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On the Conjugate and Conjugacy Class Graphs Related to a 3-generator 3-group

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Graph theory is the study of graphs, which are mathematical structures used to model pairwise relations between objects. A graph in this context is made up of vertices which are connected by edges. Many real-life problem are related to graphs, for example the electric network problem and computational linguistic. In this research, using the definition and results from the conjugacy classes, some graphs related to a 3-generator 3-group are found, together with some of their properties including the independent number, chromatic number, clique number and dominating number.

Keywords: graph, independent number, chromatic number, clique number, dominating number

1. INTRODUCTION

Graphs naturally can describe a model in real-life situation. Many practical problems can be represented using graphs. For example, graph are used to denote data organization and the flow of computation in computer science area. In addition, graph theory is widely used to study molecules in chemistry.

In recent years, many researchers have studied about the relation of graph theory and group theory. In 2005, Moreto *et al.*[1] studied about the finite groups whose conjugacy class graphs have a few vertices. A research on graphs associated to conjugacy classes of some three-generator groups have done by Mohd Noor *et al.*[2] Four types of graph is discussed in this paper including the commuting graph, non-commuting graph, conjugate graph and conjugacy classes graph. Besides, Ibrahim *et al.*[3] have also studied about the conjugacy classes of some finite metabelian groups and their related graphs.

The next section recalls some basic definitions and concepts in graph theory and group theory which are used in this paper.

2. PRELIMINARIES

In this section, some basic definitions in graph theory are included. The methodology used in this research will also be included.

Definition 2.1[4] Conjugacy Class of an Element

Let G be a finite group. Then the conjugacy class of the element a in G is given as:

$$cl(a) = \{g \in G : \text{there exist } x \in G \text{ with } g = x^{-1}ax\}.$$

Proposition 2.1[4]

The conjugacy class of the identity element is its own class, namely $cl(1) = \{1\}$.

Proposition 2.2[4]

Let a and b be two elements in a finite group G. The elements a and b are conjugate if they belong in one conjugacy class, that is $cl(a) = cl(b)$.

Proposition 2.3[4]

Suppose a is an element of a group G, then a lies in the center $Z(G)$ if and only if its conjugacy class has only one element.

Definition 2.2[5] Conjugate Graph

A conjugate graph, Γ_G^{conj}, is a graph whose vertices are non-central elements of G, that is $|V(\Gamma_G^{conj})| = |G| - |Z(G)|$ in which two vertices are adjacent if they are conjugate.

Definition 2.3[6] Conjugacy Class Graph

A conjugacy class graph, Γ_G^{cl}, is a graph whose vertices are the non-central conjugacy classes of G, i.e $|V(\Gamma_G^{cl})| = K_G - |Z(G)|$, in which K_G is the number of conjugacy classes in G. Two vertices are adjacent if their cardinalities are not coprime, which means that the greatest common divisor (gcd) of the number of vertices in the group is not equal to one.

Definition 2.4[7] Independent Number

A set of vertices in a graph is called independent if no two vertices in the set are adjacent. Meanwhile, the independent number is the number of vertices in the maximum independent set and it is denoted by $\alpha(\Gamma)$.

Definition 2.5[7] Chromatic Number

The chromatic number, denoted by $\chi(\Gamma)$, is the smallest number of colors needed to color the vertices of Γ so that no two adjacent vertices share the same color.

Definition 2.6[5] Clique Number

Clique is a complete subgraph in Γ. The clique number is the size of the largest clique in Γ and it is denoted by $\omega(\Gamma)$.

Definition 2.7[7] Dominating Number

The dominating set $X \leq V(\Gamma)$ is a set where for each v outside X, there exists x in X such that v is adjacent to x. The minimum size of X is called the dominating number and it is denoted by $\gamma(\Gamma)$.

The research scope for this paper is on a 3-generator 3-group. The classification and group presentation is given by Kim[8]. This paper focuses on some graphs associated to conjugacy classes of a 3-generator 3-group. However, only one type of 3-generator 3-group is considered as presented in (1):

$$H = < x, y, x \mid x^3 = y^3 = z^9 = 1, [x,z] = [y,z] = 1, [x,y] = z^3 >. \quad (1)$$

The order of H, $|H| = 81$ and the center of H, $Z(G) = \{ Z \mid i = 1,\ldots,8 \} = \{ 1, z, z^2, z^3, z^4, z^5, z^6, z^7, z^8\}$, thus $|Z(G)| = 9$.

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The objectives of this research are to find the two types of graphs, namely the conjugate graph and conjugacy class graph. In order to find these graphs, we first find their conjugacy classes and then we used their definitions. We also applied the definitions to determine the graphs properties.

3. RESULTS AND DISCUSSION

The main results of this paper are presented in three parts. The first part is the computation of the conjugacy classes of a 3-generator 3-group based on the given relation in (1). The second part will be the conjugate graph based on the conjugacy classes calculated in the first part and also its properties. The last part is the conjugacy class graph with the properties of graph.

3.1. THE CONJUGACY CLASSES OF 3-GENERATOR 3-GROUP

Theorem 3.1 Let $H = \langle x, y, z \mid x^3 = y^3 = z^9 = 1, [x, z] = [y, z] = 1, [x, y] = z^3 \rangle$. Then the number of conjugacy classes of H, $K(H) = 33$.

Proof: Using Definition 2.1, the conjugacy classes of H can be found. Thus, the conjugacy classes of H are determined as follows: First, let $a = 1$. By Proposition 2.1, cl(1) = {1}.

Next, we find the conjugacy class for a, cl(a) = { hah^{-1}} where $h \in H$ = {1, x, y, z, x^2, y^2, z^2, z^3, z^4, z^5, z^6, z^7, z^8, xy, xy^2, xz, xz^2, xz^3, xz^4, xz^5, xz^6, xz^7, xz^8, x^2y, x^2y^2, x^2z, x^2z^2, x^2z^3, x^2z^4, x^2z^5, x^2z^6, x^2z^7, x^2z^8, yz, yz^2, yz^3, yz^4, yz^5, yz^6, yz^7, yz^8, y^2z, y^2z^2, y^2z^3, y^2z^4, y^2z^5, y^2z^6, y^2z^7, y^2z^8, xyz, xyz^2, xyz^3, xyz^4, xyz^5, xyz^6, xyz^7, xyz^8, xy^2z, xy^2z^2, xy^2z^3, xy^2z^4, xy^2z^5, xy^2z^6, xy^2z^7, xy^2z^8, x^2yz, x^2yz^2, x^2yz^3, x^2yz^4, x^2yz^5, x^2yz^6, x^2yz^7, x^2yz^8, x^2y^2z, $x^2y^2z^2, x^2y^2z^3$, $x^2y^2z^4$, $x^2y^2z^5$, $x^2y^2z^6$, $x^2y^2z^7$, $x^2y^2z^8$ } for all a in H. The conjugate elements when a = x are determined as follows:

When $h = x$, $hah^{-1} = (x)x(x)^{-1} = x$,
when $h = y$, $hah^{-1} = (y)x(y)^{-1} = yxy^2 = yxyy = yyxz^2y$ (since $xy = yxz^3$) $= yyxyz^3 = yyyxz^3z^3 = xz^6$,
when $h = z$, $hah^{-1} = (z)x(z)^{-1} = zxz^8 = x$ (since $z^9 = 1$, thus $z^{-1} = z^8$) **note: the conjugate element of $z^n = x$ because z^n is in the center,
when $h = x^2$, $hah^{-1} = (x^2)x(x^2)^{-1} = x$,
when $h = y^2$, $hah^{-1} = (y^2)x(y^2)^{-1} = (y^2)x(y) = y^2yxz^3 = xz^3$,
when $h = xy$, $hah^{-1} = (xy)x(xy)^{-1} = (xy)x(x^2y^2z^6) = xz^6$,
when $h = xy^2$, $hah^{-1} = (xy^2)x(xy^2)^{-1} = (xy^2)x(x^2yz^3) = xz^3$,
when $h = xz$, $hah^{-1} = (xz)x(xz)^{-1} = (xz)x(x^2z^8) = x$ **note: the conjugate element of $xz^n = x$ because z^n is in the center,
when $h = x^2y$, $hah^{-1} = (x^2y)x(x^2y)^{-1} = (x^2y)x(xy^2z^3) = xxyxxyyz^3 = xyxz^3xxyyz^3 = xz^6$,
when $h = x^2y^2$, $hah^{-1} = (x^2y^2)x(x^2y^2)^{-1} = (x^2y^2)x(xyz^6) = xxyyxxyz^6 = xxyyxyxz^3z^6 = xxyyyxz^3xz^3z^6 = xz^3$,
when $h = x^2z$, $hah^{-1} = (x^2z)x(x^2z)^{-1} = (x^2z)x(xz^8) = x$ **note: the conjugate element of $x^2z^n = x$ because z^n is in the center,
when $h = yz$, $hah^{-1} = (yz)x(yz)^{-1} = (yz)x(y^2z^8) = yzxyyz^8 = yxyyzz^8 = yyxz^3y = yyxyz^3 = yyyxz^3z^3 = xz^6$ **note: the conjugate element of $yz^n = xz^6$ because z^n is in the center,
when $h = y^2z$, $hah^{-1} = (y^2z)x(y^2z)^{-1} = (y^2z)x(yz^8) = yyxyzz^8 = yyyxz^3 = xz^3$ **note: the conjugate element of $y^2z^n = xz^3$ because z^n is in the center, when $h = xyz$, $hah^{-1} = (xyz)x(xyz)^{-1} = (xyz)x(x^2y^2z^5) = xz^6$ **note: the conjugate element of $xyz^n = xz^6$ because z^n is in the center,
when $h = xy^2z$, $hah^{-1} = (xy^2z)x(xy^2z)^{-1} = (xy^2z)x(x^2yz^2) = xz^3$ **note: the conjugate element of $xy^2z^n = xz^3$ because z^n is in the center,
when $h = x^2yz$, $hah^{-1} = (x^2yz)x(x^2yz)^{-1} = (x^2yz)x(xy^2z^2) = xyxz^3zxxyyzz = xz^6$ **note: the conjugate element of $x^2yz^n = xz^6$ because z^n is in the center,
when $h = x^2y^2z$, $hah^{-1} = (x^2y^2z)x(x^2y^2z)^{-1} = (x^2y^2z)x(xyz^5) = xxyyzxxyz^5 = xxyyzxyxz^3z^5 = xxyyyxyx = xxyyyxz^3x = xz^3$ **note: the conjugate element of $x^2y^2z^n = xz^3$ because z^n is in the center.
Thus, $cl(x)$ = { x, xz^3, xz^6}. Since x, xz^3 and xz^6 belong in one conjugacy class, then by Proposition 2.2, $cl(x) = cl(xz^3) = cl(xz^6)$.

By replacing a with the rest of the elements in the set { y, z, x^2, y^2, z^2, z^3, z^4, z^5, z^6, z^7, z^8, y, xy^2, xz, xz^2, xz^3, xz^4, xz^5, xz^6, xz^7, xz^8, x^2y, x^2y^2, x^2z, x^2z^2, x^2z^3, x^2z^4, x^2z^5, x^2z^6, x^2z^7, x^2z^8, yz, yz^2, yz^3, yz^4, yz^5, yz^6, yz^7, yz^8, y^2z, y^2z^2, y^2z^3, y^2z^4, y^2z^5, y^2z^6, y^2z^7, y^2z^8, xyz, xyz^2, xyz^3, xyz^4, xyz^5, xyz^6, xyz^7, xyz^8, xy^2z, xy^2z^2, xy^2z^3, xy^2z^4, xy^2z^5, xy^2z^6, xy^2z^7, xy^2z^8, x^2yz, x^2yz^2, x^2yz^3, x^2yz^4, x^2yz^5, x^2yz^6, x^2yz^7, x^2yz^8, x^2y^2z, $x^2y^2z^2, x^2y^2z^3$, $x^2y^2z^4$, $x^2y^2z^5$, $x^2y^2z^6$, $x^2y^2z^7$, $x^2y^2z^8$}, the conjugacy classes of H are found as follows:

$cl(1) = \{1\}$,
$cl(x) = \{x, xz^3, xz^6\} = cl(xz^3) = cl(xz^6)$,
$cl(y) = \{y, yz^3, yz^6\} = cl(yz^3) = cl(yz^6)$,
$cl(z) = \{z\}$,
$cl(z^2) = \{z^2\}$,
$cl(z^3) = \{z^3\}$,
$cl(z^4) = \{z^4\}$,
$cl(z^5) = \{z^5\}$,
$cl(z^6) = \{z^6\}$,
$cl(z^7) = \{z^7\}$,
$cl(z^8) = \{z^8\}$,
$cl(x^2) = \{x^2, x^2z^3, x^2z^6\} = cl(x^2z^3) = cl(x^2z^6)$,
$cl(y^2) = \{y^2, y^2z^3, y^2z^6\} = cl(y^2z^3) = cl(y^2z^6)$,
$cl(xy) = \{xy, xyz^3, xyz^6\} = cl(xyz^3) = cl(xyz^6)$,
$cl(xy^2) = \{xy^2, xy^2z^3, xy^2z^6\} = cl(xy^2z^3) = cl(xy^2z^6)$,
$cl(xz) = \{xz, xz^4, xz^7\} = cl(xz^4) = cl(xz^7)$,
$cl(xz^2) = \{xz^2, xz^5, xz^8\} = cl(xz^5) = cl(xz^8)$,
$cl(x^2y) = \{x^2y, x^2yz^3, x^2yz^6\} = cl(x^2yz^3) = cl(x^2yz^6)$,
$cl(x^2y^2) = \{x^2y^2, x^2y^2z^3, x^2y^2z^6\} = cl(x^2y^2z^3) = cl(x^2y^2z^6)$,
$cl(x^2z) = \{x^2z, x^2z^4, x^2z^7\} = cl(x^2z^4) = cl(x^2z^7)$,
$cl(x^2z^2) = \{x^2z^2, x^2z^5, x^2z^8\} = cl(x^2z^5) = cl(x^2z^8)$,
$cl(yz) = \{yz, yz^4, yz^7\} = cl(yz^4) = cl(yz^7)$,
$cl(yz^2) = \{yz^2, yz^5, yz^8\} = cl(yz^5) = cl(yz^8)$,
$cl(y^2z) = \{y^2z, y^2z^4, y^2z^7\} = cl(y^2z^4) = cl(y^2z^7)$,
$cl(y^2z^2) = \{y^2z^2, y^2z^5, y^2z^8\} = cl(y^2z^5) = cl(y^2z^8)$,
$cl(xyz) = \{xyz, xyz^4, xyz^7\} = cl(xyz^4) = cl(xyz^7)$,
$cl(xyz^2) = \{xyz^2, xyz^5, xyz^8\} = cl(xyz^5) = cl(xyz^8)$,
$cl(xy^2z) = \{xy^2z, xy^2z^4, xy^2z^7\} = cl(xy^2z^4) = cl(xy^2z^7)$,
$cl(xy^2z^2) = \{xy^2z^2, xy^2z^5, xy^2z^8\} = cl(xy^2z^5) = cl(xy^2z^8)$,
$cl(x^2yz) = \{x^2yz, x^2yz^4, x^2yz^7\} = cl(x^2yz^4) = cl(x^2yz^7)$,
$cl(x^2yz^2) = \{x^2yz^2, x^2yz^5, x^2yz^8\} = cl(x^2yz^5) = cl(x^2yz^8)$,
$cl(x^2y^2z) = \{x^2y^2z, x^2y^2z^4, x^2y^2z^7\} = cl(x^2y^2z^4) = cl(x^2y^2z^7)$,
$cl(x^2y^2z^2) = \{x^2y^2z^2, x^2y^2z^5, x^2y^2z^8\} = cl(x^2y^2z^5) = cl(x^2y^2z^8)$. (2)

Therefore, K(H) = 33.

■

3.2.THE CONJUGATE GRAPH OF 3-GENERATOR 3-GROUP

Theorem 3.2 Let H= <x, y, z | x^3= y^3= z^9= 1, [x, z] = [y, z] = 1, [x, y] = z^3> be a 3-generator 3-group. Then, the conjugacy class graph of H, Γ_H^{conj} = $\bigcup_{i=1}^{24} K_3$, that is, a union of 24 complete graphs with three vertices.

Proof: As mentioned earlier, the order of the group H is 81 while the order of the center of the group is 9. Thus, the order of the vertices of the conjugate graph is equal to 81 – 9 = 72. By Definition 2.2, two vertices are adjacent if they are conjugate. Based on Proposition 2.2, any two elements in a group G which belong to the same conjugacy class are conjugate. Recall from the conjugacy classes of H listed in (2), the number of edges in the conjugate graph with 72 vertices is equal to 24. The conjugate graph of H is presented in Fig. 1:

The four properties of this graph are discussed as follows:

Proposition 3.1 Let H= <x, y, z | x^3= y^3= z^9= 1, [x, z] = [y, z] = 1, [x, y] = z^3> be a 3-generator 3-group. Then, the independent number of the conjugate graph of H, α(Γ_H^{conj}) = 24.

Proof: By Definition 2.4, the independent number is the number of vertices in a maximum independent set. From Figure 1, one of the maximum independent sets is {x, x^2, y, y^2, xy, xy^2, xz, xz^2, x^2y, x^2y^2, x^2z, x^2z^2, yz, yz^2, y^2z, y^2z^2, xyz, xyz^2, xy^2z, xy^2z^2, x^2yz, x^2yz^2, x^2y^2z, $x^2y^2z^2$ }. Since the number of vertices of the maximum independent set is 24, thus, the independent number of H, α(Γ_H^{conj}) = 24.

Proposition 3.2 Let H= <x, y, z | x^3= y^3= z^9= 1, [x, z] = [y, z] = 1, [x, y] = z^3> be a 3-generator 3-group. Then, the chromatic number of the conjugate graph of H, χ(Γ_H^{conj}) = 3.

Proof: Based on Definition 2.5, the chromatic number χ(Γ_H^{conj}) of the conjugate graph H is 3 since the three adjacent vertices in K_3 have different colors of vertices. ■

Proposition 3.3 Let H= <x, y, z | x^3= y^3= z^9= 1, [x, z] = [y, z] = 1, [x, y] = z^3> be a 3-generator 3-group. Then, the clique number of the conjugate graph of H, ω(Γ_H^{conj}) = 3.

Proof: The clique number is the size of the largest complete subgraph in a graph. The largest complete subgraph in the conjugate graph of H is K_3. Therefore, the clique number, ω(Γ_H^{conj}) = 3. ■

Proposition 3.4 Let H= <x, y, z | x^3= y^3= z^9= 1, [x, z] = [y, z] = 1, [x, y] = z^3> be a 3-generator 3-group. Then, the dominating number of the conjugate graph of H, denoted by γ(Γ_H^{conj}) is equal to 24.

Proof: The proof is based on Definition 2.7. The minimum size of vertex which can connect itself and other vertices in Γ_H^{conj} is 24. Thus, the dominating number, γ(Γ_H^{conj})= 24. ■

3.2 THE CONJUGACY CLASS GRAPH OF 3-GENERATOR 3-GROUP

Theorem 3.3 Let H= <x, y, z | x^3= y^3= z^9= 1, [x, z] = [y, z] = 1, [x, y] = z^3> be a 3-generator 3-group. Then, the conjugacy class graph of H, Γ_H^{Cl} = K_{24}.

Proof: By Definition 2.3, since the number of conjugacy classes of H is 33 while the order of the center of H is 9, thus the number of the vertices with non-central elements of H is 24. Notice that all conjugacy classes have size 3, so their greatest common divisor is also 3. Thus, the graph is complete. The conjugacy class graph of H is shown in Fig. 2.

The four properties of this graph are discussed as follows:

Proposition 3.5 Let H= <x, y, z | x^3= y^3= z^9= 1, [x, z] = [y, z] = 1, [x, y] = z^3> be a 3-generator 3-group. Then, the independent number of the conjugacy class graph of H, α(Γ_H^{cl}) = 1.

Proof: Based on Figure 2, since all vertices are connected, therefore the maximum independent set is 1. Thus, the independent number of H, α(Γ_H^{cl}) = 1. ■

Proposition 3.6 Let H= <x, y, z | x^3= y^3= z^9= 1, [x, z] = [y, z] = 1, [x, y] = z^3> be a 3-generator 3-group. Then, the chromatic number of the conjugacy class graph of H, χ(Γ_H^{cl}) = 24.

Proof: The chromatic number of the conjugacy class graph H is 24 since the 24 adjacent vertices in K_{24} have different colors of vertices. ■

Proposition 3.7 Let H= <x, y, z | x^3= y^3= z^9= 1, [x, z] = [y, z] = 1, [x, y] = z^3> be a 3-generator 3-group. Then, the clique number of the conjugacy class graph of H, ω(Γ_H^{cl}) = 24.

Proof: As stated in Definition 2.5, the clique number is the size of the largest complete subgraph in a graph. The largest complete subgraph of the conjugacy class graph of H is K_{24}. Therefore, the clique number, ω(Γ_H^{cl}) = 24. ■

Proposition 3.8 Let H= <x, y, z | x^3= y^3= z^9= 1, [x, z] = [y, z] = 1, [x, y] = z^3> be a 3-generator 3-group. Then, the dominating number of the conjugacy class graph of H, denoted by γ(Γ_H^{cl}) is equal to 1.

Proof: The minimum size of vertex which can connect itself and other vertices in Γ_H^{cl} is 1. Thus, the dominating number, γ(Γ_H^{cl})= 1. ■

4. CONCLUSION

As a conclusion, the number of the conjugacy classes of a 3-generator 3-group in the scope of this paper is found to be 33. The conjugate graph of H is shown to be the union of 24 complete graphs of K_3 while the conjugacy class graph is the complete graph K_{24}. Some properties of the graphs have also been determined which includes the independent number, chromatic number, clique number and dominating number.

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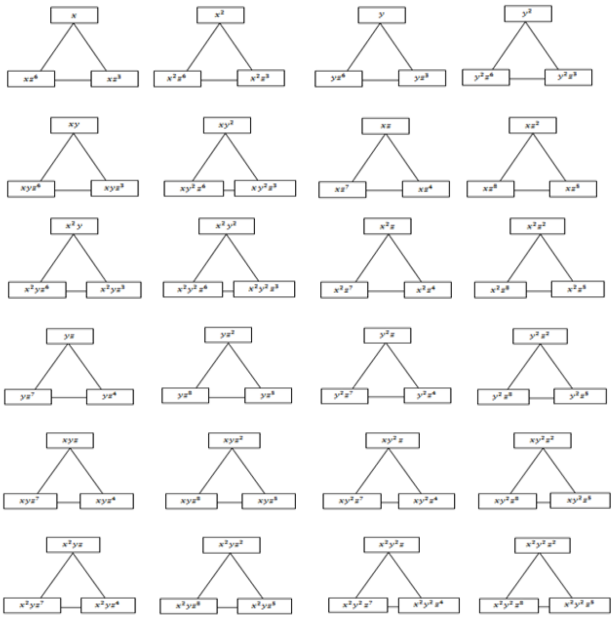

Fig. 1 The Conjugate Graph of *H*.

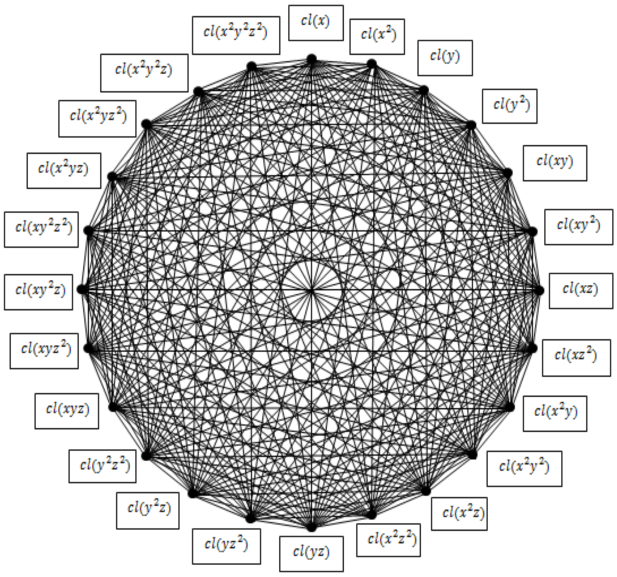

Fig. 2 The Conjugacy Class Graph of *H*.

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