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# On the Orbit of Some Metabelian Groups of Order 24 and Its Applications 

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#### Abstract

An orbit is defined as the partition of an equivalent relation of elements in a group. In order to obtain the orbit, a group action acting on the elements of the groups is considered. In this study, the orbits of some metabelian groups are found using conjugation action. The metabelian groups considered in this study are some nonabelian metabelian groups of order 24 , which are the dihedral group, $D_{12}$ as well as the semidirect products, $R=\mathbb{Z}_{3} \rtimes \mathbb{Z}_{8}$ and $S=\mathbb{Z}_{3} \rtimes Q$. The results obtained from the orbits are then applied into an extension of commutativity degree, which is the probability that a group element fixes a set where the probability is the ration of the number of orbits and the number of elements in the set. The set considered in this study is the set of all pairs of commuting elements of the groups that is in the form of $(x, y)$, where $\operatorname{lcm}(|x|,|y|)=2$. Then, the results of the orbits will also be applied into graph theory, specifically generalized conjugacy class graph where its vertices are the non-central orbits and two vertices are adjacent if the cardinality of the orbits is not coprime. Lastly, some properties of the graph which are the chromatic number and clique number are obtained.


Keywords: Commutativity degree, Graph theory, Group theory, Metabelian group

## INTRODUCTION

An orbit is defined as the partition of an equivalent relation of elements in a group [1]. To obtain the orbit, a finite group $G$ must act on a certain set $\Omega$. An orbit is also called conjugacy class, if the group action involved is conjugation action. In this paper, the orbits are applied in group as well as graph theory. In group theory, the orbits are used to calculate the commutativity degree of a group. Meanwhile, in graph theory, the orbits are used to construct the generalized conjugacy class graph, where its vertices are the non-central orbits and two vertices are connected if the cardinality of the non-central orbits are not coprime.

The probability that two random elements of a group commute is called the commutativity degree of a group. This concept is used to determine the abelianness of a group. It was introduced by Miller [2] in 1944 on some finite groups. If all elements of a group commute, then the commutativity degree is one and the group is abelian. In 1973, Gustafson [3] introduced another method of obtaining the commutativity degree, using the number of conjugacy classes. Besides that, the concept of commutativity degree has also attracted many researchers to make the extension and generalization of the concept. One of the extensions of this concept is the probability that a group element fixes
a set. This extension of commutativity degree is introduced by Omer et al. [4] and can be obtained by some group actions on a set.

Graph is a mathematical structure which is commonly used to visualize the relationship between two objects. The concept of graph theory was first introduced by Leonard Euler [5] who solved the Konigsberg bridge problem using vertices and edges. Some years later, graph theory is proven to be a very useful tool in various fields.

In mathematics, many studies have been done in order to show the relation between group and graph. For instance, non-commuting graph that shows the relation between non-central elements of a group and the commutativity of the elements was introduced by Neumann [6] in 1967. Then, in 2013, Tolue and Erfanian [7] generalized the non-commuting graph to the relative non-commuting graph from the relative commutativity degree of a group. The elements in the group excluding the centralizer of the subgroup are the vertices of the graph and two vertices are adjacent if their commutator is not equal to one. In 1990, Bertram [8] introduced a graph related to conjugacy classes, where the non-central conjugacy classes of a group are the vertices and the vertices are connected if the cardinalities of the conjugacy classes are not coprime. Recently, in 2015, Omer et al. [9] introduced the generalized conjugacy class graph whose vertices are the non-central orbits under group action on a set and the edges are constructed if the cardinalities of the non-central orbits are not coprime.

Throughout this study, the orbits of some finite groups are found using conjugation action. The groups involved are some metabelian groups of order 24 which are the dihedral group, $D_{12}$ as well as the semidirect products, $R=\mathbb{Z}_{3} \rtimes \mathbb{Z}_{8}$ and $S=\mathbb{Z}_{3} \rtimes Q$. After obtaining the orbits, the probability of a group element fixes the set is calculated and the generalized conjugacy class graphs are constructed for all those three groups. The set considered throughout this study is the set of all pairs of commuting elements of the groups that is in the form of $(x, y)$, where $\operatorname{lcm}(|x|,|y|)=2$.

## PRELIMINARIES

In this section, some fundamental concepts and definitions related to the topic are presented.

## Definition 1. [10] Metabelian Groups

A group $G$ is metabelian if the group has a normal subgroup $A$ such that $A$ and the factor group $G / A$ are abelian.

## Definition 2. [1] Group Acting on a Set

Let $G$ be a finite group and $X$ be a set. $G$ acts on $X$ if there is a function which maps $G \times X \rightarrow X$ such that

1. $(g h) x=g(h x), \forall g, h \in G, x \in X$,
2. $e \cdot x=x \cdot e, \forall x \in X$.

## Definition 3. [11] Orbit

Suppose $G$ is a finite group that acts on a set $\Omega$ and $\omega \in \Omega$. The orbit of $\omega$, denoted by $O(\omega)$, is the subset. $O(\omega)=\{g \omega \mid g \in G, \omega \in \Omega\}$.

In this study, we consider the metabelian groups act on themselves by conjugation. Hence the orbit is defined as $O(\omega)=\left\{g \omega g^{-1} \mid g \in G, \omega \in \Omega\right\}$. Thus, the orbits in this study is the set of all conjugates of the elements, which is also often called the conjugacy classes. Later on, the number of orbits will be used in computing the commutativity degree of a group.

The commutativity degree is the probability that two random elements selected from a group commute. This concept is used to determine the abelianness of a group. The exact definition of commutativity degree is given in the following definition.

## Definition 4. [2] Commutativity Degree of a Group $G$

The probability that two random elements $(x, y)$ in a group $G$ commute is called the commutativity degree of a group and is defined as:

$$
P(G)=\frac{|\{(x, y) \in G \times G \mid x y=y x\}|}{|G|^{2}} .
$$

This concept has been applied in many studies. In 1968, Erdos and Turan [12] used this concept for some symmetric groups, $S_{n}$. In 2009, Tarnauceanu [13] found the subgroup commutativity degree for some finite groups, including dihedral groups, quasi-dihedral group and generalized quaternion group. In 1973, Gustafson [3] introduced a new method to obtain the commutativity degree by using the number of conjugacy classes of a group. The method of computing is given in the following theorem:

## Theorem 1. [3]

Let $G$ be a finite group and $K(G)$ is the number of conjugacy classes of the group. Then the commutativity degree of a group, denoted by $P(G)$, is given as:

$$
P(G)=\frac{K(G)}{|G|}
$$

In 2013, Omer et al. [4] extended the concept of commutativity degree by introducing the probability that an element of a group fixes a set in which some group actions on a set are used in order to determine the probability. The definition of the probability that an element of a group fixes a set is given as follows:

## Definition 5. [4] The Probability that an Element of a Group Fixes a Set

Let $G$ be a group and $S$ be the set of all subsets of commuting elements of size two in $G$, where $G$ acts on $S$ by conjugation. Then the probability of an element of a group fixes a set is given as:

$$
P_{G}(S)=\frac{\mid\{(g, s) \mid g S=S \text { for } g \in G \text { and } s \in S\} \mid}{|G||S|}
$$

In term of the number of orbits, the probability that an element of a group fixes a set can be written as in the following definition:

Definition 6. [4] The Probability that an Element of a Group Fixes a Set
Let $G$ be a group and $S$ be a set of all commuting elements of $G$ of size two. If $G$ acts on $S$ by conjugation, then the probability that an element of a group fixes a set is given by:

$$
P_{G}(S)=\frac{K}{|S|}
$$

where $K$ is the number of orbits of $S$ in $G$.
As described in the introduction section earlier, there are various researches who relate graphs with groups. In this study, the obtained results from the orbits are applied into the generalized conjugacy class graph. The definition of the generalized conjugacy class graph is given below:

## Definition 7. [9] Generalized Conjugacy Class Graph

Let $G$ be a finite group and $\Omega$ be a set of $G$ and $A$ be a set of commuting elements in $\Omega$, i.e. $A=\{\omega \in \Omega \mid \omega g=g \omega, g \in G\}$. Then the generalized conjugacy class graph, $\Gamma_{G}^{\Omega_{c}}$ is defined as a graph whose vertices are non-central orbits under group action on a set, that is $V\left(\Gamma_{G}^{\Omega_{c}}\right)=K(\Omega)-A$. Two vertices $\omega_{1}$ and $\omega_{2}$ are connected if the cardinalities of the orbits are not coprime.

After obtaining the generalized conjugacy class graph, some graph properties are analyzed. In this study, the chromatic number and the clique number are obtained from the graphs constructed. Therefore, the definition of chromatic number and clique number are given as in the following.

## Definition 8. [14] Chromatic Number

Let $k>0$ be an integer. A $k$-vertex coloring of a graph $\Gamma$ is an assignment of $k$ colors to the vertices such that no two adjacent vertices have the same color. The vertex chromatic number, $\chi(\Gamma)$ of a graph $\Gamma$ is the minimum $k$ for which $\Gamma$ has a $k$-vertex coloring.

## Definition 9. [14] Clique Number

A subset $C$ of vertices of a graph $\Gamma$ is called a clique if the induced subgraph on $C$ is a complete graph. The maximum size of a clique is called the clique number of $\Gamma$ and is denoted by $\omega(\Gamma)$.

## MAIN RESULTS

In this section, the orbits are computed for some metabelian groups which are the dihedral group, $D_{12}$ as well as the semidirect products, $R=\mathbb{Z}_{3} \rtimes \mathbb{Z}_{8}$ and $S=\mathbb{Z}_{3} \rtimes Q$. Then, based on the results obtained, the probability that an element of a group fixes a set is determined and the generalized conjugacy class graphs are constructed. Lastly, the chromatic number and the clique number for the graphs are obtained. The set that is used throughout this study is defined formally as the following:

## Definition 10. The Set $\Omega$

The set $\Omega$ under this study is the set of all pairs of commuting elements in the form of $(x, y)$, where $x$ and $y$ are the elements of the groups $G$ and the least common multiple of the order of the elements is two. In mathematical symbols, we can write it as:

$$
\Omega=\{(x, y) \in G \times G \mid x y=y x, x \neq y, \operatorname{lcm}(|x|,|y|)=2\}
$$

The results of the orbits are given in the following lemmas:
Lemma 1. Let $G$ be the dihedral group of order 24, $D_{12}$ and $\Omega=\{(x, y) \in G \times G \mid x y=y x, x \neq y, \operatorname{lcm}(|x|,|y|)=2\}$. Then the number of elements in the set $\Omega,|\Omega|=62$. If $G$ acts on $\Omega$ by conjugaction, then the number of orbits of $\Omega$ is $K(\Omega)=12$.

Proof. Suppose $G$ is the dihedral group of order $24, D_{12}$ and $\Omega=\{(x, y) \in G \times G \mid x y=y x, x \neq y, \operatorname{lcm}(|x|,|y|)=2\}$. The group $D_{12}$ has 13 elements with order two, which are $a^{6}, b, a b, a^{2} b, a^{3} b, a^{4} b, a^{5} b, a^{6} b, a^{7} b, a^{8} b, a^{9} b, a^{10} b$ and $a^{11} b$. By using Definition 10, it can be found that there are 62 elements of the set $\Omega$ which are listed as follows.

$$
\begin{aligned}
& \Omega=\left\{\left(1, a^{6}\right),(1, b),(1, a b),\left(1, a^{2} b\right),\left(1, a^{3} b\right),\left(1, a^{4} b\right),\left(1, a^{5} b\right),\left(1, a^{6} b\right),\left(1, a^{7} b\right),\left(1, a^{8} b\right),\left(1, a^{9} b\right),\left(1, a^{10} b\right),\left(1, a^{11} b\right),\left(a^{6}, 1\right),\left(a^{6}, b\right),\right. \\
& \left(a^{6}, a b\right),\left(a^{6}, a^{2} b\right),\left(a^{6}, a^{3} b\right),\left(a^{6}, a^{4} b\right),\left(a^{6}, a^{5} b\right),\left(a^{6}, a^{6} b\right),\left(a^{6}, a^{7} b\right),\left(a^{6}, a^{8} b\right),\left(a^{6}, a^{9} b\right),\left(a^{6}, a^{10} b\right),\left(a^{6}, a^{11} b\right),(b, 1),\left(b, a^{6}\right), \\
& \left(b, a^{6} b\right),(a b, 1),\left(a b, a^{6}\right),\left(a b, a^{7} b\right),\left(a^{2} b, 1\right),\left(a^{2} b, a^{6}\right),\left(a^{2} b, a^{8} b\right),\left(a^{3} b, 1\right),\left(a^{3} b, a^{6}\right),\left(a^{3} b, a^{9} b\right),\left(a^{4} b, 1\right),\left(a^{4} b, a^{6}\right),\left(a^{4} b, a^{10} b\right), \\
& \left(a^{5} b, 1\right),\left(a^{5} b, a^{6}\right),\left(a^{5} b, a^{11} b\right),\left(a^{6} b, 1\right),\left(a^{6} b, a^{6}\right),\left(a^{6} b, b\right),\left(a^{7} b, 1\right),\left(a^{7} b, a^{6}\right),\left(a^{7} b, a b\right),\left(a^{8} b, 1\right),\left(a^{8} b, a^{6}\right),\left(a^{8} b, a^{2} b\right),\left(a^{9} b, 1\right), \\
& \left.\left(a^{9} b, a^{6}\right),\left(a^{9} b, a^{3} b\right),\left(a^{10} b, 1\right),\left(a^{10} b, a^{6}\right),\left(a^{10} b, a^{4} b\right),\left(a^{11} b, 1\right),\left(a^{11} b, a^{6}\right),\left(a^{11} b, a^{5} b\right)\right\} .
\end{aligned}
$$

Suppose $D_{12}$ acts on $\Omega$ by conjugation. By using Definition 3, the orbits are calculated and listed as the following:
i) $\quad O\left(1, a^{6}\right)=\left\{\left(1, a^{6}\right)\right\}$.
ii) $O\left(a^{6}, 1\right)=\left\{\left(a^{6}, 1\right)\right\}$.
iii) $O(1, b)=\left\{(1, b),\left(1, a^{2} b\right),\left(1, a^{4} b\right),\left(1, a^{6} b\right),\left(1, a^{8} b\right),\left(1, a^{10} b\right)\right\}$.
iv) $O(b, 1)=\left\{(b, 1),\left(a^{2} b, 1\right),\left(a^{4} b, 1\right),\left(a^{6} b, 1\right),\left(a^{8} b, 1\right),\left(a^{10} b, 1\right)\right\}$.
v) $O(1, a b)=\left\{(1, a b),\left(1, a^{3} b\right),\left(1, a^{5} b\right),\left(1, a^{7} b\right),\left(1, a^{9} b\right),\left(1, a^{11} b\right)\right\}$.
vi) $O(a b, 1)=\left\{(a b, 1),\left(a^{3} b, 1\right),\left(a^{5} b, 1\right),\left(a^{7} b, 1\right),\left(a^{9} b, 1\right),\left(a^{11} b, 1\right)\right\}$.
vii) $O\left(a^{6}, b\right)=\left\{\left(a^{6}, b\right),\left(a^{6}, a^{2} b\right),\left(a^{6}, a^{4} b\right),\left(a^{6}, a^{6} b\right),\left(a^{6}, a^{8} b\right),\left(a^{6}, a^{10} b\right)\right\}$.
viii) $O\left(b, a^{6}\right)=\left\{\left(b, a^{6}\right),\left(a^{2} b, a^{6}\right),\left(a^{4} b, a^{6}\right),\left(a^{6} b, a^{6}\right),\left(a^{8} b, a^{6}\right),\left(a^{10} b, a^{6}\right)\right\}$.
ix) $O\left(a b, a^{6}\right)=\left\{\left(a b, a^{6}\right),\left(a^{3} b, a^{6}\right),\left(a^{5} b, a^{6}\right),\left(a^{7} b, a^{6}\right),\left(a^{9} b, a^{6}\right),\left(a^{11} b, a^{6}\right)\right\}$.
х) $O\left(a^{6}, a b\right)=\left\{\left(a^{6}, a b\right),\left(a^{6}, a^{3} b\right),\left(a^{6}, a^{5} b\right),\left(a^{6}, a^{7} b\right),\left(a^{6}, a^{9} b\right),\left(a^{6}, a^{11} b\right)\right\}$.
xi) $O\left(b, a^{6} b\right)=\left\{\left(b, a^{6} b\right),\left(a^{2} b, a^{8} b\right),\left(a^{4} b, a^{10} b\right),\left(a^{6} b, b\right),\left(a^{8} b, a^{2} b\right),\left(a^{10} b, a^{4} b\right)\right\}$.
xii) $O\left(a b, a^{7} b\right)=\left\{\left(a b, a^{7} b\right),\left(a^{3} b, a^{9} b\right),\left(a^{5} b, a^{11} b\right),\left(a^{7} b, a b\right),\left(a^{9} b, a^{3} b\right),\left(a^{11} b, a^{5} b\right)\right\}$.

Therefore, the number of orbits, $K(\Omega)=12$.
Lemma 2. Let $G$ be the semidirect product, $R=\mathbb{Z}_{3} \rtimes \mathbb{Z}_{8}$ and $\Omega=\{(x, y) \in G \times G \mid x y=y x, x \neq y, \operatorname{lcm}(|x|,|y|)=2\}$. Then the number of elements in the set $\Omega,|\Omega|=2$. If $G$ acts on $\Omega$ by conjugaction, then the number of orbits of $\Omega$ is $K(\Omega)=2$.

Proof. Suppose $G$ is the semidirect product, $R=\mathbb{Z}_{3} \rtimes \mathbb{Z}_{8}$ and $\Omega=\{(x, y) \in G \times G \mid x y=y x, x \neq y, \operatorname{lcm}(|x|,|y|)=2\}$. The group $G$ has only one element with order two, which is $a^{4}$. Based on Definition $10, \Omega=\left\{\left(1, a^{4}\right),\left(a^{4}, 1\right)\right\}$. Hence, the order of the set $\Omega,|\Omega|=2$. When $G$ acts on $\Omega$ by conjugation action, the orbits are listed below:
i) $O\left(1, a^{4}\right)=\left\{\left(1, a^{4}\right)\right\}$.
ii) $O\left(a^{4}, 1\right)=\left\{\left(a^{4}, 1\right)\right\}$.

Therefore, it was found that there are two orbits and $K(\Omega)=2$.

Lemma 3. Let $G$ be the semidirect product, $S=\mathbb{Z}_{3} \rtimes Q$ and $\Omega=\{(x, y) \in G \times G \mid x y=y x, x \neq y, \operatorname{lcm}(|x|,|y|)=2\}$. Then the number of elements in the set $\Omega,|\Omega|=2$. If $G$ acts on $\Omega$ by conjugaction, then the number of orbits of $\Omega$ is $K(\Omega)=2$.

Proof. Let $G$ be the semidirect product, $S=\mathbb{Z}_{3} \rtimes Q$ and $\Omega=\{(x, y) \in G \times G \mid x y=y x, x \neq y, \operatorname{lcm}(|x|,|y|)=2\}$. The semidirect product $S$ only has one element with order two, which is $a^{6}$. Based on Definition 10, the elements of the set $\Omega$ obtained are $\Omega=\left\{\left(1, a^{6}\right),\left(a^{6}, 1\right)\right\}$. Hence, the order of the set $\Omega,|\Omega|=2$. When $G$ acts on $\Omega$ by conjugation action, the orbits calculated are listed below:
i) $O\left(1, a^{6}\right)=\left\{\left(1, a^{6}\right)\right\}$.
ii) $O\left(a^{6}, 1\right)=\left\{\left(a^{6}, 1\right)\right\}$.

Therefore, there are two orbits and $K(\Omega)=2$.
Proposition 1. Let $G$ be some nonabelian metabelian groups of order 24 and $\Omega$ be the set of all pairs of commuting elements of $G$ in the form of $(x, y)$, where $x$ and $y$ are the elements of the groups $G$ and the least common multiple of the order of the elements is two. If $G$ acts on $\Omega$ by conjugation, the probability that a group element fixes a set for the groups are given as follows:

$$
P_{G}(\Omega)=\left\{\begin{array}{l}
\frac{6}{31}, \text { if } G=D_{12} \\
1, \text { if } G=\Phi_{3 \prime \prime}{ }_{8}, \Phi_{3} \prime \prime Q
\end{array}\right.
$$

Proof. Let $G$ be the dihedral group of order $24, D_{12}$. Based on the results in Lemma 1, the number of elements of the set $\Omega,|\Omega|=62$ and the number of orbits obtained by using conjugation action, $K(\Omega)=12$. Therefore, by using Definition 5 , the probability that an element of $D_{12}$ fixes the set $\Omega$ is $P_{D_{12}}(\Omega)=\frac{12}{62}=\frac{6}{31}$.

Assume $G$ to be the semidirect products, $R=\mathbb{Z}_{3} \rtimes \mathbb{Z}_{8}$ and $S=\mathbb{Z}_{3} \rtimes Q$. Referring to Lemma 2 and Lemma 3, the number of elements of the set $\Omega,|\Omega|=2$ for both $R$ and $S$. The number of orbits obtained by conjugation action for both semidirect products, $K(\Omega)=2$. Therefore, the probability that an element of the groups fixes a set, $P_{R}(\Omega)=P_{S}(\Omega)=\frac{2}{2}=1$.

Proposition 2. Let $G$ be some nonabelian metabelian groups of order 24 and $\Omega$ be the set of all pairs of commuting elements of $G$ in the form of $(x, y)$, where $x$ and $y$ are the elements of the groups $G$ and the least common multiple of the order of the elements is two. If $G$ acts on $\Omega$ by conjugation, then the generalized conjugacy class graph, $\Gamma_{G}^{\Omega}$ of the groups are given as the following:

$$
\Gamma_{G}^{\Omega_{e}}=\left\{\begin{array}{l}
\text { complete graph, } K_{10}, \text { if } G=D_{12} \\
\text { null graph, } \mathrm{K}_{0}, \text { if } G=\Phi_{3}{ }_{3}^{\prime \prime} \mathrm{\Phi}_{8}, \mathrm{G}_{3} \prime \prime
\end{array}\right.
$$

Proof. Suppose $G$ is the dihedral group $D_{12}$. From Lemma 1, it has been found that the number of orbits, $K(\Omega)=12$ where two of the orbits are central orbits. Meanwhile the other ten orbits are non-central orbits. Hence, the number of vertices of the generalized conjugacy class graph, $V\left(\Gamma_{D_{1}}^{\Omega_{c}}\right)=12-2=10$. From the orbits that have been listed, it is clearly shown that the cardinality of all the non-central orbits is six. Hence, the greatest common divisor for all the order of orbits is six and is not coprime, making all vertices connected to each other. Therefore, the generalized conjugacy class graph, $\Gamma_{D_{12}}^{\Omega_{c}}$ is a complete graph of ten vertices, $K_{10}$. The graph is illustrated in the following figure:


Figure 1. The complete graph of ten vertices, $K_{10}$.

Suppose $G$ is the groups $R=\mathbb{Z}_{3} \rtimes \mathbb{Z}_{8}$ and $S=\mathbb{Z}_{3} \rtimes Q$. From Lemma 2 and Lemma 3, it has been found that the number of orbits for both groups, $K(\Omega)=2$. Both of the orbits found are central orbits. Hence, the vertices of the generalized conjugacy class graphs, $V\left(\Gamma_{R}^{\Omega_{C}}\right)=V\left(\Gamma_{S}^{\Omega_{C}}\right)=2-2=0$. Since the generalized conjugacy class graphs, do not have any vertex, no edges can be constructed. Therefore, the generalized conjugacy class graphs $\Gamma_{R}^{\Omega_{C}}$ and $\Gamma_{S}^{\Omega_{C}}$ are null graphs.

Proposition 3. Let $G$ be some nonabelian metabelian groups of order 24 and $\Omega$ be the set of all pairs of commuting elements of $G$ in the form of $(x, y)$, where $x$ and $y$ are the elements of the groups $G$ and the least common multiple of the order of the elements is two. If $G$ acts on $\Omega$ by conjugation, then the chromatic number for the generalized conjugacy class graph, $\chi\left(\Gamma_{G}^{\Omega_{C}}\right)$ of the groups are given as:

$$
\chi\left(\Gamma_{G}^{\Omega_{c}}\right)=\left\{\begin{array}{l}
10, \text { if } G=D_{12} \\
0, \text { if } G=\mathrm{Z}_{3}, \mathrm{Z}_{8}, \mathrm{Z}_{3}, \mathrm{Q}
\end{array}\right.
$$

Proof. Let $G$ be the dihedral group of order 24, $D_{12}$. From Proposition 2, the generalized conjugacy class graph obtained for the group $D_{12}$ is a complete graph of ten vertices, $K_{10}$. Since all of the vertices are connected to each other, each of the vertices needs to have different colors so that no two adjacent vertices share the same color. Therefore, $\chi\left(\Gamma_{D_{12}}^{\Omega_{c}}\right)=10$.

Let $G$ be the semidirect products $R=\mathbb{Z}_{3} \rtimes \mathbb{Z}_{8}$ and $S=\mathbb{Z}_{3} \rtimes Q$. Referring to Proposition 2, the generalized conjugacy class graph obtained for both of the groups are null graph. Since a null graph does not have any vertex, no color can be applied. Therefore, the chromatic number, $\chi\left(\Gamma_{R}^{\Omega_{c}}\right)=\chi\left(\Gamma_{S}^{\Omega_{c}}\right)=0$.

Proposition 4. Let $G$ be some nonabelian metabelian groups of order 24 and $\Omega$ be the set of all pairs of commuting elements of $G$ in the form of $(x, y)$, where $x$ and $y$ are the elements of the groups $G$ and the least common multiple of the order of the elements is two. If $G$ acts on $\Omega$ by conjugation, then the clique number for the generalized conjugacy class graph, $\omega\left(\Gamma_{G}^{\Omega_{c}}\right)$ of the groups are given as:

$$
\omega\left(\Gamma_{G}^{\Omega_{c}}\right)=\left\{\begin{array}{l}
10, \text { if } G=D_{12} \\
0, \text { if } G=\mathrm{Z}_{3}, \mathrm{Z}_{8}, \mathrm{Z}_{3}, \mathrm{Q}
\end{array}\right.
$$

Proof. Let $G$ be the dihedral group of order $24, D_{12}$. Based on the results in Proposition 2, the generalized conjugacy class graph obtained for the group $D_{12}$ is a complete graph of ten vertices, $K_{10}$. Clique number is the size of the largest complete subgraph that can be obtained from a graph. From a complete graph of ten vertices, the largest
complete subgraph that can be constructed is a complete graph of ten vertices itself. Therefore, the clique number, $\omega\left(\Gamma_{D_{12}}^{\Omega_{c}}\right)=10$.

Let $G$ be the semidirect products $R=\mathbb{Z}_{3} \rtimes \mathbb{Z}_{8}$ and $S=\mathbb{Z}_{3} \rtimes Q$. From Proposition 2, the generalized conjugacy class graph obtained for both of the groups are null graph. There is no complete subgraph can be obtained from a null graph. Therefore, the clique number, $\omega\left(\Gamma_{R}^{\Omega_{C}}\right)=\omega\left(\Gamma_{S}^{\Omega_{C}}\right)=0$.

## CONCLUSION

In this study, the orbits for some nonabelian metabelian groups of order 24 which are the dihedral group, $D_{12}$ as well as the semidirect products, $R=\mathbb{Z}_{3} \rtimes \mathbb{Z}_{8}$ and $S=\mathbb{Z}_{3} \rtimes Q$ are determined. These orbits are then used to obtain the probability that a group element fixes a set as well as the generalized conjugacy class graph. The results are summarized in the following table:

TABLE 1. Summary of the main results

| Metabelian Group | Number of <br> Orbits | Probability that a <br> Group Element Fixes <br> the Set $\boldsymbol{\Omega}$ | Generalized <br> Conjugacy Class <br> Graph | Chromatic <br> Number | Clique Number |
| :---: | :---: | :---: | :---: | :---: | :---: |
| $D_{12}$ | 12 | $\frac{6}{21}$ | Complete graph, | 10 | 10 |
|  |  | $K_{10}$ |  |  |  |
| $R=\mathbb{Z}_{3} \rtimes \mathbb{Z}_{8}$ | 2 | 1 | Null graph | 0 | 0 |
| $S=\mathbb{Z}_{3} \rtimes Q$ | 2 | 1 | Null graph | 0 | 0 |

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