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Nabilah Najmuddin, Nor Haniza Sarmin, and Ahmad Erfanian

Domination Polynomial of the Commuting and Noncommuting Graphs of Some Finite Nonabelian Groups

Nabilah Najmuddin[1,a)], Nor Haniza Sarmin[1,b)] and Ahmad Erfanian[2,c)]

[1] *Department of Mathematical Sciences, Faculty of Science, Universiti Teknologi Malaysia, 81310 UTM Johor Bahru, Johor, Malaysia.*
[2] *Department of Pure Mathematics, Faculty of Mathematics, Ferdowsi University of Mashhad, Azadi Square, 9177948975 Mashhad, Razavi Khorasan, Iran.*

[a)] nabilah.najmuddin@yahoo.com
[b)] nhs@utm.my
[c)] erfanian@um.ac.ir

Abstract. A dominating set S of a graph is a subset of the vertex set of the graph in which the closed neighborhood of S is the whole vertex set. A domination polynomial of a graph contains coefficients that represent the number of dominating sets in the graph. A domination polynomial is usually being obtained for common types of graphs but not for graphs associated to groups. Two types of graphs associated to groups that are used in this research are the commuting graph and the noncommuting graph. The commuting graph of a group G is a graph whose vertex set contains all noncentral elements of G and any two vertices in the set are adjacent if and only if they commute in G. Meanwhile, the noncommuting graph of a group G is a graph whose vertex set contains all noncentral elements of G and any two vertices in the set are adjacent if and only if they do not commute in G. This paper establishes the domination polynomial of the commuting and noncommuting graphs for the dihedral groups, generalized quaternion groups and quasidihedral groups.

INTRODUCTION

A graph $\Gamma = (V, E)$ consists of non-empty set of vertices V and a set of edges, E. For vertices $u, v \in V$, u and v are adjacent if they are connected by an edge, $e = (u, v) \in E$. Note that all graphs that are considered in this paper are simple, without multiple edges or loops. Open neighborhood of vertex v is defined to be the set of all vertices adjacent to v, without v, denoted by $N(v) = \{u \in V : (u, v) \in E, u \neq v\}$. If $N(v) \cup \{v\}$, then the set is called closed neighborhood, denoted by $N[v]$ [1]. A graph with empty neighborhood is called an empty graph, denoted by E_n, which contains n vertices without any edge. In graph theory, the complement of a graph Γ is the graph $\overline{\Gamma}$ on the same vertices, such that two distinct vertices of $\overline{\Gamma}$ are adjacent if and only if they are not adjacent in Γ.

Let $\Gamma_1 = (V_1, E_1)$ and $\Gamma_2 = (V_2, E_2)$ be two simple graphs. The graph $\Gamma = (V, E)$ is called the union of Γ_1 and Γ_2 if and only if the graph contains $V = V_1 \cup V_2$ and $E = E_1 \cup E_2$, and is denoted by $\Gamma = \Gamma_1 \cup \Gamma_2$. The join of Γ_1 and Γ_2, denoted as $\Gamma_1 \vee \Gamma_2$ is the graph containing all the vertices and edges like in $\Gamma_1 \cup \Gamma_2$ and each vertex of Γ_1 is also adjacent to every vertex of Γ_2. A complete graph K_n is a graph in which it contains n vertices and each pair of distinct vertices is connected by an edge. Let Γ be the union of t complete graphs K_{n_i} of n_i vertices, $1 \leq i \leq t$, then $\overline{\Gamma}$ is a multipartite graph denoted by $K_{n_1, n_2, \ldots, n_t}$ [2].

Graph polynomial is studied to describe the combinatorial information of graph. Graph polynomial is a graph property whose values are polynomials. Some examples of graph polynomials that have been introduced by other researchers are the chromatic polynomial [3], the independence polynomial [4], the cycle polynomial [5] and the matching polynomial [6]. The domination polynomial of a graph is introduced by Alikhani [7], in which its coefficients represent the number of dominating sets in the graph and the degree of polynomial is the number of vertices in the graph. The domination polynomials have been obtained for graphs such as the complete graph [7], the empty graph [7] and also for the union of complete graphs [8], but not yet for the graphs associated to groups.

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Two types of graphs associated to groups that will be considered in this research are the commuting graph and the noncommuting graph. A commuting graph Γ_G^{comm} of a group G is defined as the graph whose vertex set is the noncentral elements of G and two distinct vertices are adjacent if they commute in G [9]. The commuting graphs for the nonabelian groups that are used in this research can be expressed as the union of some complete graphs on certain number of vertices. Meanwhile, a noncommuting graph Γ_G^{nc} of a group G is a graph whose vertex set is the noncentral elements of G and two distinct vertices are adjacent if they do not commute in G [10]. For the groups considered in this research, their noncommuting graphs are expressed as multipartite graphs.

In this research, the domination polynomials are obtained for the commuting and noncommuting graphs of some finite nonabelian groups, namely, dihedral group D_{2n} of order $2n$, generalized quaternion group Q_{4n} of order $4n$ and quasidiheral group QD_{2^n} of order 2^n. These three groups can be expressed in group representations as follows:

- $D_{2n} = \langle a, b : a^n = b^2 = 1, bab = a^{-1} \rangle$, where $n \geq 3$, $n \in \mathbb{N}$.
- $Q_{4n} = \langle a, b : a^{2n} = b^4 = 1, a^n = b^2, b^{-1}ab = a^{-1} \rangle$, where $n \geq 2$, $n \in \mathbb{N}$.
- $QD_{2^n} = \langle a, b : a^{2^{n-1}} = b^2 = 1, bab^{-1} = a^{2^{n-2}-1} \rangle$, where $n \geq 4$, $n \in \mathbb{N}$.

PRELIMINARIES

The following are some preliminaries in graph theory that are needed to assist in obtaining the results of this research. First, the basic concepts on the dominating sets and the domination polynomials are given.

Definition 1 *[7] The dominating set of a graph Γ is a subset $S \subseteq V$ in which every vertex in $V - S$ is adjacent to at least one vertex in S, or equivalently $N[S] = V$. The minimum cardinality of a dominating set in Γ is called the domination number $\delta(\Gamma)$.*

Definition 2 *[7] The domination polynomial of a graph Γ, denoted by $D(\Gamma; x)$, is the polynomial whose coefficient on x^k is given by the number of dominating sets of order k in Γ. So*

$$D(\Gamma; x) = \sum_{k=\delta(\Gamma)}^{|V(\Gamma)|} d_k x^k$$

in which d_k is the number of dominating sets of size k and $\delta(\Gamma)$ is the domination number of the graph Γ.

From previous researches, some properties related to the domination polynomials are presented as in the following.

Proposition 1 *[7] Let Γ be the union of n disjoint graphs, $\Gamma_1, \Gamma_2, \ldots, \Gamma_n$. Then the domination polynomial of Γ is the product of the domination polynomials of the graph Γ_i, $1 \leq i \leq n$, expressed as follows:*

$$D(\Gamma; x) = D(\Gamma_1 \cup \Gamma_2 \cup \ldots \cup \Gamma_n; x) = D(\Gamma_1; x) \cdot D(\Gamma_2; x) \cdot \ldots \cdot D(\Gamma_n; x).$$

Proposition 2 *[7] The domination polynomial of a complete graph, K_n on n vertices is*

$$D(K_n; x) = (1 + x)^n - 1.$$

Proposition 3 *[8] The domination polynomial of the union of m complete graphs is*

$$D\left(\bigcup_{i=1}^{m} K_{n_i}; x\right) = \prod_{i=1}^{m} \left[(1 + x)^{n_i} - 1\right].$$

Proposition 4 *[7] The domination polynomial of an empty graph, E_n with n vertices is*

$$D(E_n; x) = x^n.$$

Proposition 5 *[11] Suppose that the order n of a graph Γ is the number of vertices in Γ. Let $\Gamma_1, \Gamma_2, \ldots, \Gamma_t$ be graphs of order $n_1, n_2, \ldots, n_t$ respectively. Then the domination polynomial of the join of the graphs $\Gamma_1, \Gamma_2, \ldots, \Gamma_t$ is presented as follows:*

$$D\left(\bigvee_{i=1}^{t} \Gamma_i; x\right) = \sum_{j=1}^{t-1}\left((1+x)^{\sum_{i=j+1}^{t} n_i} - 1\right) + \sum_{i=1}^{t} D(\Gamma_i; x)$$

By using Proposition 4 and Proposition 5, the domination polynomial of the multipartite graph is obtained as presented in the following lemma.

Lemma 1 *Let $\Gamma = K_{n_1,n_2,\ldots n_t}$ be a multipartite graph. Then the domination polynomial of Γ is*

$$\begin{aligned} D(K_{n_1,n_2,\ldots n_t}; x) = \; & \left[(1+x)^{n_1} - 1\right]\left[(1+x)^{n_2+n_3+\ldots+n_t} - 1\right] + \left[(1+x)^{n_2} - 1\right]\left[(1+x)^{n_3+n_4+\ldots+n_t} - 1\right] + \ldots + \\ & \left[(1+x)^{n_{t-1}} - 1\right]\left[(1+x)^{n_t} - 1\right] + x^{n_1} + x^{n_2} + \ldots + x^{n_t}. \end{aligned}$$

Proof By substituting Γ_i with the empty graph E_{n_i} in Proposition 5 and since $D(\Gamma_i; x) = D(E_{n_i}; x) = x^{n_i}$, the domination polynomial for multipartite graph is obtained as follows:

$$\begin{aligned} D(K_{n_1,n_2,\ldots n_t}; x) = \; & D(\overline{K_{n_1} \cup K_{n_2} \cup \ldots \cup K_{n_t}}; x) \\ = \; & D(\overline{K_{n_1}} \vee \overline{K_{n_2}} \vee \ldots \vee \overline{K_{n_t}}; x) \\ = \; & D(E_{n_1} \vee E_{n_2} \vee \ldots \vee K_{n_t}; x) \\ = \; & \textstyle\sum_{j=1}^{t-1}\left((1+x)^{\sum_{i=j+1}^{t} n_i} - 1\right) + \sum_{i=1}^{t} D(E_{n_i}; x) \\ = \; & \left[(1+x)^{n_1} - 1\right]\left[(1+x)^{n_2+n_3+\ldots+n_t} - 1\right] + \left[(1+x)^{n_2} - 1\right]\left[(1+x)^{n_3+n_4+\ldots+n_t} - 1\right] + \ldots + \\ & \left[(1+x)^{n_{t-1}} - 1\right]\left[(1+x)^{n_t} - 1\right] + x^{n_1} + x^{n_2} + \ldots + x^{n_t}. \end{aligned}$$

Next, some important preliminaries related to the theory of graphs on groups are given below. Some researchers have obtained the general form of the commuting and noncommuting graphs associated to some finite groups. For example, for groups that are related to this research, their commuting and noncommuting graphs are stated in the following.

Proposition 6 *[12] Suppose that D_{2n} is the dihedral group of order $2n$, where $n \geq 3$, $n \in \mathbb{N}$. Then the commuting graph of D_{2n} is*

$$\Gamma_{D_{2n}}^{comm} = \begin{cases} \bigcup_{i=1}^{n} K_1 \cup K_{n-1} & ; \quad n \text{ is odd} \\ \bigcup_{i=1}^{\frac{n}{2}} K_2 \cup K_{n-2} & ; \quad n \text{ is even} \end{cases}$$

and the noncommuting graph of D_{2n} is

$$\Gamma_{D_{2n}}^{nc} = \begin{cases} K_{\underbrace{1,1,\ldots,1}_{n \text{ times}},n-1} & ; \quad n \text{ is odd} \\ K_{\underbrace{2,2,\ldots,2}_{\frac{n}{2} \text{ times}},n-2} & ; \quad n \text{ is even.} \end{cases}$$

Proposition 7 *[12] Suppose that Q_{4n} is the generalized quaternion group of order $4n$, where $n \geq 2$, $n \in \mathbb{N}$. Then the commuting graph of Q_{4n} is*

$$\Gamma_{Q_{4n}}^{comm} = \bigcup_{i=1}^{n} K_2 \cup K_{2n-2}$$

and the noncommuting graph of Q_{4n} is

$$\Gamma^{nc}_{Q_{4n}} = K_{\underbrace{2,2,\ldots,2}_{n\ times},2n-2}.$$

Proposition 8 *[13] Suppose that QD_{2^n} is the quasidihedral group of order 2^n, where $n \geq 4$, $n \in \mathbb{N}$. Then the commuting graph of QD_{2^n} is*

$$\Gamma^{comm}_{QD_{2^n}} = \bigcup_{i=1}^{2^{n-2}} K_2 \cup K_{2^{n-1}-2}$$

and the noncommuting graph of Q_{4n} is

$$\Gamma^{nc}_{QD_{2^n}} = K_{\underbrace{2,2,\ldots,2}_{2^{n-2}\ times},2^{n-1}-2}.$$

MAIN RESULTS

This section is divided into three parts. The first part is on the domination polynomials of the commuting and noncommuting graphs of dihedral groups. Then, the domination polynomials for the same graphs associated to generalized quaternion groups are presented. Lastly, the third part contains the domination polynomials of the commuting and noncommuting graphs of quasidihedral groups.

Domination Polynomial of the Commuting and Noncommuting Graphs of Dihedral Group

The general form of the domination polynomials of the commuting and noncommuting graphs for dihedral groups are presented in the following theorems.

Theorem 1 *Supose that D_{2n} is a dihedral group, where $n \geq 3$, $n \in \mathbb{N}$, and $\Gamma^{comm}_{D_{2n}}$ is its commuting graph. Then, the domination polynomial of $\Gamma^{comm}_{D_{2n}}$ is*

$$D(\Gamma^{comm}_{D_{2n}}; x) = \begin{cases} x^n\Big[(1+x)^{n-1} - 1\Big] & ; \quad n \text{ is odd} \\ \Big(2x + x^2\Big)^{\frac{n}{2}}\Big[(1+x)^{n-2} - 1\Big] & ; \quad n \text{ is even}. \end{cases}$$

Proof Let D_{2n} be a dihedral group of order $2n$, where $n \geq 3$, $n \in \mathbb{N}$ and $\Gamma^{comm}_{D_{2n}}$ be its commuting graph.
Case 1: n is odd
From Proposition 6, $\Gamma^{comm}_{D_{2n}} = \bigcup_{i=1}^{n} K_1 \cup K_{n-1}$. By Proposition 1, Proposition 2 and Proposition 3, the domination polynomial of $\Gamma^{comm}_{D_{2n}}$ is computed as follows:

$$\begin{aligned} D(\Gamma^{comm}_{D_{2n}}; x) &= D\Big(\bigcup_{i=1}^{n} K_1 \cup K_{n-1}; x\Big) \\ &= D\Big(\bigcup_{i=1}^{n} K_1; x\Big) \cdot D\Big(K_{n-1}; x\Big) \\ &= \Big[(1+x)^1 - 1\Big]^n \Big[(1+x)^{n-1} - 1\Big] \\ &= \Big(x^n\Big)\Big[(1+x)^{n-1} - 1\Big]. \end{aligned}$$

Case 2: n is even

From Proposition 6, $\Gamma^{comm}_{D_{2n}} = \bigcup^{\frac{n}{2}} K_2 \cup K_{n-2}$. By Proposition 1, Proposition 2 and Proposition 3, the domination polynomial of $\Gamma^{comm}_{D_{2n}}$ is computed as follows:

$$\begin{aligned} D(\Gamma^{comm}_{D_{2n}}; x) = &\quad D\Big(\bigcup_{i=1}^{\frac{n}{2}} K_2 \cup K_{n-2}; x\Big) \\ = &\quad D\Big(\bigcup_{i=1}^{\frac{n}{2}} K_2; x\Big) \cdot D\Big(K_{n-2}; x\Big) \\ = &\quad \Big[(1+x)^2 - 1\Big]^{\frac{n}{2}} \Big[(1+x)^{n-2} - 1\Big] \\ = &\quad \Big[1 + 2x + x^2 - 1\Big]^{\frac{n}{2}} \Big[(1+x)^{n-2} - 1\Big] \\ = &\quad \Big(2x + x^2\Big)^{\frac{n}{2}} \Big[(1+x)^{n-2} - 1\Big]. \end{aligned}$$

Theorem 2 *Suppose that D_{2n} is a dihedral group, where $n \geq 3$, $n \in \mathbb{N}$, and $\Gamma^{nc}_{D_{2n}}$ is its noncommuting graph. Then, the domination polynomial of $\Gamma^{nc}_{D_{2n}}$ is*

$$D(\Gamma^{nc}_{D_{2n}}; x) = \begin{cases} x\Big[(1+x)^{n-1} - 1\Big] + nx + x^{n-1} + x\sum_{i=2}^{n} \Big[(1+x)^{2n-i} - 1\Big] & ; \quad n \text{ is odd} \\ \Big(2x + x^2\Big)\Big[(1+x)^{n-2} - 1\Big] + \frac{n}{2}x^2 + x^{n-2} + \Big(2x + x^2\Big)\sum_{i=2}^{\frac{n}{2}} \Big[(1+x)^{2(n-i)} - 1\Big] & ; \quad n \text{ is even}. \end{cases}$$

Proof Let D_{2n} be a dihedral group of order $2n$, where $n \geq 3$, $n \in \mathbb{N}$ and $\Gamma^{nc}_{D_{2n}}$ be its noncommuting graph.

Case 1: n is odd

From Proposition 6, $\Gamma^{nc}_{D_{2n}} = K_{\underbrace{1,1,\ldots,1}_{\text{n times}},n-1}$. Using Lemma 1, the domination polynomial of $\Gamma^{nc}_{D_{2n}}$ is established through the following:

$$\begin{aligned} D(\Gamma^{nc}_{D_{2n}}; x) = &\quad D\Big(K_{\underbrace{1,1,\ldots,1}_{\text{n times}},n-1}; x\Big) \\ = &\quad \Big[(1+x)^1 - 1\Big]\Big[(1+x)^{(n-1)+(n-1)} - 1\Big] + \Big[(1+x)^1 - 1\Big]\Big[(1+x)^{(n-2)+(n-1)} - 1\Big] + \\ &\quad \Big[(1+x)^1 - 1\Big]\Big[(1+x)^{(n-3)+(n-1)} - 1\Big] + \ldots + \Big[(1+x)^1 - 1\Big]\Big[(1+x)^{(n-1)} - 1\Big] + \\ &\quad \underbrace{x^1 + x^1 + \ldots + x^1}_{\text{n times}} + x^{n-1} \\ = &\quad [x]\Big[(1+x)^{2n-2} - 1\Big] + [x]\Big[(1+x)^{2n-3} - 1\Big] + [x]\Big[(1+x)^{2n-4} - 1\Big] + \ldots + \\ &\quad [x]\Big[(1+x)^{(n-1)} - 1\Big] + nx + x^{n-1}. \\ = &\quad (x)\Big[(1+x)^{(n-1)} - 1\Big] + nx + x^{n-1} + x\sum_{i=2}^{n} \Big[(1+x)^{2n-i} - 1\Big]. \end{aligned}$$

Case 2: n is even

From Theorem 6, $\Gamma^{nc}_{D_{2n}} = K_{\underbrace{2,2,\ldots,2}_{\frac{n}{2}\text{ times}},n-2}$. By Lemma 1, the domination polynomial of $\Gamma^{nc}_{D_{2n}}$ is computed as in the following:

$$\begin{aligned}
D(\Gamma^{nc}_{D_{2n}};x) = {} & D\Big(K_{\underbrace{2,2,\ldots,2}_{\frac{n}{2}\text{ times}},n-2};x\Big) \\
= {} & \Big[(1+x)^2-1\Big]\Big[(1+x)^{(\frac{n}{2}-1)(2)+(n-2)}-1\Big]+\Big[(1+x)^2-1\Big]\Big[(1+x)^{(\frac{n}{2}-2)(2)+(n-2)}-1\Big]+ \\
& \Big[(1+x)^2-1\Big]\Big[(1+x)^{(\frac{n}{2}-3)(2)+(n-2)}-1\Big]+\ldots+\Big[(1+x)^2-1\Big]\Big[(1+x)^{(n-2)}-1\Big]+ \\
& \underbrace{x^2+x^2+\ldots+x^2}_{\frac{n}{2}\text{ times}}+x^{n-2} \\
= {} & \Big[2x+x^2\Big]\Big[(1+x)^{2(n-2)}-1\Big]+\Big[2x+x^2\Big]\Big[(1+x)^{2(n-3)}-1\Big]+\Big[2x+x^2\Big]\Big[(1+x)^{2(n-4)}-1\Big]+ \\
& \ldots+\Big[2x+x^2\Big]\Big[(1+x)^{(n-2)}-1\Big]+\tfrac{n}{2}x^2+x^{n-2}. \\
= {} & \Big[2x+x^2\Big]\Big[(1+x)^{(n-2)}-1\Big]+\tfrac{n}{2}x^2+x^{n-2}+\Big[2x+x^2\Big]\textstyle\sum_{i=2}^{\frac{n}{2}}\Big[(1+x)^{2(n-i)}-1\Big].
\end{aligned}$$

Domination Polynomial of the Commuting and Noncommuting Graphs of Generalized Quaternion Group

The general form of the domination polynomials of the commuting and noncommuting graphs for generalized quaternion groups are presented in the following theorems.

Theorem 3 *Suppose that Q_{4n} is a generalized quaternion group, where $n \geq 2$, $n \in \mathbb{N}$ and $\Gamma^{comm}_{Q_{4n}}$ is its commuting graph. Then, the domination polynomial of $\Gamma^{comm}_{Q_{4n}}$ is*

$$D(\Gamma^{comm}_{Q_{4n}};x) = \Big(2x+x^2\Big)^n\Big[(1+x)^{2n-2}-1\Big].$$

Proof Let Q_{4n} be a generalized quaternion group of order $4n$, where $n \geq 2$, $n \in \mathbb{N}$ and $\Gamma^{comm}_{Q_{4n}}$ be its commuting graph. From Proposition 7, $\Gamma^{comm}_{Q_{4n}} = \bigcup_{i=1}^{n} K_2 \cup K_{2n-2}$. By Proposition 1, Proposition 2 and Proposition 3, the domination polynomial of $\Gamma^{comm}_{Q_{4n}}$ is computed as follows:

$$\begin{aligned}
D(\Gamma^{comm}_{Q_{4n}};x) = {} & D\Big(\bigcup_{i=1}^{n} K_2 \cup K_{2n-2};x\Big) \\
= {} & D\Big(\bigcup_{i=1}^{n} K_2;x\Big)\cdot D\Big(K_{2n-2};x\Big) \\
= {} & \Big[(1+x)^2-1\Big]^n\Big[(1+x)^{2n-2}-1\Big] \\
= {} & \Big(2x+x^2\Big)^n\Big[(1+x)^{2n-2}-1\Big].
\end{aligned}$$

Theorem 4 *Suppose that Q_{4n} is a generalized quaternion group, where $n \geq 2$, $n \in \mathbb{N}$, with $\Gamma^{nc}_{Q_{4n}}$ as its noncommuting graph. Then, the domination polynomial of $\Gamma^{nc}_{Q_{4n}}$ is*

$$D(\Gamma^{nc}_{Q_{4n}};x) = \Big(2x+x^2\Big)\Big[(1+x)^{2(n-1)}-1\Big]+nx^2+x^{2(n-1)}+\Big(2x+x^2\Big)\sum_{i=1}^{n}\Big[(1+x)^{2(2n-i)}-1\Big].$$

Proof Let Q_{4n} be a generalized quaternion group of order $4n$, where $n \geq 2$, $n \in \mathbb{N}$ and $\Gamma^{nc}_{Q_{4n}}$ be its noncommuting graph.
From Proposition 7, $\Gamma^{comm}_{Q_{4n}} = K_{\underbrace{2,2,\ldots,2}_{\text{n times}},2n-2}$. Using Lemma 1, the domination polynomial of $\Gamma^{nc}_{Q_{4n}}$ is established through

the following:

$$
\begin{aligned}
D(\Gamma^{nc}_{Q_{4n}};x) = \;& D\Big(K_{\underbrace{2,2,\ldots,2}_{\text{n times}},2n-2};x\Big)\\
=\;& \Big[(1+x)^2-1\Big]\Big[(1+x)^{(n-1)(2)+(2n-2)}-1\Big]+\Big[(1+x)^2-1\Big]\Big[(1+x)^{(n-2)(2)+(2n-2)}-1\Big]+\\
& \Big[(1+x)^2-1\Big]\Big[(1+x)^{(n-3)(2)+(2n-2)}-1\Big]+\ldots+\Big[(1+x)^2-1\Big]\Big[(1+x)^{2n-2}-1\Big]+\\
& \underbrace{x^2+x^2+\ldots+x^2}_{\text{n times}}+x^{2n-2}\\
=\;& \Big[2x+x^2\Big]\Big[(1+x)^{2(2n-2)}-1\Big]+\Big[2x+x^2\Big]\Big[(1+x)^{2(2n-3)}-1\Big]+\Big[2x+x^2\Big]\Big[(1+x)^{2(2n-4)}-1\Big]+\\
& \ldots+\Big[2x+x^2\Big]\Big[(1+x)^{2n-2}-1\Big]+nx^2+x^{2n-2}.\\
=\;& \Big[2x+x^2\Big]\Big[(1+x)^{2n-2}-1\Big]+nx^2+x^{2n-2}+\Big[2x+x^2\Big]\textstyle\sum_{i=2}^{n}\Big[(1+x)^{2(2n-i)}-1\Big].
\end{aligned}
$$

Domination Polynomial of the Commuting and Noncommuting Graphs of Quasidihedral Group

The general form of the domination polynomials of the commuting and noncommuting graphs for quasidihedral groups are presented in the following theorems.

Theorem 5 *Suppose that QD_{2^n} be a quasidihedral group, where $n \geq 4$, $n \in \mathbb{N}$ and $\Gamma^{comm}_{QD_{2^n}}$ be its commuting graph. Then, the domination polynomial of $\Gamma^{comm}_{QD_{2^n}}$ is*

$$D(\Gamma^{comm}_{QD_{2^n}};x) = \Big(2x+x^2\Big)^{2^{n-2}}\Big[(1+x)^{2^{n-1}-2}-1\Big].$$

Proof Let QD_{2^n} be a quasidihedral group of order 2^n, where $n \geq 4$, $n \in \mathbb{N}$ and $\Gamma^{comm}_{QD_{2^n}}$ be its commuting graph. From Proposition 8, $\Gamma^{comm}_{QD_{2^n}} = \bigcup_{i=1}^{2^{n-2}} K_2 \cup K_{2^{n-1}-2}$. By Proposition 1, Proposition 2 and Proposition 3, the domination polynomial of $\Gamma^{comm}_{QD_{2^n}}$ is computed as follows:

$$
\begin{aligned}
D(\Gamma^{comm}_{QD_{2^n}};x) = \;& D\Big(\bigcup_{i=1}^{2^{n-2}} K_2 \cup K_{2^{n-1}-2};x\Big)\\
=\;& D\Big(\bigcup_{i=1}^{2^{n-2}} K_2;x\Big)\cdot D\Big(K_{2^{n-1}-2};x\Big)\\
=\;& \Big[(1+x)^2-1\Big]^{2^{n-2}}\Big[(1+x)^{2^{n-1}-2}-1\Big]\\
=\;& \Big(2x+x^2\Big)^{2^{n-2}}\Big[(1+x)^{2^{n-1}-2}-1\Big].
\end{aligned}
$$

Theorem 6 *Suppose that QD_{2^n} be a quasidihedral group, where $n \geq 4$, $n \in \mathbb{N}$ and $\Gamma^{nc}_{QD_{2^n}}$ is its noncommuting graph. Then, the domination polynomial of $\Gamma^{nc}_{QD_{2^n}}$ is*

$$D(\Gamma^{nc}_{QD_{2^n}};x) = \Big(2x+x^2\Big)\Big[(1+x)^{2^{n-1}-2}-1\Big]+2^{n-2}x^2+x^{2^{n-1}-2}+\Big(2x+x^2\Big)\sum_{i=2}^{2^{n-2}}\Big[(1+x)^{2(2^{n-1}-i)}-1\Big].$$

Proof Let QD_{2^n} be a quasidihedral group of order 2^n, where $n \geq 4$, $n \in \mathbb{N}$ and $\Gamma^{nc}_{QD_{2^n}}$ be its noncommuting graph. From Proposition 8, $\Gamma^{comm}_{QD_{2^n}} = K_{\underbrace{2,2,\ldots,2}_{2^{n-2}\text{ times}},2^{n-1}-2}$. Using Lemma 1, the domination polynomial of $\Gamma^{nc}_{QD_{2^n}}$ is established through the following:

$$
\begin{aligned}
D(\Gamma^{nc}_{QD_{2^n}};x) = \;& D\Big(K_{\underbrace{2,2,\ldots,2}_{2^{n-2}\text{times}},2^{n-1}-2};x\Big)\\
= \;& \Big[(1+x)^2-1\Big]\Big[(1+x)^{(2^{n-2}-1)(2)+(2^{n-1}-2)}-1\Big]+\Big[(1+x)^2-1\Big]\Big[(1+x)^{(2^{n-2}-2)(2)+(2^{n-1}-2)}-1\Big]+\\
& \Big[(1+x)^2-1\Big]\Big[(1+x)^{(2^{n-2}-3)(2)+(2^{n-1}-2)}-1\Big]+\ldots+\Big[(1+x)^2-1\Big]\Big[(1+x)^{2^{n-1}-2}-1\Big]+\\
& \underbrace{x^2+x^2+\ldots+x^2}_{2^{n-2}\text{ times}}+x^{2^{n-1}-2}\\
= \;& \Big[2x+x^2\Big]\Big[(1+x)^{2(2^{n-1}-2)}-1\Big]+\Big[2x+x^2\Big]\Big[(1+x)^{2(2^{n-1}-3)}-1\Big]+\Big[2x+x^2\Big]\Big[(1+x)^{2(2^{n-1}-4)}-1\Big]+\\
& \ldots+\Big[2x+x^2\Big]\Big[(1+x)^{2^{n-1}-2}-1\Big]+2^{n-2}x^2+x^{2^{n-1}-2}.\\
= \;& \Big[2x+x^2\Big]\Big[(1+x)^{2^{n-1}-2}-1\Big]+2^{n-2}x^2+x^{2^{n-1}-2}+\Big[2x+x^2\Big]\sum_{i=2}^{2^{n-2}}\Big[(1+x)^{2(2^{n-1}-i)}-1\Big].
\end{aligned}
$$

CONCLUSION

In this paper, the general form of domination polynomials for the commuting and noncommuting graphs are established for the dihedral groups, the generalized quaternion groups and the quasidihedral groups. The degree of each domination polynomial obtained is the number of vertices of the graphs that depends on the values of n. By definition of the domination polynomial of graph, the smallest number for the power of x is the domination number of the graph. Therefore, in this paper, the domination numbers of the commuting graph of dihedral groups are $n+1$ when n is odd and $\frac{n}{2}+1$ when n is even. The domination number of the noncommuting graph of dihedral groups are 1 when n is odd and 2 when n is even. Meanwhile, from the domination polynomial obtained for the generalized quaternion group, the domination number for its commuting graph is $n+1$ and the domination number for its noncommuting graph is 2. Furthermore, from the domination polynomial obtained for the quasidihedral group, the domination number for its commuting graph is $2^{n-2}+1$ and the domination number for its noncommuting graph is 2.

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