Southeast Asian Bulletin of Mathematics (2018) 42: 729–746

Southeast Asian
Bulletin of
Mathematics

Recent Updates on Homological Invariants of Bieberbach Groups*

N. H. Sarmin and H. I. Mat Hassim
Dept. of Mathematical Sciences, Faculty of Science, Universiti Teknologi Malaysia, 81310 UTM Johor Bahru, Johor, Malaysia
Email: nhs@utm.my; hazzirah@utm.my

S. A. Mohammad
Faculty of Computer and Mathematical Sciences, Universiti Teknologi MARA, 85000 Segamat, Johor, Malaysia
Email: sitiafiqah@uitm.edu.my

N. Mohd Idrus, R. Masri and Y. T. Tan
Department of Mathematics, Faculty of Science and Mathematics, Universiti Pendidikan Sultan Idris, 35900, Tanjung Malim, Perak, Malaysia
Email: {norashiqin, rohaidah}@fsmt.upsi.edu.my; yeeting07@yahoo.com.my

Received 15 September 2016
Accepted 24 January 2018

Communicated by Intan Muchtadi

AMS Mathematics Subject Classification(2000): 12G05

Abstract. Homological invariants or homological functors of groups have their roots in algebraic K-theory and homotopy theory. They were first used in algebraic topology, but are common in many areas of mathematics. The homological invariants are also used in group cohomology to classify abelian group extensions. Researches on homological invariants have grown intensively over the years. In this paper, recent updates of the homological invariants of Bieberbach groups with certain point groups will be presented. Furthermore, some of the homological invariants of a Bieberbach group of dimension six with quaternion point group of order eight are computed.

*Supported in part by GUP grant (13H79), RACE Grant Scheme (Vote no: 2012-0133-101-62), UTM Zamalah Scholarship, Postdoctoral in University of Leeds, MyPhD Scholarship.

Keywords: Homological invariants; Bieberbach groups; Point groups.

1. Introduction

Homological invariants are useful tools in mathematical areas such as in algebraic number theory, block theory of group algebras and classification of finite simple groups. A Bieberbach group is a torsion free crystallographic group in which all elements are of infinite orders except its identity. It is an extension of a free abelian group, L of finite rank by a finite group, P. In other words, there is a short exact sequence $L \overset{\phi}{\to} G \overset{\sigma}{\to} P$ such that $G/\phi(L) \cong P$ where ϕ is a monomorphism and σ is an epimorphism. Furthermore, the image of ϕ is the kernel of σ.

Being one of the torsion free crystallographic groups, Bieberbach group has the symmetry structure. It widely covers the enumeration of the symmetry structure which can be formed in a crystal and as for that a crystal has a close relation to group theory and geometry. One visualizes a crystal as an interlocking system of atoms that can move indefinitely in any way filling up all space and at the same time developing the regular pattern. Our mathematical approach is to replace the regular pattern by the group G with rigid motions on Euclidean space that preserve it. When one contemplate a crystal as a regular pattern, all solid motions which transform this pattern into itself are symmetry transform in which it can be given a group structure such as homological invariants. Each subgroup of orthogonal group is called a point group, because it leaves the origin invariant. Thus, any point group which transforms a lattice into itself is called a crystallographic point group [8].

A technique developed by Blyth and Morse [4] which is $\nu(G)$ will be used where

$$\nu(G) = \langle G, G^{\varphi} | R, R^{\varphi}, [g, h^{\varphi}]^{x} = [g^{x}, (h^{x})^{\varphi}] = [g, h^{\varphi}]^{x^{\varphi}}, \forall x, g, h \in G \rangle$$

where $g^h = h^{-1}gh$ and $[g, h] = g^{-1}g^h$. This technique is applied in which the polycyclic presentations were constructed by using their matrix representation and then were tested to be satisfying its consistency relations. It is necessary to show the consistency of the presentation before its homological invariants can be determined.

Researches on the homological invariants of various groups have gained significant interest over the years. Algebraic K-theory and homotopy theory were the origin of these studies while extending the ideas of Whitehead [22]. It was started by the study of the nonabelian tensor square which is one of the homological invariants by Brown and Loday [5]. Then, Brown *et al.* [6] introduced the homological invariants of a group G in Figure 1. (see Figure 1).

In this paper, the homological invariants which are considered are the nonabelian tensor squares $(G \otimes G)$, the kernel of the nonabelian tensor square $(J(G))$, $\nabla(G)$, the exterior square $(G \wedge G)$ and the Schur multiplier $(M(G))$. Furthermore, the computation of the central subgroup of a Bieberbach group

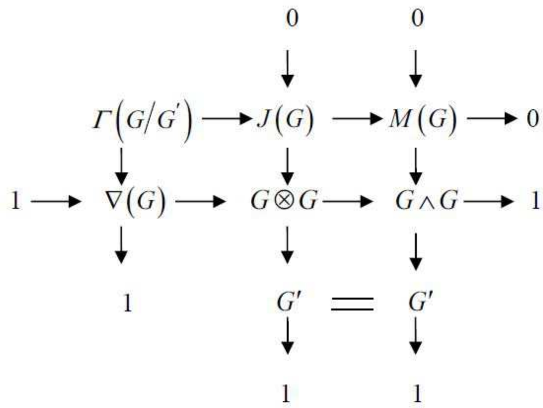

Figure 1. The commutative diagram of the homological invariants.

of dimension six with quaternion point group of order eight will be shown. All definitions and propositions that are used throughout this research are stated in the following.

Definition 1.1. [1] *The nonabelian tensor square of a group G, denoted as $G \otimes G$, is generated by the symbols $g \otimes h$ for all $g, h \in G$ subject to the relations*

$$gh \otimes k = (g^h \otimes k^h)(h \otimes k) \text{ and } g \otimes hk = (g \otimes k)(g^k \otimes h^k)$$

for all $g, h, k \in G$, where $g^h = h^{-1}gh$.

The group G acts naturally on the nonabelian tensor squares by $(g \otimes k)^h = (g^h \otimes k^h)$ and there exists a homomorphism $\kappa : G \otimes G \to G'$ defined by $\kappa(g \otimes h) = [g, h]$ where $[g, h] = g^{-1}h^{-1}gh$. Another homological invariant, known as $J(G)$, which is the kernel of the nonabelian tensor square is defined as in the following:

Definition 1.2. [2, 15] *Let G be a group. Then, $J(G)$ is the kernel of the homomorphism of the nonabelian tensor square with $\kappa : G \otimes G \to G'$ and kernel(κ) $= \{x \in G | kernel(\kappa) = 1\}$.*

Next, a central subgroup of $G \otimes G$, known as $\nabla(G)$, is defined as follows.

Definition 1.3. [2] *Let G be a group. Then, $\nabla(G)$ is the subgroup of $J(G)$ generated by the elements $x \otimes x$ for all $x \in G$.*

Besides, another three homological invariants are stated in the following definitions.

Definition 1.4. [2] *The exterior square of a group G, denoted as $G \wedge G$, is defined to be*

$$G \wedge G = {}^{(G \otimes G)}\!/_{\nabla(G)}.$$

For g and h in G, the coset $(g \otimes h)\nabla(G)$ is denoted by $g \wedge h$.

Definition 1.5. [2] *The Schur multiplier of a group G, $M(G)$, is defined to be*

$$M(G) = {}^{J(G)}\!/_{\nabla(G)}.$$

Definition 1.6. [7] **(Polycyclic Presentation)** *Let F_n be a free group on generators $g_i, \ldots, g_n$ and R be a set of relations of group F_n. The relations of a polycyclic presentation ${}^{F_n}\!/_R$ have the form:*

$$\begin{aligned} g_i^{e_i} &= g_{i+1}^{x_{i,i+1}} \cdots g_n^{x_{i,n}} && \text{for } i \in I, \\ g_j^{-1} g_i g_j &= g_{j+1}^{y_{i,j,j+1}} \cdots g_n^{y_{i,j,n}} && \text{for } j < i, \\ g_j g_i g_j^{-1} &= g_{j+1}^{z_{i,j,j+1}} \cdots g_n^{z_{i,j,n}} && \text{for } j < i \text{ and } j \notin I \end{aligned}$$

for some $I \subseteq \{1, \ldots n\}$, certain exponents $e_i \in \mathbb{N}$ for $i \in I$ and $x_{i,j}, y_{i,j,k}, z_{i,j,k} \in \mathbb{Z}$ for all i, j and k.

Blyth and Morse [4] proved that if G is polycyclic, then $G \otimes G$ is polycyclic. Hence, $G \otimes G$ has a consistent polycyclic presentation. The following definition is needed in order to check the polycyclic presentation of a group that has been constructed using the above definition satisfies all the consistency relations in order to compute its homological invariants later.

Definition 1.7. [7] **(Consistent Polycyclic Presentation)** *Let G be a group generated by $g_1, \ldots, g_n$ and the consistency relations in G can be evaluated in the polycyclic presentation of G using the collection from the left as in the following:*

$$\begin{aligned} g_k(g_j g_i) &= (g_k g_j) g_i && \text{for } k > j > i, \\ (g_j^{e_j}) g_i &= g_j^{e_j - 1}(g_j g_i) && \text{for } j > i, j \in I, \\ g_j(g_i^{e_i}) &= (g_j g_i) g_i^{e_i - 1} && \text{for } j > i, i \in I, \\ (g_i^{e_i}) g_i &= g_i (g_i^{e_i}) && \text{for } i \in I, \\ g_j &= (g_j g_i^{-1}) g_i && \text{for } j > i, i \notin I \end{aligned}$$

for some $I \subseteq \{1, \ldots, n\}$, $e^i \in \mathbb{N}$. Then, G is said to be given by a consistent polycyclic presentation.

Theorem 1.8. [16] *Let G be a group. Then, the map $\sigma : G \otimes G \to [G, G^{\varphi}] \lhd \nu(G)$ defined by $\sigma(g \otimes h) = [g, h^{\varphi}]$ for all g, h in G is an isomorphism.*

The theorem above indicates that $G \otimes G$ is isomorphic to $[G, G^{\varphi}]$. Hence, in this paper, $[G, G^{\varphi}]$ is computed in order to find the nonabelian tensor square of a group.

To find the exterior square of a group, $G \wedge G$, the next theorem that indicates $G \wedge G$ is isomorphic to $[G, G^{\varphi}]_{\tau(G)}$ is stated. Before that, the following definition is needed.

Definition 1.9. [4] *Let G be any group. Then $\tau(G)$ is defined to be the quotient group ${}^{\nu(G)}/_{\sigma(\nabla(G))}$, where $\sigma : G \otimes G \to [G, G^{\varphi}]$ is as defined in Theorem 1.8.*

Theorem 1.10. [4] *Let G be any group. Then, the map*

$$\hat{\sigma} : G \wedge G \to [G, G^{\varphi}]_{\tau(G)} \lhd \tau(G)$$

defined by $\hat{\sigma}(g \wedge h) = [g, h^{\varphi}]_{\tau(G)}$ is an isomorphism.

Hence, in this research, $[G, G^{\varphi}]_{\tau(G)}$ is computed in order to find the exterior square of a group. Since $\tau(G)$ is a subgroup of $\nu(G)$, $[g, h^{\varphi}]_{\tau(G)}$ coincides with $[g, h^{\varphi}]$. Therefore, in this paper, for simplification, $[g, h^{\varphi}]$ is used instead of $[g, h^{\varphi}]_{\tau(G)}$.

Theorem 1.11. [4] *Let G be a polycyclic group with a polycyclic generating sequence $g_1, \ldots, g_k$. Then $[G, G^{\varphi}]$, a subgroup of $\nu(G)$, is given by*

$$[G, G^{\varphi}] = \langle [g_i, g_i^{\varphi}], [g_i^{\delta}, (g_j^{\varphi})^{\varepsilon}], [g_i, g_j^{\varphi}][g_j, g_i^{\varphi}] \rangle$$

and $[G, G^{\varphi}]_{\tau(G)}$, a subgroup of $\tau(G)$, is given by

$$[G, G^{\varphi}]_{\tau(G)} = \langle [g_i^{\delta}, (g_j^{\varphi})^{\varepsilon}], [g_j^{\varepsilon}, (g_i^{\varphi})^{\delta}] \rangle$$

for $1 \leq i < j \leq k$, *where*

$$\varepsilon = \begin{cases} 1 & \text{if} \quad |\mathfrak{g}_i| < \infty, \\ \pm 1 & \text{if} \quad |\mathfrak{g}_i| = \infty \end{cases}$$

and

$$\delta = \begin{cases} 1 & \text{if} \quad |\mathfrak{g}_j| < \infty, \\ \pm 1 & \text{if} \quad |\mathfrak{g}_j| = \infty. \end{cases}$$

Theorem 1.12. [3] *Let G be a group such that G^{ab} is finitely generated. Assume that G^{ab} is the direct product of the cyclic groups $\langle x_i G' \rangle$, for $i = 1, \ldots, s$ and set $E(G)$ to be $\langle [x_i, x_j^{\varphi}] | i < j \rangle [G, G'^{\varphi}]$. Then the following hold:*

(i) *$\nabla(G)$ is generated by the elements of the set $\{[x_i, x_i^{\varphi}], [x_i, x_j^{\varphi}][x_j, x_i^{\varphi}] | 1 \leq i < j \leq s\}$;*

(ii) *$[G, G^{\varphi}] = \nabla(G)E(G)$.*

The next theorem is used to identify the order of the elements.

Theorem 1.13. [9] *Let G and H be groups and let $g \in G$. Suppose ϕ is a homomorphism from G onto H. If $\phi(g)$ has finite order then $|\phi(g)|$ divides $|g|$. Otherwise the order of $\phi(g)$ equals the order of g.*

In the next theorem, some properties of the ordinary tensor product are presented.

Theorem 1.14. [23] *Let A, B and C be abelian groups and C_0 the infinite cyclic group. Consider the ordinary tensor product of two abelian groups. Then*

(i) *$C_0 \otimes A \cong A$,*

(ii) *$C_0 \otimes C_0 \cong C_0$,*

(iii) *$C_n \otimes C_m \cong C_{gcd(n,m)}$, for $n, m \in \mathbb{Z}$, and*

(iv) *$A \otimes (B \times C) = (A \otimes B) \times (A \otimes C)$.*

Theorem 1.15. [6] *Let G and H be groups such that there is an epimorphism $\eta : G \to H$. Then there exists an epimorphism $\alpha : G \otimes G \to H \otimes H$ defined by $\alpha(g \otimes h) = \eta(g) \otimes \eta(h)$.*

2. Recent Advancements on Homological Invariants of Bieberbach Groups

This section provides some results on the homological invariants of Bieberbach groups with certain point groups, namely C_2, C_3, C_5, D_4 and S_3. The presentation of the groups can be found in Crystallographic Algorithms and Table

(CARAT) package [24]. The following are results of some homological invariants for two Bieberbach groups with point group C_2 found by Masri [9] in which their nonabelian tensor squares are abelian and the results can be extended to find the nonabelian tensor squares for this group of the same family up to dimension n. The research was then extended by Mat Hassim [10].

Theorem 2.1. [9, 10] *Let $B_1(n)$ and $B_2(n)$ be the first and second Bieberbach groups of dimension n with cyclic point group of order two. Then their polycyclic presentations are*

$$B_1(n) = \langle a, l_1, l_2, ...l_n | a^2 = l_2, {}^a l_1 = l_1^{-1}, {}^a l_j = l_j, {}^{l_i} l_j = l_j \rangle,$$
$$B_2(n) = \langle a, l_1, l_2, ...l_n | a^2 = l_3, {}^a l_1 = l_2, {}^a l_3 = l_3, {}^a l_k = l_k, {}^{l_i} l_j = l_j \rangle$$

for all $1 \leq i \leq j \leq n$ and $k = 4, 5, ...n$ respectively. Then, Table 1 summarized the results for their homological invariants.

Table 1: The Homological Invariants of two families of Bieberbach Groups with Point Group C_2

$B_i(n)$	$i = 1$	$i = 2$
$B_i(n) \otimes B_i(n)$	$C_2^{2n-3} \times C_4 \times C_0^{(n-1)^2+1}$ for $n \geq 2$	$C_2 \times C_0^{(n-1)^2+1}$ for $n \geq 3$
$\nabla(B_i(n))$	$C_0^{\frac{n(n-1)}{2}} \times C_4 \times C_2^{n-1}$ for $n \geq 3$	$C_0^{\frac{n(n-1)}{2}}$ for $n \geq 4$
$B_i(n) \wedge B_i(n)$	$C_0^{1+\frac{(n-2)(n-1)}{2}} \times C_2^{n-2}$ for $n \geq 3$	$C_0^{1+\frac{(n-2)(n-1)}{2}} \times C_2$ for $n \geq 4$
$M(B_i(n))$	$C_0^{\frac{(n-2)(n-1)}{2}} \times C_2^{n-2}$	$C_0^{\frac{(n-2)(n-1)}{2}} \times C_2$

The results for other homological invariants of $B_i(n)$ in which their nonabelian tensor squares are not abelian can be summarized in Table 2.

Table 2: The Homological Invariants of two families of Bieberbach Groups with Point Group C_2 with Nonabelian Nonabelian Tensor Square

$B_i(n)$	$i = 3, n = 3$	$i = 4, n = 5$
$B_i(n) \otimes B_i(n)$	Nonabelian	Nonabelian
$J(B_i(n))$	$C_0^2 \times C_4^2 \times C_2^3$	$C_0^9 \times C_2^2$
$M(B_i(n))$	C_0	$C_0^3 \times C_2^2$
$B_i(n) \wedge B_i(n)$	Nonabelian	Nonabelian

Theorem 2.2. [10] *Let $G_i(n)$ be the i^{th} Bieberbach groups of dimension n with cyclic point group of order three. Then, the following table are the results for the homological invariants of these groups.*

Table 3: Homological Invariants of all Bieberbach Groups with Point Group C_3

$G_i(n)$	$G_i(n) \otimes G_i(n)$	$\nabla(G_i(n)$	$J(G_i(n))$	$M(G_i(n))$	$G_i(n) \wedge G_i(n)$
$i = 1, n = 3$	Nonabelian	$C_0 \times C_3^2$	$C_0^2 \times C_3^2$	C_0	Nonabelian
$i = 2, n = 4$	Nonabelian	$C_0^3 \times C_3^3$	$C_0^5 \times C_3^4$	$C_0^2 \times C_3$	Nonabelian
$i = 3, n = 4$	Nonabelian	C_0^3	C_0^5	C_0^2	Nonabelian
$i = 4, n = 5$	Nonabelian	$C_0 \times C_3^5$	$C_0^5 \times C_3^5$	C_0^4	Nonabelian
$i = 6, n = 5$	Nonabelian	C_0^6	C_0^{10}	C_0^4	Nonabelian
$i = 7, n = 7$	Nonabelian	$C_0^3 \times C_7^3$	$C_0^8 \times C_3^9$	$C_0^5 \times C_3^2$	Nonabelian
$i = 8, n = 6$	Nonabelian	$C_0^3 \times C_3^3$	$C_0^8 \times C_3^3$	C_0^5	Nonabelian
$i = 10, n = 6$	Nonabelian	C_0^{10}	C_0^{17}	C_0^7	Nonabelian

Theorem 2.3. [10] *Let $H_i(n)$ be the i^{th} Bieberbach groups of dimension n with cyclic point group of order five. Then, the results for the homological invariants of these groups are given in Table 4.*

Table 4: Homological Invariants of all Bieberbach Groups with Point Group C_5

$H_i(n)$	$H_i(n) \otimes H_i(n)$	$\nabla(H_i(n)$	$J(H_i(n))$	$M(H_i(n))$	$H_i(n) \wedge H_i(n)$
$i = 1, n = 5$	Nonabelian	$C_0 \times C_5^2$	$C_0^3 \times C_5^2$	C_0^2	Nonabelian
$i = 2, n = 6$	Nonabelian	$C_0^3 \times C_3^5$	$C_0^6 \times C_4^5$	$C_0^3 \times C_5$	Nonabelian
$i = 3, n = 6$	Nonabelian	C_0^3	C_0^6	C_0^3	Nonabelian

Further research on Bieberbach groups had grabbed interests of some other researchers which include Mohd Idrus [11, 12], Mohammad *et al.* [13], Wan Mohd Fauzi [21] and Tan *et al.* [18]. They computed some homological invariants for Bieberbach groups with nonabelian point group, namely the dihedral point group of order eight. The homological invariants of these groups are presented in the following theorem.

Theorem 2.4. [11, 12, 13, 21, 18] *Let $D_i(n)$ be the i^{th} Bieberbach groups of dimension n with dihedral point group of order eight. Then, Table 5 shows the results of some homological invariants of the groups.*

Table 5: The Homological Invariants of Some Bieberbach Groups with Point Group D_4

$D_i(n)$	$i = 1, n = 4$	$i = 2, n = 5$	$i = 3, n = 5$	$i = 4, n = 5$
$D_i(n) \otimes D_i(n)$	Nonabelian	Nonabelian	Nonabelian	Nonabelian
$\nabla(D_i(n))$	$C_8^2 \times C_4$	$C_8^2 \times C_4$	$C_8^2 \times C_4^2 \times C_2^2$	$C_8^2 \times C_4^2 \times C_2^2$
$J(D_i(n))$	$C_8^2 \times C_4$	$C_8^2 \times C_4 \times C_0$	$C_8^2 \times C_4^2 \times C_2^2$	$C_8^2 \times C_4^2 \times C_2^2 \times C_0$
$M(D_i(n))$	$1_\otimes$	C_0	C_0	C_0
$D_i(n) \wedge D_i(n)$	Nonabelian	Nonabelian	Nonabelian	Nonabelian

In 2014 onwards, Tan *et al.* [19, 17, 20] focused on Bieberbach group with symmetric point group of order six and the following theorem summarized the homological invariants that have been computed by them.

Theorem 2.5. [19, 17, 20] *Let $S_i(n)$ be the i^{th} Bieberbach groups of dimension n with symmetric point group of order six. Then, Tables 6 and 7 show the results for the homological invariants of these groups.*

Table 6: The Homological Invariants of Bieberbach Groups with Point Group S_3 for $n \geq 4$

$S_i(n)$	$i = 1, n \geq 4$	$i = 2, n \geq 4$
$S_i(n) \otimes S_i(n)$	Nonabelian	Nonabelian
$\nabla(S_i(n))$	$C_0^{\frac{(n-2)(n-3)}{2}} \times C_2^{n-3} \times C_4$	$C_0^{\frac{(n-2)(n-3)}{2}} \times C_2^{n-3} \times C_3^{n-2} \times C_4$
$J(S_i(n))$	$C_0^{(n-3)^2} \times C_2^{2(n-3)} \times C_4$	$C_0^{(n-3)^2} \times C_2^{2(n-3)} \times C_3^{(2n-5)} \times C_4$
$M(S_i(n))$	$C_0^{\frac{(n-3)(n-4)}{2}} \times C_2^{n-3}$	$C_0^{\frac{(n-3)(n-4)}{2}} \times C_2^{n-3} \times C_3^{n-3}$
$S_i(n) \wedge S_i(n)$	Nonabelian	Nonabelian

Table 7: The Homological Invariants of Bieberbach Groups with Point Group S_3 for $n \geq 5$

$S_i(n)$	$i = 3, n \geq 5$	$i = 2, n \geq 5$
$S_i(n) \otimes S_i(n)$	Nonabelian	Nonabelian
$\nabla(S_i(n))$	$C_0^{\frac{(n-2)(n-3)}{2}}$	$C_0^{\frac{(n-2)(n-3)}{2}} \times C_3^{n-2}$
$J(S_i(n))$	$C_0^{(n-3)^2} \times C_2^2$	$C_0^{(n-3)^2} \times C_2^2 \times C_3^{(2n-5)}$
$M(S_i(n))$	$C_0^{\frac{(n-3)(n-4)}{2}} \times C_2^2$	$C_0^{\frac{(n-3)(n-4)}{2}} \times C_2^2 \times C_3^{n-3}$
$S_i(n) \wedge S_i(n)$	Nonabelian	Nonabelian

3. Main Results

The isomorphism types of space groups were designed by Opgenorth *et al.* [14] and the tables were given in CARAT package [24] to enable the user to construct and recognize space groups. Based on [24, 14], a Bieberbach group of dimension six with point group Q_8 is isomorphic to one of these groups, namely G_1 where

$$G_1 = \langle a_0, a_1, l_1, l_2, l_3, l_4, l_5, l_6 \rangle, \tag{1}$$

where

$$a_0 = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & -\frac{1}{4} \\ 0 & 0 & 0 & 0 & 0 & 1 & \frac{1}{4} \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad a_1 = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

$$l_1 = \begin{bmatrix} 1&0&0&0&0&0&1 \\ 0&1&0&0&0&0&0 \\ 0&0&1&0&0&0&0 \\ 0&0&0&1&0&0&0 \\ 0&0&0&0&1&0&0 \\ 0&0&0&0&0&1&0 \\ 0&0&0&0&0&0&1 \end{bmatrix}, \quad l_2 = \begin{bmatrix} 1&0&0&0&0&0&0 \\ 0&1&0&0&0&0&1 \\ 0&0&1&0&0&0&0 \\ 0&0&0&1&0&0&0 \\ 0&0&0&0&1&0&0 \\ 0&0&0&0&0&1&0 \\ 0&0&0&0&0&0&1 \end{bmatrix},$$

$$l_3 = \begin{bmatrix} 1&0&0&0&0&0&0 \\ 0&1&0&0&0&0&0 \\ 0&0&1&0&0&0&1 \\ 0&0&0&1&0&0&0 \\ 0&0&0&0&1&0&0 \\ 0&0&0&0&0&1&0 \\ 0&0&0&0&0&0&1 \end{bmatrix}, \quad l_4 = \begin{bmatrix} 1&0&0&0&0&0&0 \\ 0&1&0&0&0&0&0 \\ 0&0&1&0&0&0&0 \\ 0&0&0&1&0&0&1 \\ 0&0&0&0&1&0&0 \\ 0&0&0&0&0&1&0 \\ 0&0&0&0&0&0&1 \end{bmatrix},$$

$$l_5 = \begin{bmatrix} 1&0&0&0&0&0&0 \\ 0&1&0&0&0&0&0 \\ 0&0&1&0&0&0&0 \\ 0&0&0&1&0&0&0 \\ 0&0&0&0&1&0&1 \\ 0&0&0&0&0&1&0 \\ 0&0&0&0&0&0&1 \end{bmatrix}, \text{ and } l_6 = \begin{bmatrix} 1&0&0&0&0&0&1 \\ 0&1&0&0&0&0&0 \\ 0&0&1&0&0&0&0 \\ 0&0&0&1&0&0&0 \\ 0&0&0&0&1&0&0 \\ 0&0&0&0&0&1&1 \\ 0&0&0&0&0&0&1 \end{bmatrix}.$$

$G_1 = \langle a_0, a_1, l_1, l_2, l_3, l_4, l_5, l_6 \rangle$ has been shown to be isomorphic to $Q_1(6) = \langle a, b, c, l_1, l_2, l_3, l_4, l_5, l_6 \rangle$ with $c = a_0^2 l_6^{-1}$. Hence, the polycyclic presentation of $Q_1(6)$ is established as

$$\begin{aligned} Q_1(6) = \langle a, b, c, l_1, l_2, l_3, l_4, l_5, l_6 \,|\, & a^2 = cl_6, b^2 = c, c^2 = l_5^{-1} l_6^{-1}, b^a = bc^{-1} l_5^{-1}, \\ & c^a = c, c^b = c, l_1^a = l_3, l_1^b = l_4, l_1^c = l_1^{-1}, l_2^a = l_4, l_2^b = l_3^{-1}, l_2^c = l_2^{-1}, \\ & l_3^a = l_1^{-1}, l_3^b = l_2, l_3^c = l_3^{-1}, l_4^a = l_2^{-1}, l_4^b = l_1^{-1}, l_4^c = l_4^{-1}, l_5^a = l_5, l_5^b = l_6, \\ & l_5^c = l_5, l_6^a = l_6, l_6^b = l_5, l_6^c = l_6, l_2^{l_1} = l_2, l_3^{l_1} = l_3, l_4^{l_1} = l_4, l_5^{l_1} = l_5, \\ & l_6^{l_1} = l_6, l_3^{l_2} = l_3, l_4^{l_2} = l_4, l_5^{l_2} = l_5, l_6^{l_2} = l_6, l_4^{l_3} = l_4, l_5^{l_3} = l_5, l_6^{l_3} = l_6, \\ & l_5^{l_4} = l_5, l_6^{l_4} = l_6, l_6^{l_5} = l_6, l_2^{l_1^{-1}} = l_2, l_3^{l_1^{-1}} = l_3, l_4^{l_1^{-1}} = l_4, l_5^{l_1^{-1}} = l_5, \\ & l_6^{l_1^{-1}} = l_6, l_3^{l_2^{-1}} = l_3, l_4^{l_2^{-1}} = l_4, l_5^{l_2^{-1}} = l_5, l_6^{l_2^{-1}} = l_6, l_4^{l_3^{-1}} = l_4, l_5^{l_3^{-1}} = l_5, \\ & l_6^{l_3^{-1}} = l_6, l_5^{l_4^{-1}} = l_5, l_6^{l_4^{-1}} = l_6, l_6^{l_5^{-1}} = l_6 \rangle. \end{aligned} \tag{2}$$

Next, all relations as given in (2) have been shown to satisfy the five consistency relations as given in Definition 1.7, therefore $Q_1(6)$ has a consistent polycyclic presentation.

In Lemma 3.1 and Lemma 3.2, the computations of the derived subgroup and abelianization of $Q_1(6)$ are presented, respectively.

Lemma 3.1. *Let $Q_1(6)$ be a Bieberbach group of dimension six with quaternion* *point group of order eight. Then, the derived subgroup of* $Q_1(6)$ *is* $(Q_1(6))' = \langle cl_5, l_1l_3^{-1}, l_1l_4, l_5l_6^{-1}\rangle$.

Proof. By the relations of $Q_1(6)$, $Q_1(6)$ is generated by $a, b, c, l_1, l_2, l_3, l_4, l_5$, and l_6. There are 72 commutators of $Q_1(6)$, namely $[a,b]$, $[a,c]$, $[b,c]$, $[a,l_1]$, $[a,l_2]$, $[a,l_3]$, $[a,l_4]$, $[a,l_5]$, $[a,l_6]$, $[b,l_1]$, $[b,l_2]$, $[b,l_3]$, $[b,l_4]$, $[b,l_5]$, $[b,l_6]$, $[c,l_1]$, $[c,l_2]$, $[c,l_3]$, $[c,l_4]$, $[c,l_5]$, $[c,l_6]$, $[b,a]$, $[c,a]$, $[c,b]$, $[l_1,a]$, $[l_2,a]$, $[l_3,a]$, $[l_4,a]$, $[l_5,a]$, $[l_6,a]$, $[l_1,b]$, $[l_2,b]$, $[l_3,b]$, $[l_4,b]$, $[l_5,b]$, $[l_6,b]$, $[l_1,c]$, $[l_2,c]$, $[l_3,c]$, $[l_4,c]$, $[l_5,c]$, $[l_6,c]$, $[l_1,l_2]$, $[l_1,l_3]$, $[l_1,l_4]$, $[l_1,l_5]$, $[l_1,l_6]$, $[l_2,l_3]$, $[l_2,l_4]$, $[l_2,l_5]$, $[l_2,l_6]$, $[l_3,l_4]$, $[l_3,l_5]$, $[l_3,l_6]$, $[l_4,l_5]$, $[l_4,l_6]$, $[l_5,l_6]$, $[l_2,l_1]$, $[l_3,l_1]$, $[l_4,l_1]$, $[l_5,l_1]$, $[l_6,l_1]$, $[l_3,l_2]$, $[l_4,l_2]$, $[l_5,l_2]$, $[l_6,l_2]$, $[l_4,l_3]$, $[l_5,l_3]$, $[l_6,l_3]$, $[l_5,l_4]$, $[l_6,l_4]$, and $[l_6,l_5]$. By collecting and excluding repeated terms, the derived subgroup of $Q_1(6)$ is found to be

$$\begin{aligned}(Q_1(6))' =&\langle cl_5, l_1l_3^{-1}, l_2l_4^{-1}, l_1l_3, l_2l_4, l_1l_4^{-1}, l_2l_3, l_2^{-1}l_3, l_1l_4, l_5l_6^{-1}, l_5^{-1}l_6, l_1^2, l_2^2, l_3^2, l_4^2,\\ &c^{-1}l_5^{-1}, l_3l_1^{-1}, l_4l_2^{-1}, l_3^{-1}l_1^{-1}, l_4^{-1}l_2^{-1}, l_4^{-1}l_1, l_3^{-1}l_2^{-1}, l_3^{-1}l_2, l_4^{-1}l_1^{-1}, l_6l_5^{-1}, l_6l_5^{-1},\\ &l_6^{-1}l_5, l_1^{-2}, l_2^{-2}, l_3^{-2}, l_4^{-2}\rangle.\end{aligned}$$

However,

$$\begin{aligned} l_1^{-2} =&(l_1^2)^{-1}; l_2^{-2} = (l_2^2)^{-1}; l_3^{-2} = (l_3^2)^{-1}; l_4^{-2} = (l_4^2)^{-1},\\ c^{-1}l_5^{-1} =&(l_5c)^{-1} = (cl_5)^{-1} \qquad \text{since } c \text{ and } l_5 \text{ commutes},\\ l_3l_1^{-1} =&(l_1l_3^{-1})^{-1}; l_4l_2^{-1} = (l_2l_4^{-1})^{-1}; l_4^{-1}l_1 = l_1l_4^{-1}; l_3^{-1}l_2 = (l_2^{-1}l_3)^{-1};\\ l_3^{-1}l_1^{-1} =&(l_1l_3)^{-1}; l_4^{-1}l_2^{-1} = (l_2l_4)^{-1}; l_3^{-1}l_2^{-1} = (l_2l_3)^{-1};\\ l_4^{-1}l_1^{-1} =&(l_1l_4)^{-1}; l_6l_5^{-1} = (l_5l_6^{-1})^{-1}; l_6^{-1}l_5 = (l_5^{-1}l_6)^{-1};\\ l_1^2 =&(l_1l_3)(l_1l_3^{-1}); l_2^2 = (l_2l_4)(l_2l_4^{-1}); l_3^2 = (l_2l_3)(l_2^{-1}l_3); l_4^2 = (l_1l_4^{-1})^{-1}(l_1l_4);\\ l_5^{-1}l_6 =&(l_5l_6^{-1})^{-1}; l_1l_4^{-1} = (l_1l_4)^{-1}(l_1l_3^{-1})(l_1l_3); l_2l_4 = (l_1l_4)(l_1l_3)^{-1}(l_2l_3);\\ l_2l_3 =&(l_2l_4^{-1})(l_1l_4)(l_1l_3^{-1})^{-1}; l_2^{-1}l_3 = (l_2l_4^{-1})^{-1}(l_1l_4)^{-1}(l_1l_3);\\ l_2l_4^{-1} =&(l_2^{-1}l_3)^{-1}(l_1l_3)(l_1l_4)^{-1}.\end{aligned}$$

Hence,

$$(Q_1(6))' = \langle cl_5, l_1l_3^{-1}, l_1l_3, l_1l_4, l_5l_6^{-1}\rangle. \qquad ■$$

Lemma 3.2. *Let* $Q_1(6)$ *be a Bieberbach group of dimension six with quaternion point group of order eight. Then, the abelianization of* $Q_1(6)$ *is found to be*

$$(Q_1(6))^{ab} = \langle a(Q_1(6))', b(Q_1(6))', l_1(Q_1(6))'\rangle \cong C_0 \times C_2^2.$$

Proof. The abelianization of $Q_1(6)$, $(Q_1(6))^{ab}$ is the factor group ${Q_1(6)}/{(Q_1(6))'}$. Then, using Lemma 3.1,

$$(Q_1(6))^{ab} = Q_1(6)/(Q_1(6))'$$

$$
\begin{aligned}
&=\langle a,b,c,l_1,l_2,l_3,l_4,l_5,l_6\rangle/(Q_1(6))'\\
&=\langle a(Q_1(6))',b(Q_1(6))',c(Q_1(6))',l_1(Q_1(6))',l_2(Q_1(6))',l_3(Q_1(6))',\\
&\quad l_4(Q_1(6))',l_5(Q_1(6))',l_6(Q_1(6))'\rangle.
\end{aligned}
$$

However, some of these generators can be written in terms of the other generators. By the relations of $Q_1(6)$, $b^2=c$. This implies that

$$(b(Q_1(6))')^2=c(Q_1(6))'.$$

Next,

$$
\begin{aligned}
l_1(Q_1(6))' &= l_3(Q_1(6))', &&\text{since } l_1l_3^{-1}\in(Q_1(6))'\\
l_2(Q_1(6))' &= l_4(Q_1(6))', &&\text{since } l_2l_4^{-1}\in(Q_1(6))'\\
l_5(Q_1(6))' &= l_6(Q_1(6))', &&\text{since } l_5l_6^{-1}\in(Q_1(6))'
\end{aligned}
$$

Moreover, by using $a^2=cl_6$ and $b^2=c$, then $a^2=b^2l_6$ implies that $l_6=b^{-2}a^2$. Hence

$$
\begin{aligned}
l_6(Q_1(6))' =&(b(Q_1(6))')^{-2}(a(Q_1(6))')^2,\\
l_1(Q_1(6))' =&l_4(Q_1(6))' \quad \text{since } l_1l_4^{-1}\in(Q_1(6))'\\
=&l_2(Q_1(6))' \quad \text{since } l_4(Q_1(6))'=l_2(Q_1(6))',\ l_2=l_1
\end{aligned}
$$

Therefore, the independent generators of $(Q_1(6))^{ab}$ are $a(Q_1(6))', b(Q_1(6))'$, and $l_1(Q_1(6))'$. Next, the order of $a(Q_1(6))', b(Q_1(6))'$, and $l_1(Q_1(6))'$ are determined. $l_1(Q_1(6))'$ is of order two since $l_1^2\in(Q_1(6))'$. Besides, $b(Q_1(6))'$ is of order two since $b^2=c$ and $c\in(Q_1(6))'$. However, $a(Q_1(6))'\cap c(Q_1(6))'\cap l_6(Q_1(6))'$ is not trivial since $a^2(Q_1(6))'=cl_6(Q_1(6))'$ by the relations of $Q_1(6)$ that implies $a^2(Q_1(6))'=b^2l_6(Q_1(6))'\to(a(Q_1(6))')^2=(b(Q_1(6))')^2l_6(Q_1(6))')$. Suppose the order of $a(Q_1(6))'$ is finite. Then, for an arbitrary integer r, $a^r(Q_1(6))'$ is trivial, which implies that $a^r\in(Q_1(6))'$. However, this is a contradiction since by the relations of $Q_1(6)$, a^r cannot be written as any element in $(Q_1(6))'$ due to the fact that a is independent of b, l_1, l_2, l_3, l_4 and l_5. Hence, since all nontrivial elements in $(Q_1(6))'$ have infinite order, thus $a(Q_1(6))'$ has infinite order. Therefore,

$$(Q_1(6))^{ab}=\langle a(Q_1(6))',b(Q_1(6))',l_1(Q_1(6))'\rangle\cong C_0\times C_2^2.$$

This lemma shows that $(Q_1(6))^{ab}$ is finitely generated. ■

Next, the computation of a homological invariant which is the central subgroup of the nonabelian tensor square for $Q_1(6)$, denoted as $\nabla(Q_1(6))$, is presented in the following theorem.

Theorem 3.3. *Let $Q_1(6)$ be a Bieberbach group of dimension six with quaternion point group of order eight. Then, the central subgroup of $Q_1(6)$, denoted as $\nabla(Q_1(6))$ is given as*

$$\begin{aligned}\nabla(Q_1(6)) &= \langle a\otimes a, b\otimes b, l_1\otimes l_1, (a\otimes b)(b\otimes a), (a\otimes l_1)(l_1\otimes a), (b\otimes l_1)(l_1\otimes b)\rangle \\ &\cong C_0\times C_2^4\times C_4.\end{aligned}$$

Proof. By Lemma 3.2, the abelianization of $Q_1(6)$ is generated by $a(Q_1(6))'$ which is of infinite order and $b(Q_1(6))'$, $l_1(Q_1(6))'$ which are of order two. Then, based on Theorem 1.12(1), $\nabla(Q_1(6))$ is generated by $[a, a^\varphi]$, $[b, b^\varphi]$, $[l_1, l_1^\varphi]$, $[a, b^\varphi][b, a^\varphi]$, $[a, l_1^\varphi][l_1, a^\varphi]$ and $[b, l_1^\varphi][l_1, b^\varphi]$. Since $\nabla(Q_1(6))$ is a subgroup of $Q_1(6)\otimes Q_1(6)$ and the mapping $\sigma : Q_1(6)\otimes Q_1(6)\to [Q_1(6), Q_1(6)^\varphi]$ defined by $\sigma(g\otimes h) = [g, h^\varphi]$ for all $g, h\in Q_1(6)$ is an isomorphism by Theorem 1.8. Thus,

$$\begin{aligned}\nabla(Q_1(6)) =&\langle a\otimes a, b\otimes b, l_1\otimes l_1, (a\otimes b)(b\otimes a), (a\otimes l_1)(l_1\otimes a),\\ &(b\otimes l_1)(l_1\otimes b)\rangle.\end{aligned}$$

Next, the order of each generator for $\nabla(Q_1(6))$ is computed. The abelianization of $Q_1(6)$ is denoted by $(Q_1(6))^{ab}$ with has mapping $\eta : Q_1(6)\to (Q_1(6))^{ab}$. Since $(Q_1(6))^{ab}$ is finitely generated, then its nonabelian tensor square is just the ordinary tensor product of $Q_1(6)^{ab}$. By using Theorem 1.14,

$$\begin{aligned}&(Q_1(6))^{ab}\otimes (Q_1(6))^{ab}\\ \cong&(C_0\times C_2\times C_2)\otimes (C_0\times C_2\times C_2)\\ =&(C_0\otimes (C_0\times C_2\times C_2))\times (C_2\otimes (C_0\times C_2\times C_2))\times (C_2\otimes (C_0\times C_2\times C_2))\\ =&(C_0\otimes C_0)\times (C_0\otimes C_2)\times (C_0\otimes C_2)\times (C_2\otimes C_0)\times (C_2\otimes C_2)\times (C_2\otimes C_2)\\ &\times (C_2\otimes C_0)\times (C_2\otimes C_2)\times (C_2\otimes C_2)\\ \cong&C_0\times C_2\times C_2\times C_2\times C_2\times C_2\times C_2\times C_2\times C_2\\ =&C_0\times C_2^8.\end{aligned}$$

Then, Lemma 3.2 provides that $(Q_1(6))^{ab}$ is generated by $\eta(a)$ of order infinite and $\eta(b)$, $\eta(l_1)$ are of order two. Again, by Theorem 1.14,

$$\langle \eta(a)\otimes \eta(a)\rangle \cong C_0.$$

By Theorem 1.15, there is a natural epimorphism $\alpha : Q_1(6)\otimes Q_1(6)\to Q_1(6)^{ab}\otimes Q_1(6)^{ab}$. Therefore, the image $\alpha(a\otimes a) = \eta(a)\otimes\eta(a)$ has infinite order. Thus by Theorem 1.13, $a\otimes a$ has also infinite order. By the relations of $Q_1(6)$, $[b, b^\varphi]^4 = [b^2, b^{2\varphi}] = 1$. This means that the order of $[b, b^\varphi]$ divides four. Hence the order of $b\otimes b$ is four. By Theorem 1.14,

$$\begin{aligned}\langle \eta(l_1)\otimes \eta(l_1)\rangle &\cong C_2,\\ \langle \eta(a)\otimes \eta(b)\rangle &\cong C_2,\\ \langle \eta(b)\otimes \eta(a)\rangle &\cong C_2,\end{aligned}$$

$$\langle \eta(a) \otimes \eta(l_1) \rangle \cong C_2,$$
$$\langle \eta(l_1) \otimes \eta(a) \rangle \cong C_2,$$
$$\langle \eta(b) \otimes \eta(l_1) \rangle \cong C_2,$$
$$\langle \eta(l_1) \otimes \eta(b) \rangle \cong C_2.$$

So, the order of $l_1 \otimes l_1, (a \otimes b)(b \otimes a), (a \otimes l_1)(l_1 \otimes a)$ and $(b \otimes l_1)(l_1 \otimes b)$ are two. Finally, the desired result is obtained, namely

$$\begin{aligned}\nabla(Q_1(6)) &= \langle a \otimes a, b \otimes b, l_1 \otimes l_1, (a \otimes b)(b \otimes a), (a \otimes l_1)(l_1 \otimes a), (b \otimes l_1)(l_1 \otimes b) \rangle \\ &\cong C_0 \times C_4 \times C_2 \times C_2 \times C_2 \times C_2 \\ &= C_0 \times C_2^4 \times C_4.\end{aligned}$$

■

Thus, a homological invariant namely the $\nabla(G)$ for a Bieberbach group of dimension six with quaternion point group of order eight has been computed manually.

The computation of $J(Q_1(6))$ is presented in the following theorem.

Theorem 3.4. *Let $Q_1(6)$ be a Bieberbach group of dimension six with quaternion point group of order eight. Then, $J(Q_1(6))$ is given as*

$$\begin{aligned}&J(Q_1(6)) \\ =&\langle a \otimes a, b \otimes b, l_1 \otimes l_1, a \otimes c, (a \otimes b)(b \otimes a), (a \otimes l_1)(l_1 \otimes a), \\ &(a \otimes l_2)(l_2 \otimes a), (b \otimes l_1)(l_1 \otimes b), (b \otimes l_2)(l_2 \otimes b), (b \otimes l_5)(l_5 \otimes b) \rangle. \\ \cong& {C_0}^4 \times C_2^5 \times C_4.\end{aligned}$$

Proof. By Definition 1.2, $J(Q_1(6))$ has a mapping $\kappa : Q_1(6) \otimes Q_1(6) \to Q_1(6)'$ defined by $\kappa(g \otimes h) = [g, h] = 1$, for all $g, h \in Q_1(6)$. The nonabelian tensor square of $Q_1(6)$ is generated by

$$\begin{aligned}&Q_1(6) \otimes Q_1(6) \\ =&\langle a \otimes a, b \otimes b, l_1 \otimes l_1, a \otimes b, a \otimes c, a \otimes l_1, a \otimes l_2, b \otimes l_1, b \otimes l_2, b \otimes l_5, \\ &b \otimes l_6, c \otimes l_1, c \otimes l_2, (a \otimes b)(b \otimes a), (a \otimes l_1)(l_1 \otimes a), (a \otimes l_2)(l_2 \otimes a), \\ &(b \otimes l_1)(l_1 \otimes b), (b \otimes l_2)(l_2 \otimes b), (b \otimes l_5)(l_5 \otimes b), (c \otimes l_1)(l_1 \otimes c), (c \otimes l_2)(l_2 \otimes c) \rangle.\end{aligned}$$

By the relations of $Q_1(6)$ and the derived subgroup, the image of each generator can be computed as in the following:

$$\begin{aligned}\kappa(a \otimes a) &= [a, a] = a^{-1}a^{-1}aa = 1, \\ \kappa(b \otimes b) &= [b, b] = b^{-1}b^{-1}bb = 1, \\ \kappa(l_1 \otimes l_1) &= [l_1, l_1] = l_1^{-1}l_1^{-1}l_1l_1 = 1, \\ \kappa(a \otimes b) &= [a, b] = a^{-1}b^{-1}ab = l_5c = cl_5 \neq 1,\end{aligned}$$

$$\begin{aligned}
\kappa(a \otimes c) &= [a, c] = a^{-1}c^{-1}ac = 1, \\
\kappa(a \otimes l_1) &= [a, l_1] = a^{-1}l_1^{-1}al_1 = l_3^{-1}l_1 \neq 1, \\
\kappa(a \otimes l_2) &= [a, l_2] = a^{-1}l_2^{-1}al_2 = l_4^{-1}l_2 \neq 1, \\
\kappa(b \otimes l_1) &= [b, l_1] = b^{-1}l_1^{-1}bl_1 = l_4^{-1}l_1 \neq 1, \\
\kappa(b \otimes l_2) &= [b, l_2] = b^{-1}l_2^{-1}bl_2 = l_2l_3 \neq 1, \\
\kappa(b \otimes l_5) &= [b, l_5] = b^{-1}l_5^{-1}bl_5 = l_6^{-1}l_5 \neq 1, \\
\kappa(b \otimes l_6) &= [b, l_6] = b^{-1}l_6^{-1}bl_6 = l_5^{-1}l_6 \neq 1, \\
\kappa(c \otimes l_1) &= [c, l_1] = c^{-1}l_1^{-1}cl_1 = l_1^2 \neq 1, \\
\kappa(c \otimes l_2) &= [c, l_2] = c^{-1}l_2^{-1}cl_2 = l_2^2 \neq 1,
\end{aligned}$$

Then, since κ is a homomorphism,

$$\begin{aligned}
\kappa((a \otimes b)(b \otimes a)) &= \kappa(a \otimes b)\kappa(b \otimes a) \\
&= [a, b][b, a] \\
&= [a, b][a, b]^{-1} \\
&= 1,
\end{aligned}$$

$$\begin{aligned}
\kappa((a \otimes l_1)(l_1 \otimes a)) &= \kappa(a \otimes l_1)\kappa(l_1 \otimes a) \\
&= [a, l_1][l_1, a] \\
&= [a, l_1][a, l_1]^{-1} \\
&= 1, \\
\kappa((a \otimes l_2)(l_2 \otimes a)) &= \kappa(a \otimes l_2)\kappa(l_2 \otimes a) \\
&= [a, l_2][l_2, a] \\
&= [a, l_2][a, l_2]^{-1} \\
&= 1, \\
\kappa((b \otimes l_1)(l_1 \otimes b)) &= \kappa(b \otimes l_1)\kappa(l_1 \otimes b) \\
&= [b, l_1][l_1, b] \\
&= [b, l_1][b, l_1]^{-1} \\
&= 1, \\
\kappa((b \otimes l_2)(l_2 \otimes b)) &= \kappa(b \otimes l_2)\kappa(l_2 \otimes b) \\
&= [b, l_2][l_2, b] \\
&= [b, l_2][b, l_2]^{-1} \\
&= 1, \\
\kappa((b \otimes l_5)(l_5 \otimes b)) &= \kappa(b \otimes l_5)\kappa(l_5 \otimes b) \\
&= [b, l_5][l_5, b] \\
&= [b, l_5][b, l_5]^{-1}
\end{aligned}$$

$$\begin{aligned}
&= 1, \\
\kappa((c \otimes l_1)(l_1 \otimes c)) &= \kappa(c \otimes l_1)\kappa(l_1 \otimes c) \\
&= [c, l_1][l_1, c] \\
&= [c, l_1][c, l_1]^{-1} \\
&= 1, \\
\kappa((c \otimes l_2)(l_2 \otimes c)) &= \kappa(b \otimes l_2)\kappa(l_2 \otimes b) \\
&= [c, l_2][l_2, c] \\
&= [c, l_2][c, l_2]^{-1} \\
&= 1.
\end{aligned}$$

But, $(c \otimes l_1)(l_1 \otimes c) = ((b \otimes l_1)(l_1 \otimes b))^2$.

$$\begin{aligned}
[c, l_1^{\varphi}][l_1, c^{\varphi}] &= [b^2, l_1^{\varphi}][l_1, b^{2\varphi}] \\
&= ([b, l_1^{\varphi}][l_1, b^{\varphi}])^2 \\
&= ((b \otimes l_1)(l_1 \otimes b))^2.
\end{aligned}$$

$(c \otimes l_2)(l_2 \otimes c) = ((b \otimes l_2)(l_2 \otimes b))^2$.

$$\begin{aligned}
[c, l_2^{\varphi}][l_2, c^{\varphi}] &= [b^2, l_2^{\varphi}][l_2, b^{2\varphi}] \\
&= ([b, l_2^{\varphi}][l_2, b^{\varphi}])^2 \\
&= ((b \otimes l_2)(l_2 \otimes b))^2.
\end{aligned}$$

Therefore, by collecting all images that are equal to 1 according to the definition of the kernel,

$$\begin{aligned}
J(Q_1(6)) = \langle & a \otimes a, b \otimes b, l_1 \otimes l_1, a \otimes c, (a \otimes b)(b \otimes a), (a \otimes l_1)(l_1 \otimes a), \\
& (a \otimes l_2)(l_2 \otimes a), (b \otimes l_1)(l_1 \otimes b), (b \otimes l_2)(l_2 \otimes b), (b \otimes l_5)(l_5 \otimes b)\rangle.
\end{aligned}$$

Therefore, based on the order of the generators found in Theorem 3.3 it is found that $a \otimes a$ has infinite order. Besides, $a \otimes c, (a \otimes l_2)(l_2 \otimes a)$ and $(b \otimes l_5)(l_5 \otimes b)$ are also of infinite order since $\eta(a)$ is of infinite order and c, l_2 and l_5 are not independent. Meanwhile, the order of $b \otimes b$ is four whereas $l_1 \otimes l_1, (a \otimes b)(b \otimes a), (a \otimes l_1)(l_1 \otimes a), (b \otimes l_1)(l_1 \otimes b)$ are two.

By Theorem 1.14,

$$\begin{aligned}
\langle \eta(b) \otimes \eta(l_2) \rangle &\cong C_2, \\
\langle \eta(l_2) \otimes \eta(b) \rangle &\cong C_2.
\end{aligned}$$

So, the order of and $(b \otimes l_2)(l_2 \otimes b)$ is also two. Finally, the desired result is obtained, namely

$$\begin{aligned}
J(Q_1(6)) = \langle & a \otimes a, b \otimes b, l_1 \otimes l_1, a \otimes c, (a \otimes b)(b \otimes a), (a \otimes l_1)(l_1 \otimes a), \\
& (a \otimes l_2)(l_2 \otimes a), (b \otimes l_1)(l_1 \otimes b), (b \otimes l_2)(l_2 \otimes b), (b \otimes l_5)(l_5 \otimes b)
\end{aligned}$$

$$\cong C_0 \times C_4 \times C_2 \times C_0 \times C_2 \times C_2 \times C_0 \times C_2 \times C_2 \times C_0$$
$$= {C_0}^4 \times C_2^5 \times C_4. \qquad \blacksquare$$

The homological invariants such as $\nabla(G)$ and $J(G)$ of a Bieberbach group of dimension six with quaternion point group of order eight have been computed. On top of that, the abelianization of the group is found to be finitely generated. As a result, $\nabla(G)$, $J(G)$ and $M(G)$ are found to be abelian since they are in the center of the nonabelian tensor square of the group.

References

[1] M.R. Bacon, L.C. Kappe, The nonabelian tensor square of a 2-generator p-group of class 2, *Arch. Math* **61** (1993) 508–516.

[2] M.R. Bacon, L.C. Kappe, On capable p-group of nilpotency class two, *Illinois Journal of Math.* **47** (2003) 49–62.

[3] R.D. Blyth, F. Fumagalli, M. Morigi, Some structural results on the non-abelian tensor square of groups, *Journal of Group Theory* **13** (2010) 83–94.

[4] R.D. Blyth, R.F. Morse, Computing the nonabelian tensor square of polycyclic groups, *Journal of Algebra* **321** (2009) 2139–2148.

[5] R. Brown, J.L. Loday, Van Kampen theorems for diagrams of spaces, *Topology* **26** (1987) 311–335.

[6] R. Brown, D.L. Johnson, E.F. Robertson, Some computations of non-abelian tensor products of groups, *Journal of Algebra* **111** (1987) 177–202.

[7] B. Eick, W. Nickel, Computing Schur multiplicator and tensor square of polycyclic group, *Journal of Algebra* **320** (1) (2008) 927–944.

[8] T. Janssen, *Crystallographic Groups*, American Elsevier Publishing Company, New York, 1973.

[9] R. Masri, *The Nonabelian Tensor Squares of Certain Bieberbach Groups with Cyclic Point Group of Order Two*, Ph.D. Thesis, Universiti Teknologi Malaysia, 2009.

[10] H.I. Mat Hassim, *The Homological Functors of Bieberbach Groups with Cyclic Point Groups of Order Two, Three and Five*, Ph.D. Thesis, Universiti Teknologi Malaysia, 2014.

[11] N. Mohd Idrus, *Bieberbach Groups with Finite Point Groups*, Ph.D. Thesis, Universiti Teknologi Malaysia, 2011.

[12] N. Mohd Idrus, W.N.F. Wan Mohd Fauzi, R. Masri, Y.T. Tan, N.H. Sarmin, H.I. Mat Hassim, The central subgroup of nonabelian tensor square of the third Bieberbach group with dihedral point group, *International Journal of Applied Mathematics and Statistics* **53** (4) (2015) 104–109.

[13] S.A. Mohammad, N.M Mohd Ali, N.H. Sarmin, N. Mohd Idrus, R. Masri, On some homological functors of a Bieberbach group of dimension four with dihedral point group of order eight, *AIP Conference Proceedings* **1602** (2010) 754–759.

[14] J. Opgenorth, W. Plesken, T. Schulz, Crystallographic algorithms and tables, *Acta Cryst.* (1998) 517–531.

[15] D.J.S. Robinson, *A Course in the Theory of Groups*, Graduate Texts in Mathematics, Springer-Verlag New York Heidelberg Berlin, 1982.

[16] N.R. Rocco, On a construction related to the nonabelian tensor square of a group, *Bol. Soc. Brasil. Mat. (N. S.)* **22** (1) (1991) 63–79.

[17] Y.T. Tan, R. Masri, N. Mohd Idrus, W.N.F. Wan Mohd Fauzi, N.H. Sarmin, H.I. Mat Hassim, The central subgroup of the nonabelian tensor square of Bieberbach

group of dimension six with symmetric point group of order six, *International Journal of Applied Mathematics and Statistics* **53** (4) (2015) 98–103.

[18] Y.T. Tan, N. Mohd Idrus, R. Masri, N.H. Sarmin, H.I. Mat Hassim, Homological functor of a torsion free crystallographic group of dimension five with nonabelian point group of order eight, *AIP Conf. Proc.* **1602** (2014) 691–695.

[19] Y.T. Tan, N. Mohd Idrus, R. Masri, W.N.F. Wan Mohd Fauzi, N.H. Sarmin, On the torsion free crystallographic group with symmetric point group on three elements, In: *Proceeding of 2nd International Science Postgraduate Conference 2014 (ISPC2014)*, 2014.

[20] Y.T. Tan, N. Mohd Idrus, R. Masri, W.N.F. Wan Mohd Fauzi, N.H. Sarmin, H.I. Mat Hassim, The nonabelian tensor square of a Bieberbach group with symmetric point group of order six, *Jurnal Teknologi* **78** (1) (2016) 189–193.

[21] W.N.F. Wan Mohd Fauzi, N. Mohd Idrus, R. Masri, Y.T. Tan, N.H. Sarmin, H.I. Mat Hassim, A homological functor of the second Bieberbach group with dihedral point group, *International Journal of Applied Mathematics and Statistics* **53** (5) (2015) 73–76.

[22] J.H.C. Whitehead, A certain exact sequence, *Annals of Mathematics* **52** (1) (1950) 51–110.

[23] A.J. Zomorodian, *Topology for Computing*, Cambridge University Press, 2005.

[24] The CARAT homepage, http://wwwb.math.rwth-aachen.de/carat/.