Southeast Asian Bulletin of Mathematics (2020) 44: 763–768

Southeast Asian
Bulletin of
Mathematics

The Harary Index of the Non-commuting Graph for Dihedral Groups*

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Received 2 April 2019
Accepted 7 June 2020

Communicated by Edy Tri Baskoro

AMS Mathematics Subject Classification(2000): 05C12, 05C40

Abstract. Assume G is a non-abelian group which consists a set of vertices, $V = \{v_1, v_2, ..., v_n\}$ and a set of edges, $E = \{e_1, e_2, ..., e_m\}$ where n and m are the positive integers. The non-commuting graph of G, denoted by Γ_G, is the graph of vertex set $G - Z(G)$, whose vertices are non-central elements, in which $Z(G)$ is the center of G and two distinct vertices are adjacent if and only if they do not commute. In addition, the Harary index of a graph Γ_G is the half-sum of the elements in the reciprocal distance of D_{ij} where D_{ij} the distance between vertex i and vertex j. In this paper, the Harary index of the non-commuting graph for dihedral groups is determined and its general formula is developed.

Keywords: Harary index; Non-commuting graph; Dihedral groups.

1. Introduction

A graph Γ is a finite nonempty set V of objects called vertices together with a

*This research is supported by the UTM Fundamental Research Grant Vote No. 20H70.

set E of 2-element subsets of V called edges. The points or nodes are called the vertices, while the lines or links are called edges. Each edge of V is commonly denoted by uv or vu, where u and v are the vertices of a graph Γ. The edge e is said to join vertex u and vertex v if $e = uv$ [2]. The number of edges is the size of Γ and the number of vertices in a graph Γ is the order of Γ. A graph is normally written as $\Gamma = (V, E)$ which indicates that a graph Γ has vertex set V and edge set E [2]. A graph with no loop and multiple edges is called a simple graph.

In this paper, a type of graph associated to group namely the non-commuting graph is considered. In [8], the non-commuting graph which denoted as Γ_G is a graph which consists non-central vertices and two distinct vertices are joined by an edge if and only if they do not commute. The non-commuting graph is also known as the complement of the commuting graph. In group theory, the center of a group G, denoted as $Z(G)$ is the subset of elements in G that commute with every element of G, written as $Z(G) = \{a \in G | ax = xa, \forall x \in G\}$ (see [10]). According to Gallian [3], the conjugacy class of element a is the set $cl = \{xax^{-1} | x \in G\}$ and the element a and b are conjugate if $xax^{-1} = b$ for some $x \in G$. The number of conjugacy classes of a is denoted as $k(G)$.

In addition, the topological indices such as the Wiener index and Harary index have been studied by many researchers. The idea of the Harary index is originally come from the Wiener index in a different form of structure. Some properties of graph theory are used in calculating the topological indices, which are the connectivity and the valencies of the graph. Furthermore, the Harary index of the non-commuting graph for a type of group, namely the dihedral group, is discussed in this paper. It is a group of symmetries of a regular polygon where all elements are determined by rotations and reflections, which denoted as D_{2n}. Only the non-abelian dihedral groups are considered in this research. In [4], it is expressed in a presentation as follows:

$$D_{2n} =< a, b | a^n = b^2 = 1, bab = a^{-1} >,$$

where $n \geq 3$.

Furthermore, the Harary index has been studied by many researchers with different scope and objectives. The Harary index of a graph G, denoted as H has first been introduced by Plavsic et al. [7] in 1993 while the performance of the Wiener index and the Harary index in alkanes in terms of chemical properties are compared in [7]. Most of the researchers focus on finding the upper bound and lower bound for different types of graphs (see [12, 13, 5]).

In [7], the Harary index is defined as a half-sum of the elements in the reciprocal distance matrix D^r which is based on the concept of the reciprocal distance, written as follows:

$$H = \frac{1}{2} \sum_{i=1}^{N} \sum_{j=1}^{N} (D^r)_{ij}.$$

By replacing all matrix elements $(D)_{ij}$ which represent the shortest distances between vertices i and j, in the distance matrix D of a graph G by their reciprocals, the reciprocal distance matrix D^r is obtained [7], written as follows:

$$(D^r)_{ij} = \frac{1}{(D)_{ij}}; \qquad i \neq j.$$

The above definition of the Harary index of a graph G is parallel to the definition of the Wiener index in [11],

$$W = \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N}(D)_{ij},$$

where $(D)_{ij}$ represents the off-diagonal elements of the distance matrix D of G.

According to Plavsic et al. [7], distance matrix D is not suitable for the applications in physics and chemistry due to elements related to the distant sites are associated with large entries. Therefore, the reciprocal distance matrix is introduced instead.

This paper is divided into three sections which are the introduction section, the preliminaries, and the Harary index of the non-commuting graph for dihedral groups.

2. Preliminaries

In determining the general formula for the Harary index of the non-commuting graph associated to dihedral group, some propositions which are the previous results on graph theory and group theory are considered. Hence, this section explains on some propositions that are used to prove the main theorem which involves the number of conjugacy classes, the center of the group G, and the non-commuting graph of D_{2n}.

Proposition 2.1. [9] *Let G be a dihedral group of order $2n$ and $k(G)$ be the number of the conjugacy classes of G. Then,*

$$k(G) = \begin{cases} \frac{n+6}{2} & \text{if} \quad n \text{ is even,} \\ \frac{n+3}{2} & \text{if} \quad n \text{ is odd.} \end{cases}$$

Proposition 2.2. [9] *Let G be a dihedral group, $D_{2n} = \langle a, b | a^n = b^2 = 1, bab = a^{-1}\rangle$ where $n \geq 3, n \in \mathbb{N}$ and $Z(G)$ is the center of G. Then,*

$$Z(G) = \begin{cases} \{1, a^{\frac{n}{2}}\} & \text{if} \quad n \text{ is even,} \\ \{1\} & \text{if} \quad n \text{ is odd.} \end{cases}$$

Proposition 2.3. [1] *Let G be a finite group and Γ_G be the non-commuting graph of G. Then,*

$$2|E(\Gamma_G)| = |G|^2 - k(G)|G|,$$

where $k(G)$ is the number of conjugacy classes of G.

Proposition 2.4. [6] *Let G be the dihedral group, $D_{2n} = \langle a, b | a^n = b^2 = 1, bab = a^{-1} \rangle$ of order $2n$ where $n \geq 3, n \in \mathbb{N}$ and let Γ_G^{NC} be the non-commuting graph of G. Then,*

$$\Gamma_G^{NC} = \begin{cases} K_{\underbrace{1,\,1,\,\ldots\,,\,1}_{n\ times},n-1} & \text{if} \quad n \text{ is odd,} \\ K_{\underbrace{2,\,2,\,\ldots\,,\,2}_{\frac{n}{2}\ times},n-2} & \text{if} \quad n \text{ is even.} \end{cases}$$

Next, in finding the Harary index, some information from the graphs are needed which are the degree of the vertex and distance between two vertices where represent the valencies of the graph and the connectivity of the graph, respectively. Degree of a vertex v is the number of edge at $v, deg(v) = |E(v)|$. Meanwhile, the distance between two vertices, denoted as d_{ij}, is the shortest path from vertex v_i to vertex v_j where $i, j = \{1, 2, ..., n\}$ and n is the total number of vertices of the graph Γ. Some examples on calculating the Harary index of the non-commuting graph for the dihedral groups of certain order are stated in the next section and followed by the development of its general form.

3. The Harary Index of the Non-commuting Graph for D_{2n}

In this section, some calculation of the Harary index of the non-commuting graph for the dihedral groups of order six and eight are shown and the general formula for the Harary index of the non-commuting graph associated to the dihedral group is presented.

Example 3.1. Let G be a dihedral group of order six where $n = 3$. Then, the non-commuting graph of D_6 is a multipartite graph, $K_{1,1,1,2}$ and by Prop. 2.3, $|E(G)| = 9$. Since there are two vertices that are not adjacent to each other, then the Harary index of the non-commuting graph of D_6 is,

$$H(\Gamma_G) = 9 + \frac{1}{2} \times 1 = 9.5$$

Example 3.2. Let G be a dihedral group of order eight where $n = 4$. Then, the non-commuting graph of D_8 is a multipartite graph, $K_{2,2,2}$ and by Prop. 2.3, $|E(G)| = 12$. Since there are three pairs of vertices that are not adjacent to each other, then the Harary index of the non-commuting graph of D_8 is,

$$H(\Gamma_G) = 12 + \frac{1}{2} \times 3 = 13.5$$

Next, the following theorem is the general formula of the Harary index of the non-commuting graph for the dihedral groups.

Theorem 3.3. *Let G be the dihedral group of order $2n$ where $n \geq 3, n \in \mathbb{N}$. Then, the Harary index of the non-commuting graph of G is stated as follows:*

$$H(\Gamma_G) = \begin{cases} \frac{1}{4}[(n-2)(7n-3)+n] & \textit{if} \quad n \textit{ is even}, \\ \frac{1}{4}(n-1)(7n-2) & \textit{if} \quad n \textit{ is odd}. \end{cases}$$

Proof. By the definition of Harary index,

$$H = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}(D^r)_{ij} = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\left[\frac{1}{d_{ij}}\right],$$

where n is the number of vertices of the non-commuting graph.

The Harary index of the non-commuting graph for dihedral group where n is odd is the same as half of the total of all entries in its distance matrix, D^r. The entries of D^r in this case is either 1 if two vertices are connected or $\frac{1}{2}$ if two vertices are not adjacent to each other. Hence, by Props. 2.1 and 2.3, for n is even and $n \geq 4$,

$$\begin{aligned} H &= |E(\Gamma_G)| + \left(\frac{1}{2} \times \frac{n}{2}\right) + \left(\frac{(n-2)(n-3)}{2} \times \frac{1}{2}\right) \\ &= \frac{3}{2}n(n-2) + \frac{n}{4} + \frac{(n-2)(n-3)}{4} \\ &= \frac{1}{4}\left[6n(n-2) + n + (n-2)(n-3)\right] \\ &= \frac{1}{4}\left[(n-2)(7n-3) + n\right], \end{aligned}$$

for n is odd and $n \geq 3$,

$$\begin{aligned} H &= |E(\Gamma_G)| + \frac{1}{2}\left[\frac{(n-1)(n-2)}{2}\right] \\ &= \frac{3n}{2}(n-1) + \frac{1}{4}(n-1)(n-2) \\ &= \frac{1}{2}(n-1)\left(3n + \frac{1}{2}(n-2)\right) \\ &= \frac{1}{2}(n-1)\left(\frac{7}{2}n - 1\right) \\ &= \frac{1}{4}(n-1)(7n-2). \end{aligned}$$

Therefore,

$$H(\Gamma_G) = \begin{cases} \frac{1}{4}[(n-2)(7n-3)+n] & \text{if} \quad n \text{ is even}, \\ \frac{1}{4}(n-1)(7n-2) & \text{if} \quad n \text{ is odd}. \end{cases}$$

■

4. Conclusion

In this paper, the general formula of the Harary index for the non-commuting graph associated to dihedral group in terms of n is found, where n is half of the order of the group G. One of the advantages of this formula is to predict the physicochemical properties of chemical compound in the faster way.

Acknowledgement. The first author is thankful to Universiti Teknologi Malaysia for the Zamalah scholarship.

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