The Energy of Conjugacy Class Graphs of Some Metabelian Groups

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Abstract. The energy of a graph of a group G is the sum of all absolute values of the eigenvalues of the adjacency matrix. An adjacency matrix is a square matrix where the rows and columns consist of 0 or 1-entry depending on the adjacency of the vertices of the graph. A conjugacy class graph is a graph whose vertex set is the non-central conjugacy classes of the group. Two vertices are connected if their orders are not coprime. Meanwhile, a group G is said to be metabelian if there exists a normal subgroup H in G such that both H and the factor group G/H are abelian. In this research, the energy of the conjugacy class graphs for all nonabelian metabelian groups of order 24 are determined. The computations are assisted by Groups, Algorithm and Programming (GAP) and Maple2016 software. The results show that the energy of graphs of the groups in the study must be an even integer in the case that the energy is rational.

Keywords: Adjacency matrix; Conjugacy class; Conjugacy class graph; Energy of graph; Metabelian

INTRODUCTION

According to Woods [1], the study on the energy of general simple graphs was first defined by Gutman in 1978 inspired from the Hückel Molecular Orbital Theory proposed by Hückel in 1930. Hückel Molecular Orbital Theory has been used by chemists in approximating the energies related to $\pi-$electron orbitals in conjugated hydrocarbon. Later in 1956, Gunthard and Primas realized that the Hückel method is actually using the first degree polynomial of the adjacency matrix of a certain graph.

There are a few researchers that have studied on the energy of graphs. For example, in 2004, Zhou [2] and Balakrishnan [3] studied the characteristics of energy of graphs while Bapat and Pati [4] proved that the energy of a graph is never an odd integer in their research. In the same year, Yu *et al*. [5] considered the new upper bounds for the energy of graphs. Furthermore, the properties that the energy of a graph is never the square root of an odd integer has been proven by Pirzada and Gutman [6] in 2008. For the conjugacy class graphs, Bertram *et al.* [7] have introduced a graph related to conjugacy classes of groups in the year 1990.

The metabelian groups of order at most 24 have been determined by Rahman and Sarmin [8] in 2012. Ever since the metabelian groups have been determined, there are not as many researchers that make use of the groups and extend its application. In 2011, Mohd [9] has chosen the nonabelian metabelian groups of order at most 24 to find its commutativity degree. Then, Halim [10] extended the application of the nonabelian metabelian groups of order at most 24 to find its nth commutativity degree in 2013. Further in 2014, Hassan [11] deepened the scope by finding

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the relative commutativity degree for only cyclic subgroup of all nonabelian metabelian groups of order at most 24. Later in 2015, Sarmin *et al.* [12] researched on the conjugacy classes of metabelian groups of order at most 24.

In this paper, ten metabelian groups of order 24, determined by Rahman and Sarmin [8], are considered. The groups are $S_3 \times \mathbb{Z}_4$, $S_3 \times \mathbb{Z}_2 \times \mathbb{Z}_2$, $D_4 \times \mathbb{Z}_3$, $\mathbb{Q} \times \mathbb{Z}_3$, $A_4 \times \mathbb{Z}_2$, $(\mathbb{Z}_6 \times \mathbb{Z}_2) \rtimes \mathbb{Z}_2$, D_{12}, $\mathbb{Z}_2 \times (\mathbb{Z}_3 \rtimes \mathbb{Z}_4)$, $\mathbb{Z}_3 \rtimes \mathbb{Z}_8$ and $\mathbb{Z}_3 \rtimes \mathbb{Q}$.

PRELIMINARIES

The followings are some definitions that are used in this work.

Definition 1 [13]
A group G is a metabelian if there exists a normal subgroup $A \triangleleft G$ such that both A and G/A are abelian.

Definition 2 [14]
Let a and b be elements of a group G. We say that a and b are conjugate in G (and call b a conjugate of a) if $xax^{-1} = b$ for some x in G. The conjugacy class of a is the set $cl(a) = \{x^{-1}ax \mid x \in G\}$.
A conjugacy class in a group G is called non-central if it is not contained in the center of G.

Definition 3 [7]
Let G denote a finite group. A conjugacy class graph, denoted as Γ_G^{CC}, is a graph with the vertices $V = (v_1, \ldots v_n)$ represented by the non-central conjugacy classes of G. Two vertices v_1 and v_2 are connected if the order of the vertices, $|v_1|$ and $|v_2|$ have a common prime divisor.

Definition 4 [15]
For any graph G, the energy of the graph is defined as $\varepsilon(G) = \sum_{i=1}^{n} |\lambda_i|$ where $\lambda_i, \ldots \lambda_n$ are the eigenvalues of the adjacency matrix of G.

MAIN RESULTS

In this section, the main results are presented.

Theorem 1 Let $G = S_3 \times \mathbb{Z}_4$. Then, the energy of the conjugacy class graph of G, $\varepsilon(\Gamma_G) = 12$.

Proof Let $G = S_3 \times \mathbb{Z}_4$. Then, the adjacency matrix A for the conjugacy class graph $\Gamma_G = \bigcup_{i=1}^{2} K_4$ is given in the following:

$$A = \begin{bmatrix} 0&1&0&0&1&1&0&0 \\ 1&0&0&0&1&1&0&0 \\ 0&0&0&1&0&0&1&1 \\ 0&0&1&0&0&0&1&1 \\ 1&1&0&0&0&1&0&0 \\ 1&1&0&0&1&0&0&0 \\ 0&0&1&1&0&0&0&1 \\ 0&0&1&1&0&0&1&0 \end{bmatrix}.$$

Thus, the characteristic polynomial of A is given as in the following:

$$f(\lambda) = \lambda^8 - 12\lambda^6 - 16\lambda^5 + 30\lambda^4 + 96\lambda^3 + 100\lambda^2 + 48\lambda + 9.$$

Hence, the eigenvalues are found to be $\lambda = 3$ with multiplicity 2 and $\lambda = -1$ with multiplicity 6. Therefore, the energy of the conjugacy class graph for $S_3 \times \mathbb{Z}_4$ is $\varepsilon(\Gamma_G) = \sum_{i=1}^{n} |\lambda_i| = 12$. ■

Theorem 2 Let $G = S_3 \times \mathbb{Z}_2 \times \mathbb{Z}_2$. Then, the energy of the conjugacy class graph of G, $\varepsilon(\Gamma_G) = 12$.

Proof Let $G = S_3 \times \mathbb{Z}_2 \times \mathbb{Z}_2$. Then, the adjacency matrix A for the conjugacy class graph $\Gamma_G = \bigcup_{i=1}^{2} K_4$ is given in the following:

$$A = \begin{bmatrix} 0&0&0&1&0&0&1&1 \\ 0&0&1&0&1&1&0&0 \\ 0&1&0&0&1&1&0&0 \\ 1&0&0&0&0&0&1&1 \\ 0&1&1&0&0&1&0&0 \\ 0&1&1&0&1&0&0&0 \\ 1&0&0&1&0&0&0&1 \\ 1&0&0&1&0&0&1&0 \end{bmatrix}.$$

Thus, the characteristic polynomial of A is given as in the following:

$$f(\lambda) = \lambda^8 - 12\lambda^6 - 16\lambda^5 + 30\lambda^4 + 96\lambda^3 + 100\lambda^2 + 48\lambda + 9.$$

Hence, the eigenvalues are found to be $\lambda = 3$ with multiplicity 2 and $\lambda = -1$ with multiplicity 6. Therefore, the energy of the conjugacy class graph for $S_3 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ is $\varepsilon(\Gamma_G) = \sum_{i=1}^{n} |\lambda_i| = 12$. ■

Theorem 3 Let $G = D_4 \times \mathbb{Z}_3$. Then, the energy of the conjugacy class graph of G, $\varepsilon(\Gamma_G) = 16$.

Proof Let $G = D_4 \times \mathbb{Z}_3$. Then, the adjacency matrix A for the conjugacy class graph $\Gamma_G = K_9$ is given in the following:

$$A = \begin{bmatrix} 0&1&1&1&1&1&1&1&1 \\ 1&0&1&1&1&1&1&1&1 \\ 1&1&0&1&1&1&1&1&1 \\ 1&1&1&0&1&1&1&1&1 \\ 1&1&1&1&0&1&1&1&1 \\ 1&1&1&1&1&0&1&1&1 \\ 1&1&1&1&1&1&0&1&1 \\ 1&1&1&1&1&1&1&0&1 \\ 1&1&1&1&1&1&1&1&0 \end{bmatrix}.$$

Thus, the characteristic polynomial of A is given as in the following:

$$f(\lambda) = \lambda^9 - 36\lambda^7 - 168\lambda^6 - 378\lambda^5 - 504\lambda^4 - 420\lambda^3 - 216\lambda^2 - 63\lambda - 8.$$

Hence, the eigenvalues are found to be $\lambda = 8$ with multiplicity 1 and $\lambda = -1$ with multiplicity 8. Therefore, the energy of the conjugacy class graph for $D_4 \times \mathbb{Z}_3$ is $\varepsilon(\Gamma_G) = \sum_{i=1}^{n} |\lambda_i| = 16$. ■

Theorem 4 Let $G = \mathbb{Q} \times \mathbb{Z}_3$. Then, the energy of the conjugacy class graph of G, $\varepsilon(\Gamma_G) = 16$.

Proof Let $G = \mathbb{Q} \times \mathbb{Z}_3$. Then, the adjacency matrix A for the conjugacy class graph $\Gamma_G = K_9$ is given in the following:

$$A=\begin{bmatrix}0&1&1&1&1&1&1&1&1\\1&0&1&1&1&1&1&1&1\\1&1&0&1&1&1&1&1&1\\1&1&1&0&1&1&1&1&1\\1&1&1&1&0&1&1&1&1\\1&1&1&1&1&0&1&1&1\\1&1&1&1&1&1&0&1&1\\1&1&1&1&1&1&1&0&1\\1&1&1&1&1&1&1&1&0\end{bmatrix}.$$

Thus, the characteristic polynomial of A is given as in the following:

$$f(\lambda)=\lambda^9-36\lambda^7-168\lambda^6-378\lambda^5-504\lambda^4-420\lambda^3-216\lambda^2-63\lambda-8.$$

Hence, the eigenvalues are found to be $\lambda=8$ with multiplicity 1 and $\lambda=-1$ with multiplicity 8. Therefore, the energy of the conjugacy class graph for $\mathbb{Q}\times\mathbb{Z}_3$ is $\varepsilon(\Gamma_G)=\sum_{i=1}^{n}|\lambda_i|=16.$ ■

Theorem 5 Let $G=A_4\times\mathbb{Z}_2$. Then, the energy of the conjugacy class graph of G, $\varepsilon(\Gamma_G)=8$.

Proof Let $G=A_4\times\mathbb{Z}_2$. Then, the adjacency matrix A for the conjugacy class graph $\Gamma_G=K_4\cup K_2$ is given in the following:

$$A=\begin{bmatrix}0&0&1&0&1&1\\0&0&0&1&0&0\\1&0&0&0&1&1\\0&1&0&0&0&0\\1&0&1&0&0&1\\1&0&1&0&1&0\end{bmatrix}.$$

Thus, the characteristic polynomial of A is given as in the following:

$$f(\lambda)=\lambda^6-7\lambda^4-8\lambda^3+3\lambda^2+8\lambda+3.$$

Hence, the eigenvalues are found to be $\lambda=3$ and $\lambda=1$ with multiplicity 1 and $\lambda=-1$ with multiplicity 4. Therefore, the energy of the conjugacy class graph for $A_4\times\mathbb{Z}_2$ is $\varepsilon(\Gamma_G)=\sum_{i=1}^{n}|\lambda_i|=8.$ ■

Theorem 6 Let $G=(\mathbb{Z}_6\times\mathbb{Z}_2)\rtimes\mathbb{Z}_2$. Then, the energy of the conjugacy class graph of G, $\varepsilon(\Gamma_G)=12$.

Proof Let $G=(\mathbb{Z}_6\times\mathbb{Z}_2)\rtimes\mathbb{Z}_2$. Then, the adjacency matrix A for the conjugacy class graph $\Gamma_G=K_7$ is given in the following:

$$A=\begin{bmatrix}0&1&1&1&1&1&1\\1&0&1&1&1&1&1\\1&1&0&1&1&1&1\\1&1&1&0&1&1&1\\1&1&1&1&0&1&1\\1&1&1&1&1&0&1\\1&1&1&1&1&1&0\end{bmatrix}$$

Thus, the characteristic polynomial of A is given as in the following:

$$f(\lambda)=\lambda^7-21\lambda^5-70\lambda^4-105\lambda^3-84\lambda^2-35\lambda-6.$$

Hence, the eigenvalues are found to be $\lambda=6$ with multiplicity 1 and $\lambda=-1$ with multiplicity 6. Therefore, the energy of the conjugacy class graph for $(\mathbb{Z}_6\times\mathbb{Z}_2)\rtimes\mathbb{Z}_2$ is $\varepsilon(\Gamma_G)=\sum_{i=1}^{n}|\lambda_i|=12.$ ■

Theorem 7 Let G=D_{12}. Then, the energy of the conjugacy class graph of G, $\varepsilon(\Gamma_G)=12$.

Proof Let G=D_{12}. Then, the adjacency matrix A for the conjugacy class graph $\Gamma_G = K_7$ is given in the following:

$$A=\begin{bmatrix} 0&1&1&1&1&1&1\\ 1&0&1&1&1&1&1\\ 1&1&0&1&1&1&1\\ 1&1&1&0&1&1&1\\ 1&1&1&1&0&1&1\\ 1&1&1&1&1&0&1\\ 1&1&1&1&1&1&0 \end{bmatrix}.$$

Thus, the characteristic polynomial of A is given as in the following:

$$f(\lambda)=\lambda^7-21\lambda^5-70\lambda^4-105\lambda^3-84\lambda^2-35\lambda-6.$$

Hence, the eigenvalues are found to be $\lambda=6$ with multiplicity 1 and $\lambda=-1$ with multiplicity 6. Therefore, the energy of the conjugacy class graph for D_{12} is $\varepsilon(\Gamma_G)=\sum_{i=1}^{n}|\lambda_i|=12$. ■

Theorem 8 Let G= $\mathbb{Z}_2\times(\mathbb{Z}_3\rtimes\mathbb{Z}_4)$. Then, the energy of the conjugacy class graph of G, $\varepsilon(\Gamma_G)=12$.

Proof Let G= $\mathbb{Z}_2\times(\mathbb{Z}_3\rtimes\mathbb{Z}_4)$. Then, the adjacency matrix A for the conjugacy class graph $\Gamma_G=\bigcup_{i=1}^{2}K_4$ is given in the following:

$$A=\begin{bmatrix} 0&1&0&1&0&0&0&1\\ 1&0&0&1&0&0&0&1\\ 0&0&0&0&1&1&1&0\\ 1&1&0&0&0&0&0&1\\ 0&0&1&0&0&1&1&0\\ 0&0&1&0&1&0&1&0\\ 0&0&1&0&1&1&0&0\\ 1&1&0&1&0&0&0&0 \end{bmatrix}.$$

Thus, the characteristic polynomial of A is given as in the following:

$$f(\lambda)=\lambda^8-12\lambda^6-16\lambda^5+30\lambda^4+96\lambda^3+100\lambda^2+48\lambda+9.$$

Hence, the eigenvalues are found to be $\lambda=3$ with multiplicity 2 and $\lambda=-1$ with multiplicity 6. Therefore, the energy of the conjugacy class graph for $\mathbb{Z}_2\times(\mathbb{Z}_3\rtimes\mathbb{Z}_4)$ is $\varepsilon(\Gamma_G)=\sum_{i=1}^{n}|\lambda_i|=12$. ■

Theorem 9 Let G= $\mathbb{Z}_3\rtimes\mathbb{Z}_8$. Then, the energy of the conjugacy class graph of G, $\varepsilon(\Gamma_G)=12$.

Proof Let G= $\mathbb{Z}_3\rtimes\mathbb{Z}_8$. Then, the adjacency matrix A for the conjugacy class graph $\Gamma_G=\bigcup_{i=1}^{2}K_4$ is given in the following:

$$A = \begin{bmatrix} 0&0&1&1&1&0&0&0 \\ 0&0&0&0&0&1&1&1 \\ 1&0&0&1&1&0&0&0 \\ 1&0&1&0&1&0&0&0 \\ 1&0&1&1&0&0&0&0 \\ 0&1&0&0&0&0&1&1 \\ 0&1&0&0&0&1&0&1 \\ 0&1&0&0&0&1&1&0 \end{bmatrix}.$$

Thus, the characteristic polynomial of A is given as in the following:

$$f(\lambda) = \lambda^8 - 12\lambda^6 - 16\lambda^5 + 30\lambda^4 + 96\lambda^3 + 100\lambda^2 + 48\lambda + 9 .$$

Hence, the eigenvalues are found to be $\lambda = 3$ with multiplicity 2 and $\lambda = -1$ with multiplicity 6. Therefore, the energy of the conjugacy class graph for $\mathbb{Z}_3 \rtimes \mathbb{Z}_8$ is $\varepsilon(\Gamma_G) = \sum_{i=1}^{n} |\lambda_i| = 12$. ■

Theorem 10 Let G= $\mathbb{Z}_3 \rtimes \mathbb{Q}$. Then, the energy of the conjugacy class graph of G, $\varepsilon(\Gamma_G) = 12$.

Proof Let G= $\mathbb{Z}_3 \rtimes \mathbb{Q}$. Then, the adjacency matrix A for the conjugacy class graph $\Gamma_G = K_7$ is given in the following:

$$A = \begin{bmatrix} 0&1&1&1&1&1&1 \\ 1&0&1&1&1&1&1 \\ 1&1&0&1&1&1&1 \\ 1&1&1&0&1&1&1 \\ 1&1&1&1&0&1&1 \\ 1&1&1&1&1&0&1 \\ 1&1&1&1&1&1&0 \end{bmatrix}.$$

Thus, the characteristic polynomial of A is given as in the following:

$$f(\lambda) = \lambda^7 - 21\lambda^5 - 70\lambda^4 - 105\lambda^3 - 84\lambda^2 - 35\lambda - 6 .$$

Hence, the eigenvalues are found to be $\lambda = 6$ with multiplicity 1 and $\lambda = -1$ with multiplicity 6. Therefore, the energy of the conjugacy class graph for G= $\mathbb{Z}_3 \rtimes \mathbb{Q}$ is $\varepsilon(\Gamma_G) = \sum_{i=1}^{n} |\lambda_i| = 12$. ■

CONCLUSION

In this paper, the energy of conjugacy class graphs for all nonabelian metabelian groups of order 24 are determined. The results are summarized in the following table:

TABLE 1 The energy of the conjugacy class graph for all nonabelian metabelian groups of order 24

No	Groups	Conjugacy Class Graph	Energy
1	$S_3 \times \mathbb{Z}_4$	$\bigcup_{i=1}^{2} K_4$	12
2	$S_3 \times \mathbb{Z}_2 \times \mathbb{Z}_2$	$\bigcup_{i=1}^{2} K_4$	12
3	$D_4 \times \mathbb{Z}_3$	K_9	16
4	$\mathbb{Q} \times \mathbb{Z}_3$	K_9	16
5	$A_4 \times \mathbb{Z}_2$	$K_4 \cup K_2$	8
6	$(\mathbb{Z}_6 \times \mathbb{Z}_2) \rtimes \mathbb{Z}_2$	K_7	12
7	D_{12}	K_7	12
8	$\mathbb{Z}_2 \times (\mathbb{Z}_3 \rtimes \mathbb{Z}_4)$	$\bigcup_{i=1}^{2} K_4$	12
9	$\mathbb{Z}_3 \rtimes \mathbb{Z}_8$	$\bigcup_{i=1}^{2} K_4$	12
10	$\mathbb{Z}_3 \rtimes \mathbb{Q}$	K_7	12

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