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The Multiplicative Degree of Some Finite Groups

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Let G be a finite group. The multiplicative degree of G is one of the extensions of the commutativity degree that can be obtained by considering the properties of the subgroup of G. This extension is defined as the probability that the product of a pair of elements chosen randomly from G, is in a subgroup H of G. In this paper, the general formula for the multiplicative degree of some finite groups is provided. Besides, the multiplicative degree for two different subgroups in two different groups is also obtained.

Keywords: finite group; multiplicative degree; commutativity degree; probability

I. INTRODUCTION

In mathematics, there is a branch related with the study of uncertainty named probability. This probability can be applied to another field of mathematics such as group theory. The results obtained are very interesting since the calculation dealt with regular properties of elements of a certain set.

In this article only the finite groups, G are considered. The study of commutativity degree, denoted as $P(G)$ become more interesting as many researchers computed this probability in many ways. For example, (Abd Manaf *et al.*, 2010) computed the number of conjugacy classes in order to determine the commutativity degree of finite p-groups of nilpotency class two. The method of determine the commutativity degree of G by using the conjugacy class was first discovered by (Gustafson, 1973). Furthermore, (Omer *et al.,* 2012) found a formula for the commutativity degree of a group in terms of centralizers and they provided some upper bounds of the commutativity degree for dihedral groups.

There are also many researchers who worked on the extensions of the commutativity degree of G such as (Abdul Hamid *et al.*, 2013) where the relative commutativity degree of some dihedral groups is determined. Besides, (Abd Rhani *et al.*, 2016), introduced the subset relative degree where it is defined as the probability for a subset X to be a subgroup of a group G. This probability is a useful formula to determine how close a subset is to be a subgroup of G. In (Abd Rhani *et al.*, 2016), some upper and lower bounds of this probability are found, when the groups are presented in two cases, namely when their subset is a semi-group and a monoid. Furthermore, (Mohd Ali *et al.*, 2018) found the subset relative degree of two distinct sets in a finite group G.

In 2016, a new definition that is motivated from the relative commutativity degree, called the multiplicative degree of a group G was defined by (Abd Rhani *et al.*, 2016) and it is defined as the probability that the product of two elements chosen randomly from a group G, is in a subgroup H. In (Abd Rhani *et al.*, 2016), the multiplicative degree of some of dihedral groups is found. Furthermore, in (Mohd Ali et al., 2017) obtained the multiplicative degree of cyclic subgroups for non-abelian metabelian groups of order less than 24. The metabelian group is the group that are close to be abelian, approximately, every abelian group is metabelian, but not every metabelian group is abelian.

Hence, this article is organized as follows: the introduction is in the first part. The second part provides some concepts, definitions that will be used in this article. The third part is the methodology which includes some

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definitions and propositions and the last part is the main results and conclusion.

II. PRELIMINARIES

In this section, some beneficial definitions on group theory that are required in this study are provided starting with the definition of the dihedral groups.

Definition 2.1. (Dummit, 2004) For $n \geqslant 3$, the n-th dihedral group is defined as a group consists of rigid motions of a regular n-gon, denoted by D_n. The dihedral groups, D_n of order $2n$ can be presented in a form of generators and relations given as follows:

$$D_n = \langle a, b \mid a^n = b^2 = e, ba = a^{-1}b \rangle.$$

Definition 2.2. (Judson, 2015) Let A be a finite set $\{1,2,\ldots,n\}$. The permutations of a set A is the symmetric group on n letters and it is denoted by S_n.

Definition 2.3. (Dummit, 2004) The quaternion group, Q_8, is defined by

$$Q_8 = \{1, -1, i, -i, j, -j, k, -k\}$$

with the multiplicative operator, • as follows:

$$1 \cdot a = a \cdot 1 = a, \text{for all } a \in Q_8$$
$$(-1)\cdot(-1) = 1, (-1)\cdot a = a\cdot(-1) = -a, \text{for all } a \in Q_8$$
$$i \cdot i = j \cdot j = k \cdot k = -1$$
$$i \cdot j = k, j \cdot i = -k$$
$$j \cdot k = i, k \cdot j = -i$$
$$k \cdot i = j, i \cdot k = -j.$$

III. METHODOLOGY

This research begins with studying some basic concepts, definitions, computing methods and existing results of the commutativity degree of G and its generalizations. The general formula of multiplicative of G is determined by using the definition as follows:

Definition 3.1. (Abd Rhani *et al.*, 2016) Let G be a finite group and H be any subgroup of G. The multiplicative degree of G, denoted as $P_{mul}(H,G)$, is defined as:

$$P_{mul}(H,G) = \frac{\left|\{(x, y \in G \times G) : xy \in H\}\right|}{|G|^2}.$$

Based on Definition 3.1, there are four cases that can be considered in order to compute the multiplicative degree of G namely

(i) $x \in H$, $y \in H$ and $xy \in H$,

(ii) $x \in H$, $y \in G \setminus H$ and $xy \in H$,

(iii) $x \in G \setminus H$, $y \in H$ and $xy \in H$,

(iv) $x \in G \setminus H$, $y \in G \setminus H$ and $xy \in H$.

Case (i) is trivial since if $x \in H$, $y \in H$ then $xy \in H$. Case (ii) and (iii) are not possible. This is because, if $x \in H$ and $y \in G \setminus H$, then it is claimed $xy \notin H$. If $xy \in H$, then $y = x^{-1}(xy) \in H$ which is a contradiction. This similar method can be used for the case $x \in G \setminus H$ and $y \in H$, where $x = (xy)y^{-1} \in H$ which is also a contradiction. Thus, only Case (i) and Case (iv) are considered in computing the multiplicative degree of G. (Abd Rhani *et al.*, 2016) have defined the multiplicative degree of G for the case (iv) as follows.

Definition 3.2. (Abd Rhani *et al.*, 2016) Let G be a finite group and H be any subgroup of G. Then

$$P_{mul_2}(H,G) = \frac{\left|\{(x, y \in G \setminus H \times G \setminus H) : xy \in H\}\right|}{|G|^2}.$$

From Definition 3.2, they obtained the following two propositions.

Proposition 3.3. (Abd Rhani *et al.*, 2016) Let D_n be the dihedral group of order $2n$, where $n \geqslant 3$. Suppose H is a cyclic subgroup of D_n of order n. Then $P_{mul_2}(H, D_n) = \frac{1}{4}$.

Proposition 3.4. (Abd Rhani *et al.*, 2016) Let D_n be the dihedral group of order $2n$, where $n \geqslant 6$ and n is even. Let H be a cyclic subgroup of D_n of order n . Then $P_{mul_2}(H, D_n) = \frac{3}{16}$.

In the next section, new results of the multiplicative degree of some finite groups together with the multiplicative degree of two different subgroups in two different groups are presented.

IV. MAIN RESULTS

In this part, a general formula the multiplicative degree of G is found. Besides, the multiplicative degree for two different subgroups in two different groups is also obtained.

A. The General Formula for the Multiplicative Degree of Some Finite Groups

In this section, the general formula for the multiplicative degree of G is provided. The general formula of $P_{mul}(H,G)$, is found when the two cases mentioned in Part 3 are added. Therefore, $P_{mul}(H,G)$, can also be written as $P_{mul}(H,G)=P_{mul_1}(H,G)+P_{mul_2}(H,G)$, where the definitions of $P_{mul_1}(H,G)$, is stated as in Definition 4.1.

Definition 4.1. Let G be a finite group and H be any subgroup of G. Then $P_{mul_1}(H,G)=\frac{\left|\{(x,y\in H\times H):xy\in H\}\right|}{|G|^2}$.

The following example illustrates Definition 4.1

Example 4.1 Let Q_8 be the quaternion group of order 8, $Q_8=\{1,\ -1,\ i,\ -i,\ j,\ -j,\ k,\ -k\}$. Let H be a subgroup of Q_8, where $H=\{1,\ -1\}$. Firstly, the value of $P_{mul_1}(H,Q_8)$ is computed by using Definition 4.1. Suppose $x,y\in H$. If x = y, then $xy=1\in H$. If either x or y is 1 or –1, then $xy=-1\in H$.

Thus, $P_{mul_1}(H,Q_8)=\frac{4}{64}=\frac{1}{16}$.

Theorem 4.2 gives the multiplicative degree of G where both elements x and y are in the subgroup of G.

Theorem 4.2. Let G be a finite group and H be any subgroup of G. Then $P_{mul_1}(H,G)=\frac{|H|^2}{|G|^2}$.

Proof. Suppose H is any subgroup of G. By definition of the subgroup, for every $x,y\in H$, then $\left|\{(x,y)\in H\times H:xy\in H\}\right|=|H|^2$. By Definition 4.1, $P_{mul_1}(H,G)=\frac{|H|^2}{|G|^2}$. □

The result obtained in Example 4.1 is the same with the result obtained by using Theorem 4.2 which is $P_{mul_1}(H,G)=P_{mul_1}(H,Q_8)=\frac{2^2}{8^2}=\frac{1}{16}$.

Next, by using Definition 3.2, the following example is illustrated for the same group and subgroup from Example 4.1.

Example 4.2 Let Q_8 be the quaternion group of order eight, $Q_8=\{1,\ -1,\ i,\ -i,\ j,\ -j,\ k,\ -k\}$. Let H be a subgroup of Q_8, where $H=\{1,\ -1\}$. Firstly, $P_{mul_2}(H,Q_8)$) is computed by using Definition 3.2. Suppose $x,y\in Q_8\setminus H$. If $x\in G\setminus H$ and $y=x^{-1}$, then $xy=1\in H$. Meanwhile, if $x\in G\setminus H$ and y = x, then $xy=-1\in H$. Otherwise, $xy\notin H$. Therefore, $P_{mul_2}(H,Q_8)=\frac{12}{64}=\frac{3}{16}$.

Theorem 4.3 provides the multiplicative degree of G where both elements x and y are in G but not in the subgroup H.

Theorem 4.3. Let G be a finite group and H be any subgroup of G. Then $P_{mul_2}(H,G)=\frac{|H|(|G|-|H|)}{|G|^2}$.

Proof. Suppose H is any subgroup of a finite group G. Let $M=\{(x,y)\in G\setminus H\times G\setminus H:xy\in H\}$. Suppose a is an arbitrary fixed element in H. Then for every element $x\in G\setminus H$, there is $(ax^{-1},x)\in G\setminus H$ because if $ax^{-1}\in H$, then $(ax^{-1})^{-1}\in H$ since $a\in H$. Thus, $(xa^{-1})a=x\in H$ which is contradiction. Moreover, for $(ax^{-1})x=a\in H$. Hence $(ax^{-1},x)\in M$ which implies $|M|=|H||G\setminus H|=|H|(|G|-|H|)$. By Definition 3.2, $P_{mul_2}(H,G)=\frac{|M|}{|G|^2}$. Hence, $P_{mul_2}(H,G)=\frac{|H|(|G|-|H|)}{|G|^2}$.

The result in Example 4.2 is the same with the result obtained by using Theorem 4.3 which is $P_{mul_2}(H,G)=P_{mul_2}(H,Q_8)=\frac{2(8-2)}{8^2}=\frac{12}{8^2}=\frac{3}{16}$.

The following corollary gives the general formula for the multiplicative degree of a finite group.

Corollary 4.4. Let G be a finite group and H be any subgroup of G. Then $P_{mul}(H,G)=\frac{|H|}{|G|}$.

Proof. The multiplicative degree of a finite group is the summation of the two cases which is $P_{mul}(H,G)=P_{mul_1}(H,G)+P_{mul2}(H,G)$. By Theorem 4.2 and Theorem 4.3, $P_{mul}(H,G)=\frac{|H|^2}{|G|^2}+\frac{|H|(|G|-|H|)}{|G|^2}$. Therefore, $P_{mul}(H,G)=\frac{|H|}{|G|}$. □

To illustrate Corollary 4.4, the following example can be considered.

Example 4.3 Let Q_8 be the quaternion group of order 8, $Q_8=\{1, -1, \text{i}, -\text{i}, \text{j}, -\text{j}, \text{k}, -\text{k}\}$ and $H=\{1, -1\}$. The value of $P_{mul}(H,Q_8)$ can be calculated by adding up the results obtained in Example 4.1 and Example 4.2. It follows that $P_{mul}(H,Q_8)=P_{mul_1}(H,Q_8)=P_{mul_2}(H,Q_8)$ which implies $\frac{1}{16}+\frac{3}{16}=\frac{4}{16}=\frac{1}{4}$.

The result is the same with the result obtained by using the formula in Corollary 4.4 which is $P_{mul}(H,Q_8)=\frac{2}{8}=\frac{1}{4}$.

B. The Multiplicative Degree of Two Different Subgroups in Two Different Group

This subsection provides the result of the multiplicative degree of two different subgroups in two different groups. The result is given in Corollary 4.5.

Corollary 4.5. Let H_1 and H_2 be two subgroups of two finite groups G_1 and G_2, respectively. Then $P_{mul}(H_1\times H_2,G_1\times G_2)=P_{mul}(H_1,G_1)P_{mul}(H_2,G_2)$.

Proof. Suppose H_1 and H_2 are two subgroups of two finite groups G_1 and G_2 , respectively. Let $G_1\times G_2=\{(g_1,g_2):g_1\in G_1,g_2\in G_2\}$ and $H_1\times H_2=\{(h_1,h_2):h_1\in H_1,h_2\in H_2\}$. Suppose $G=G_1\times G_2$ $H=H_1\times H_2$. Then $P_{mul}(H_1\times H_2,G_1\times G_2)=P_{mul}(H,G)=\frac{|\{(x,y)\in G\times G:xy\in H\}|}{|G|^2}$. Hence, $(x,y)\in G\times G$, which implies $(x,y)\in(G_1\times G_2)\times(G_1\times G_2)$. Let $x=(m_1,n_1)$ and $y=(m_2,n_2)$. Now for $(x,y)\in H$, there is also $(x,y)\in G$. Here $(m_1,n_1)(m_2,n_2)\in G$, which is $(m_1,n_1)(m_2,n_2)\in G_1\times G_2$. Thus, $(m_1m_2,n_1n_2)\in G_1\times G_2$. Hence $(m_1m_2)\in G_1$ and $(n_1n_2)\in G_2$. Here $(m_1,m_2)\in R_1=\{(m_1,m_2)\in G_1\times G_1:m_1m_2\in G_1\}$ and $(n_1,n_2)\in R_2=\{(n_1,n_2)\in G_2\times G_2:n_1n_2\in G_2\}$. If $R=\{(x,y)\in G\times G:xy\in H\}$, then $R=R_1\times R_2$. Thus $P_{mul}(H,G)=\frac{|R|}{|G|^2}=\frac{|R|}{|G_1\times G_2|^2}=\frac{|R_1\times R_2|}{|G_1\times G_2|^2}=\frac{|R_1|\times|R_2|}{|G_1|^2\times|G_2|^2}=\frac{|R_1|}{|G_1|^2}\times\frac{|R_2|}{|G_2|^2}$ which implies $P_{mul}(H_1,G_1)P_{mul}(H_2,G_2)$.

Therefore,

$P_{mul}(H_1\times H_2,G_1\times G_2)=P_{mul}(H_1,G_1)P_{mul}(H_2,G_2)$. □

The following is an example of the multiplicative degree of two different subgroups in symmetric group of order six and dihedral group of order eight.

Example 4.4 Let G_1 be the symmetric group of order six, S_3 = {(1) , (12) , (13) , (23) , (123) , (132)} and G_2 be the dihedral group of order eight, $D_4=\{e,a,a^2,a^3,b,ab,a^2b,a^3b\}$. Let H_1 and H_2 be two subgroups of S_3 and D_4, where H_1 = {(1), (123), (132)} and $H_2=\{e,a,a^2,a^3\}$, respectively. Then

$G=S_3\times D_4=\{((1),e),((1),a)((1),a^2)((1),a^3)((1),b)((1),ab),$
$((1),a^2b),((1),a^3b),((12),e),((12),a),((12),a^2),((12),a^3),((12),b),$
$((12),ab),((12),a^2b),((12),a^3b),((13),e),((13),a),((13),a^2),$
$((13),a^3),((13),b),((13),ab),((13),a^2b),((13),a^3b),((23),e),$
$((23),a),((23),a^2),((23),a^3),((23),b),((23),ab),((23),a^2b),$

$((23),a^3b),((123),e),((123),a),((123),a^2),((123),a^3),((123),b),$ $((123),ab),((123),a^2b),((123),a^3b),((132),e),((132),a),$ $((132),a^2),((132),a^3),((132),b),((132),ab),((132),a^2b),$ $((132),a^3b)\}$ and $H=H_1\times H_2=\{((1),e),((1),a),((1),a^2),$ $((1),a^3),((123),e),((123),a),((123),a^2),((123),a^3),((132),e),$ $((132),a),((132),a^2),((132),a^3)\}$.

Firstly, the value of $P_{mul}(H_1\times H_2,S_3\times D_4)$ is computed by using Definition 3.1. If $x\in H$ with $y\in H$, then $xy\in H$. Furthemore, if $x=((1),b),((1),ab),((1),a^2b),((1),a^3b),$ $((123),b),((123),ab),((123),a^2b),((123),a^3b),((132),b),$ $((132),ab),((132),a^2b)$ and $((132),a^3b)$ with $y=((1),b),$ $((1),ab),((1),a^2b),((1),a^3b),((123),b),((123),ab),((123),a^2b),$ $((123),a^3b),((132),b),((132),ab),(((132)),a^2b)$ and $((132),a^3b),$ then $xy\in H$. Also, if $x=((12),e),((12),a),((12),a^2),$ $((12),a^3),((13),e),((13),a),((13),a^2),((13),a^3),((23),e),$ $((23),a),((23),a^2)$ and $((23),a^3)$ with $y=((12),e),((12),a),$ $((12),a^2),((12),a^3),((13),e),((13),a),((13),a^2),((13),a^3),$ $((23),e),((23),a),((23),a^2)$ and $((23),a^3)$, then $xy\in H$.

Besides, if $x=((12),b),((12),ab),((12),a^2b),((12),a^3b),$ $((13),b),((13),ab),((13),a^2b),((13),a^3b),((23),b),((23),ab),$ $((23),a^2b)$ and $((23),a^3b)$ with $y=((12),b),((12),ab),$ $((12),a^2b),((12),a^3b),((13),b),((13),ab),((13),a^2b),((13),a^3b)$ $((23),b),((23),ab),((23),a^2b)$ and $((23),a^3b)$, then $xy\in H$. Otherwise, $xy\notin H$. Thus, $P_{mul}(H_1\times H_2,S_3\times D_4)=\frac{576}{2304}=\frac{1}{4}.$

The result of $P_{mul}(H_1\times H_2,S_3\times D_4)$ is the same with the result of $P_{mul}(H_1,S_3)P_{mul}(H_2,D_4)$. By using Definition 3.1, the value of $P_{mul}(H_1,S_3)$ is computed. Here, $H_1\times H_1=\{((1),(1)),((1),(123)),((1),(132)),((123),(1)),$ $((123),(123)),((123),(132)),((132),(1)),((132),(123)),$ $((132),(132))\}$. If $x\in H_1$, with $y\in H_1$, then $xy\in H_1$. Also, if $x\in S_3\setminus H_1$ $y\in S_3\setminus H_1$, then $xy\in H_1$. Otherwise, $xy\notin H_1$. Therefore, $P_{mul}(H_1,S_3)=\frac{18}{36}=\frac{1}{2}.$

Next, the value of $P_{mul}(H_2,D_4)$ is computed. Here, $H_2\times H_2=\{(e,e),(e,a),(e,a^2),(e,a^3),(a,e),(a,a),(a,a^2),(a,a^3),$ $(a^2,e),(a^2,a),(a^2,a^2),(a^2,a^3),(a^3,e),(a^3,a),(a^3,a^2),(a^3,a^3)\}.$ If $x\in H_2$, with $y\in H_2$, then $xy\in H_2$. Also, if $x\in D_4\setminus H_2$ $y\in D_4\setminus H_2$, then $xy\in H_2$. Otherwise, $xy\notin H_2$. Therefore, $P_{mul}(H_2,D_4)=\frac{32}{64}=\frac{1}{2}.$

This implies

$$P_{mul}(H_1\times H_2,S_3\times D_4)=\left(\frac{1}{2}\times\frac{1}{2}\right)=\frac{1}{4}.$$

Therefore,

$$P_{mul}(H_1\times H_2,S_3\times D_4)=P_{mul}(H_1,S_3)P_{mul}(H_2,D_4).$$

The following corollary is to compare Corollary 4.4 with Corollary 4.5.

Corollary 4.6. Let H_1 and H_2 be two subgroups of two finite groups G_1 and G_2, respectively. Then $P_{mul}(H_1\times H_2,G_1\times G_2)=P_{mul}(H_1,G_1)P_{mul}(H_2,G_2)$.

Proof. Suppose H_1 and H_2 are two subgroups of two finite groups G_1 and G_2, respectively. Here, $H_1\times H_2$ is a subgroup of $G_1\times G_2$. Then $P_{mul}(H_1\times H_2,G_1\times G_2)=$ $\frac{|H_1\times H_2|}{|G_1\times G_2|}=\frac{|H_1|\times|H_2|}{|G_1|\times|G_2|}=\frac{|H_1|}{|G_1|}\times\frac{|H_2|}{|G_2|}$ which implies $P_{mul}(H_1,G_1)P_{mul}(H_2,G_2)$.

Therefore,

$$P_{mul}(H_1\times H_2,G_1\times G_2)=P_{mul}(H_1,G_1)P_{mul}(H_2,G_2).$$

V. CONCLUSION

In this article, a general formula of the multiplicative degree of some finite groups is found. Besides, the multiplicative degree for two different subgroups in two different groups is also obtained.

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