamma_{G}^{\Omega}\right)$ and $E\left(\Gamma_{G}^{\Omega}\right)$, respectively. The vertices of $V\left(\Gamma_{G}^{\Omega}\right)$ are non-elements in $\Omega$ but not in $A$, while the number of edges are $\left|E\left(\Gamma_{G}^{\Omega}\right)\right|=\sum_{i=1}^{\left|\nu\left(\Gamma_{G}^{\Omega}\right)\right|}\binom{v_{i}}{2}$, where $v$ is the size of orbit under group action $G$ on $\Omega$. Two vertices $v_{1}, v_{2}$ are adjacent in $\Gamma_{G}^{\Omega}$ if one of the following conditions are satisfied:
i. If there exists $g \in G$ such that $g v_{1}=v_{2}$,
ii. If the vertices of $\Gamma_{G}^{\Omega}$ are conjugate, that is $v_{1}=g^{\nu_{2}}$.

## MAIN RESULTS

The main results on the orbit graph of 3-generator 2-groups of order 16 are discussed in this section.

## Theorem 3.

Let $H_{1}$ be a 3-generator 2-group of order 16 such that

$$
H_{1}=\left\langle x, y, z \mid x^{2}=y^{2}=z^{4}=1,[x, z]=[y, z]=1,[x, y]=z^{2}\right\rangle,
$$

Let $\Omega$ be the set of all ordered pairs $(a, b)$ in such that $\operatorname{lcm}(|a|,|b|)=2, a b=b a$ and $a \neq b$. If $H_{1}$ acts on $\Omega$ by conjugation, then the orbit graph $\Gamma_{H_{1}}^{\Omega}$ is the union of 12 complete graph of two vertices, $\bigcup_{i=1}^{12}$

## Proof:

There are eight elements of order 1 and 2 which are $1, x, y, y z^{2}, x y z, x y z^{3}, x z^{2}$ and $z^{2}$. From the eight elements, there are 32 pairs of elements starting with;

$$
(1, x),(1, y),(1, x y z), \ldots,\left(z^{2}, x z^{2}\right)
$$

All the pairs of elements are commuting to each other. Thus, $|\Omega|=32$. Through manual computation, the number of orbits, $K(\Omega)$ under conjugation action of $H_{1}$ on $\Omega$ is 20 as listed below:

$$
\begin{aligned}
& \text { 1. } o((1, x))=\left\{(1, x),\left(1, x z^{2}\right)\right\} \\
& \text { 2. } o((1, x y z))=\left\{(1, x y z),\left(1, x y z^{3}\right)\right\} \\
& \text { 3. } o((1, y))=\left\{(1, y),\left(1, y z^{2}\right)\right\} \\
& \text { 4. } o\left(\left(1, z^{2}\right)\right)=\left\{\left(1, z^{2}\right)\right\} \\
& \text { 5. } o\left(\left(x, x z^{2}\right)\right)=\left\{\left(x, x z^{2}\right)\right\} \\
& \text { 6. } o\left(\left(x, z^{2}\right)\right)=\left\{\left(x, z^{2}\right),\left(x z^{2}, z^{2}\right)\right\} \\
& \text { 7. } o\left(\left(y, y z^{2}\right)\right)=\left\{\left(y, y z^{2}\right)\right\} \\
& \text { 8. } o\left(\left(y, z^{2}\right)\right)=\left\{\left(y, z^{2}\right),\left(y z^{2}, z^{2}\right)\right\} \\
& \text { 9. } o\left(\left(x y z, x y z^{3}\right)\right)=\left\{\left(x y z, x y z^{3}\right)\right\} \\
& \text { 10. } o\left(\left(x y z, z^{2}\right)\right)=\left\{\left(x y z, z^{2}\right),\left(x y z^{3}, z^{2}\right)\right\}
\end{aligned}
$$

11. $o((x, 1))=\left\{(x, 1),\left(x z^{2}, 1\right)\right\}$
12. $o((x y z, 1))=\left\{(x y z, 1),\left(x y z^{3}, 1\right)\right\}$
13. $o((y, 1))=\left\{(y, 1),\left(y z^{2}, 1\right)\right\}$
14. $o\left(\left(z^{2}, 1\right)\right)=\left\{\left(z^{2}, 1\right)\right\}$
15. $o\left(\left(x z^{2}, x\right)\right)=\left\{\left(x z^{2}, x\right)\right\}$
16. $o\left(\left(z^{2}, x\right)\right)=\left\{\left(z^{2}, x\right),\left(z^{2}, x z^{2}\right)\right\}$
17. $o\left(\left(y z^{2}, y\right)\right)=\left\{\left(y z^{2}, y\right)\right\}$
18. $o\left(\left(z^{2}, y\right)\right)=\left\{\left(z^{2}, y\right),\left(z^{2}, y z^{2}\right)\right\}$
19. $o\left(\left(x y z^{3}, x y z\right)\right)=\left\{\left(x y z^{3}, x y z\right)\right\}$
20. $o\left(\left(z^{2}, x y z\right)\right)=\left\{\left(z^{2}, x y z\right),\left(z^{2}, x y z^{3}\right)\right\}$

Based on the definition of orbit graph, the vertices of the graph is 12 since the order of $A$ is 8 . Two vertices are adjacent if the vertices of the graphs are conjugated to each other. Thus, $\Gamma_{H_{1}}^{\Omega}$ consists of 12 complete components of $K_{2}$. Hence, $\Gamma_{H_{1}}^{\Omega}=\bigcup_{i=1}^{12}$

## Theorem 4.

Let $H_{2}$ be a 3-generator 2-group of order 16 as shown below:

$$
H_{2}=\left\langle x, y, z \mid x^{4}=y^{2}=z^{2}=1,[x, z]=[y, z]=1, x^{y}=x^{3}\right\rangle .
$$

Let $\Omega$ be the set of all ordered pairs $(a, b)$ in $H_{2} \times H_{2}$ such that $l \mathrm{~cm}(|a|,|b|)=2, a b=b a$ and $a \neq b$. If $H_{2}$ acts on $\Omega$ by conjugation, then the orbit graph $\Gamma_{H_{2}}^{\Omega}$ is the union of 42 complete graph of two vertices, $\bigcup_{i=1}^{42}$.

## Proof:

There are 12 elements of order 1 and 2 which are $1, x^{2}, y, z, x y, x^{2} y, x^{3} y, x^{2} z, y z, x y z, x^{2} y z$ and $x^{3} y z$. A total of 100 pairs of elements are formed from these 12 elements starting with;

$$
\left(1, x^{2}\right),(1, y), \ldots,\left(x^{3} y z, x y z\right)
$$

All the pairs of elements are commuting to each other. Thus, $|\Omega|=100$. Through manual computation, the number of orbits, $K(\Omega)$ under conjugation action of $H_{2}$ on $\Omega$ is 54 and the order of $A$ is 12 . Therefore, the vertices of the orbit graph of $H_{2}$ is 42 . Since $\Gamma_{H_{2}}^{\Omega}$ consists of 42 complete component of $K_{2}$, hence, $\Gamma_{H_{2}}^{\Omega}=\bigcup_{i=1}^{42}$

## Theorem 5.

Let $H_{3}$ be a 3-generator 2-group of order 16 as shown below:

$$
H_{3}=\left\langle x, y, z \mid x^{4}=y^{2}=z^{2}=1,[x, y]=z,[x, z]=[y, z]=1\right\rangle .
$$

Let $\Omega$ be the set of all ordered pairs $(a, b)$ in $H_{3} \times H_{3}$ such that $\operatorname{lcm}(|a|,|b|)=2, a b=b a$ and $a \neq b$. If $H_{3}$ acts on $\Omega$ by conjugation, then the orbit graph $\Gamma_{H_{3}}^{\Omega}$ is the union 22 of complete graph of two vertices, $\bigcup_{i=1}^{22}$

## Proof:

There are eight elements of order 1 and 2 which are $1, x^{2}, y, z, x^{2} y, x^{2} z, y z$ and $x^{2} y z$. A total of 56 pairs of elements are formed from these eight elements starting with;

$$
\left(1, x^{2}\right),(1, y), \ldots,\left(x^{2} y z, y z\right)
$$

All the pairs of elements are commuting to each other. Thus, $|\Omega|=56$. Based on manual computation, the number of orbits, $K(\Omega)$ under conjugation action of $H_{3}$ on $\Omega$ is 34 and the order of $A$ is 12 . Therefore, the vertices of the orbit graph of $H_{3}$ is 22. Since $\Gamma_{H_{3}}^{\Omega}$ consists of 22 complete component of $K_{2}$, hence, $\Gamma_{H_{3}}^{\Omega}=\bigcup_{i=1}^{22}$

## CONCLUSION

This paper focuses on three types of 3-generator 2 -groups of order 16 which are Type 1 , Type 2 and Type 3 . Based on Theorem 3, 4 and 5, the orbit graphs for each type turned out to be the union of complete graphs of degree 2 as shown in Table 1.

TABLE 1 Orbit graph of 3-generator 2-groups.

| Group | Type of Orbit Graph |
| :---: | :---: |
| Type 1: $H_{1}$ | $\bigcup_{i=1}^{12}$ |
| Type 2 $: H_{2}$ | $\bigcup_{i=1}^{42}$ |
| Type 3: $H_{3}$ | $\bigcup_{i=1}^{22}$ |

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