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Energy of Cayley Graphs for Alternating Groups

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Abstract. Let G be a finite group and S be a subset of G where S does not include the identity of G and is inverse closed. A Cayley graph of a finite group G with respect to the subset S is a graph where the vertices are the elements of G and two vertices a and b in G are adjacent if ab^{-1} are in the set S. For a simple graph, the energy of a graph can be determined by its eigenvalues. Let Γ be a simple graph. Then by the summation of the absolute values of the eigenvalues of the adjacency matrix of the graph, its energy can be determined. This paper presents the Cayley graphs of alternating groups with respect to the subset S of valency 1 and 2. From the Cayley graphs, the eigenvalues are computed by using some properties of special graphs and then used to compute their energy.

Keywords: Energy of graph; Cayley graph; Alternating groups.

1. Introduction

The study on the energy of general simple graphs was first defined by Gutman [11] in 1978 inspired from the Huckel Molecular Orbital (HMO) theory proposed in 1930s. The Huckel Molecular Orbital theory has been used by chemists in approximating the energies related with π-electron orbitals in conjugated hydrocarbon. In 1956, the facts that the Huckel method is using the first degree polynomial of the adjacency matrix of a certain graph was realized by Gunthard and Primas [14].

After several years, Gutman has generalized the definition of the energy for all finite simple graphs which is the summation of the positive values of the eigenvalues of the adjacency matrix of the graphs. An adjacency matrix of a graph denoted as $A(\Gamma)$ is a square matrix where the rows and columns are indexed by the vertices of the graph Γ. The (i, j)-entry of the matrix is entered as 0 if two distinct vertices are not adjacent to each other and is entered as 1 if the vertices are adjacent to each other [4]. In 2004, Bapat and Pati [5] proved that the energy of a graph is never an odd integer while the properties that the energy of a graph is never the square root of an odd integer has been proven by Pirzada and Gutman [17] in 2008. The studies on the properties of the energy of graphs have also been studied by many researchers through the years (see [12, 13]).

Meanwhile, a Cayley graph of a finite group G with respect to a subset S of G, denoted as $Cay(G, S)$ is a graph where the vertices are the elements of G and two distinct vertices x and y are joined by an edge if $x = sy$ for s in S. The subset S is referred to as the inverse-closed set with no identity element of G [6]. The study on Cayley graphs was initiated by A. Cayley in 1878 to explain the idea of abstract groups described by a set of generators. The theory has advanced into a significant branch in algebraic graph theory and there are many problems regarding Cayley graphs that have been studied by graph theorists.

In addition, the theory of Cayley graphs relates with many problems in algebra such as the classification, the isomorphism and the enumeration of Cayley graphs [16]. In 1993, Lakshmivarahan et al. [15] have analyzed the symmetries in the interconnection networks of a variety of Cayley graphs of permutation groups. The types of symmetry analyzed consist of vertex and edge transitivity, distance regularity and distance transitivity. Meanwhile in 2000, Friedman [8] has shown in his study that among all sets of $n-1$ transpositions which generate the symmetric groups, the Cayley graph associated to set $S = \{(1, n), (2, n), ..., (n-1, n)\}$ has the highest eigenvalues.

Meanwhile, Konstantinova [18] in 2008, has surveyed the historical changes of some problems on Cayley graphs such as the Hamiltonicity and diameter problems. The author also included various uses of Cayley graphs in solving combinatorial, graph theoretical and applied problems. Further in 2012, Adiga and Ariamanesh [2] have determined the number of undirected Cayley graphs of symmetric group and alternating groups up to isomorphism.

Another interesting study on graphs is on the spectrum of the graphs. The first mathematician who considered the spectrum of Cayley graphs was Babai [3] in 1979 by using algebraic graph theory techniques. This exciting topic has been receiving increased number of attention from many researchers. In 1981, Diaconis et al. [7] have introduced the computations of the spectrum of Cayley graphs via the character table of the related groups. Following that, in 2012, Krakovski and Mohar [19] have presented their findings on the spectrum of Cayley graphs on the symmetric group generated by the set of transpositions where the spectrum contains all integers from $-(n-1)$ to $n-1$ (except 0 if $n = 2$ or $n = 3$).

Recently in 2016, Abdollahi et al. [1] have presented on the groups of whose undirected Cayley graphs are determined by their spectrum. Latest in 2018, Ghorbani and Nowroozi in their papers [9, 10] have presented their studies on the spectrum of Cayley graphs related to groups of order pqr, $2pq$ and $3pq$ where $p > q > r > 2$ are prime numbers in terms of their character tables.

Therefore, this paper intends to reveal the energy of the Cayley graphs for alternating groups, A_n with respect to the subset S of valency 1 and 2. The methodology consists of constructing the Cayley graphs with respect to the subsets, finding their eigenvalues and finally computing the energy of the respected Cayley graphs.

This paper is structured as follows: in Section 1, some introductions and previous studies on the topics are explained. In Section 2, the results on the Cayley graphs of the alternating groups are presented in term of theorems and the energy of the Cayley graphs are presented in several theorems in Section 3.

2. The Cayley Graphs of Alternating Groups

In this section, we present our results, namely the Cayley graphs of the alternating groups A_n with respect to the subsets S of valency one and two. The Cayley graphs are presented in the following theorems.

Theorem 2.1. *Let A_n be the alternating group on n symbols where $n \geq 3$. Then the Cayley graph of A_n with respect to a subset S, denoted as $Cay(A_n, S)$, where $S \subset A_n$, $e \notin S$, $S^{-1} = S$ and $|S| = 1$ is $(\frac{n!}{4})K_2$.*

Proof. Assume that $S = \{a\}$, where $a \in A_n$. Then we have $a^{-1} = a$ or equivalently $a^2 = e$. If $g \in A_n$ is an arbitrary vertex, then we can see that g is adjacent to ag, because $g(ag)^{-1} = gg^{-1}a^{-1} = a^{-1} = a \in S$. Since $Cay(A_n, S)$ is $1-$regular, g is adjacent to only ag and therefore all elements of A_n will split into two disjoint subsets $\{g_1, g_1, \ldots, g_k\}$ and $\{ag_1, ag_2, \ldots, ag_k\}$, where $k = \frac{n!}{4}$ and g_i is adjacent to ag_i for $i = 1, 2, \ldots, k$. Hence, $Cay(A_n, S) = (\frac{n!}{4})K_2$. ■

Theorem 2.2. *Let A_n be the alternating group and $n \geq 4$. Let $S \subset A_n$, such that $|S| = 2$, $S = \{a, a^{-1}\}$ where $o(a) = m$ and $m \neq 2$. Then the Cayley graph of A_n with respect to a subset S, denoted as $Cay(A_n, S)$ is $(\frac{n!}{2m})C_m$.*

Proof. Suppose that $S = \{a, a^{-1}\}$ and $o(a) = m$ where $m \neq 2$. Thus $a = a^{-1}$ and elements $e, a, a^2, \ldots, a^{m-1}$ are all distinct. It is easy to see that we have the following cycle of length m:

$$e \longleftrightarrow a \longleftrightarrow a^2 \longleftrightarrow a^3 \longleftrightarrow \cdots \longleftrightarrow a^{m-2} \longleftrightarrow a^{m-1} \longleftrightarrow e.$$

Note that $ae^{-1} = a \in S$, $a^{i+1}(a^i)^{-1} = a^{i+1}a^{-i} = a \in S$ and $a^{m-1}e^{-1} = a^{m-1} = a^{-1} \in S$. Now, for any element $g \in A_n$ and $g \neq e, a, a^2, \ldots, a^{m-1}$, we have the following cycle of length m:

$$g \longleftrightarrow ag \longleftrightarrow a^2g \longleftrightarrow a^3g \longleftrightarrow \cdots \longleftrightarrow a^{m-2}g \longleftrightarrow a^{m-1}g \longleftrightarrow g.$$

Since $(a^{i+1}g)(a^ig)^{-1} = a^{i+1}gg^{-1}a^{-i} = a \in S$ for $i = 0, 1, 2, \ldots, m-1$. Moreover, $(a^{m-1}g)g^{-1} = a^{m-1} = a^{-1} \in S$. Since $Cay(A_n, S)$ is $2-$regular, we should have $Cay(A_n, S) = (\frac{n!}{2m})C_m$, where $m > 2$ and $n \geq 4$.

Note that choosing the cycles C_m can be found as the following:

Suppose that $S = \{a, a^{-1}\}$ and $o(a) = m \neq 2$. Then we have a subgroup $H =< a >= \{e, a, a^2, \ldots, a^{m-1}\}$. Since $[A_n, H] = \frac{|A_n|}{|H|} = \frac{\frac{n!}{2}}{m} = \frac{n!}{2m}$, there are distinct elements $e = g_1, g_2, \ldots, g_t$ such that $A_n = \bigcup_{i=1}^{t} Hg_i$, where g_iH's are all disjoint and for $i \neq 1$, $g_i \notin H$. Now, we can see that we have t cycles of length m as the following:

$$\begin{aligned} g_1 = e &: e \longleftrightarrow a \longleftrightarrow a^2 \longleftrightarrow \cdots \longleftrightarrow a^{m-2} \longleftrightarrow a^{m-1} \longleftrightarrow e \\ g_2 &: g_2 \longleftrightarrow ag_2 \longleftrightarrow a^2g_2 \longleftrightarrow \cdots \longleftrightarrow a^{m-2}g_2 \longleftrightarrow a^{m-1}g_2 \longleftrightarrow g_2 \\ g_3 &: g_3 \longleftrightarrow ag_3 \longleftrightarrow a^2g_3 \longleftrightarrow \cdots \longleftrightarrow a^{m-2}g_3 \longleftrightarrow a^{m-1}g_3 \longleftrightarrow g_3 \\ g_t &: g_t \longleftrightarrow ag_t \longleftrightarrow a^2g_t \longleftrightarrow \cdots \longleftrightarrow a^{m-2}g_t \longleftrightarrow a^{m-1}g_t \longleftrightarrow g_t \end{aligned}$$

Note that $Hg_i = \{g_i, ag_i, a^2g_i, \ldots, a^{m-1}g_i\}$, $i = 0, 1, 2, \cdots t$. Hence, $Cay(A_n, S) = tC_m = (\frac{n!}{2m})C_m$. ■

Theorem 2.3. *Let A_n be the alternating group and $n \geq 4$. Let $S = \{a, b\} \subset A_n$, such that $a \neq b$, $o(a) = o(b) = 2$. Then $Cay(A_n, S)$ is $(\frac{n!}{4m})C_{2m}$, where $m = o(ab)$.*

Proof. Assume that $o(ab) = m$. Then we will have the following cycle of length $2m$:

$$\begin{aligned} &e \longleftrightarrow b \longleftrightarrow ab \longleftrightarrow b(ab) \longleftrightarrow (ab)^2 \longleftrightarrow b(ab)^2 \longleftrightarrow (ab)^3 \longleftrightarrow b(ab)^3 \longleftrightarrow \cdots \\ &\cdots \longleftrightarrow (ab)^{m-2} \longleftrightarrow b(ab)^{m-2} \longleftrightarrow (ab)^{m-1} \longleftrightarrow b(ab)^{m-1} \longleftrightarrow (ab)^m = e \end{aligned}$$

Put $H = \{e, b, ab, b(ab), (ab)^2, b(ab)^2, \ldots, b(ab)^{m-1}\}$. Then one can prove that H is a subgroup of A_n of order $2m$ (Hint: Prove that $H =< b, (ab) >$).

Since $[A_n : H] = \frac{|A_n|}{|H|} = \frac{n!/2}{2m} = \frac{n!}{4m} = t$, we have $A_n = \bigcup_{i=1}^{t} Hg_i$, where $Hg_i = He = Hg_1 = \cdots = Hg_t$ are distinct right cosets of H in A_n. Thus, $g_i \notin H$, for $i = 2, 3, \ldots, t$. Now we can see that the following is a cycle of length $2m$ for all $i = 0, 1, 2, \ldots, t$:

$$\begin{aligned} &g_i \longleftrightarrow bg_i \longleftrightarrow (ab)g_i \longleftrightarrow b(ab)g_i \longleftrightarrow (ab)^2g_i \longleftrightarrow \cdots \\ &\cdots \longleftrightarrow b(ab)^{m-2}g_i \longleftrightarrow (ab)^{m-1}g_i \longleftrightarrow b(ab)^{m-1}g_i \longleftrightarrow (ab)^mg_i = g_i \end{aligned}$$

Hence, $Cay(A_n, S) = tC_m = (\frac{n!}{4m})C_{2m}$. ■

3. The Energy of the Cayley Graphs of Alternating Groups

In this section, we present our main results, namely the energy of the Cayley graphs of the alternating groups A_n with respect to the subsets S of valency one and two. The energy of the Cayley graphs are presented in the following theorems.

Theorem 3.1. *Let A_n be the alternating group on n symbols where $n \geq 3$ and $Cay(A_n, S)$ be the Cayley graph of A_n with respect to a subset S, where $S \subset A_n$, $e \notin S$, $S^{-1} = S$ and $|S| = 1$, Then the energy of the Cayley graph, denoted as $\varepsilon(Cay(A_n, S))$ is $\frac{n!}{2}$.*

Proof. Consider the alternating group of order $\frac{n!}{2}$, A_n. By Theorem 2.1, $Cay(A_n, S) = (\frac{n!}{4})K_2$ where $|S| = 1$. Since the adjacency spectrum of a complete graph, K_n is $\{(n-1)^1, (-1)^{n-1}\}$. Then the eigenvalues of $(\frac{n!}{4})K_2$ are $\lambda_i = \frac{n!}{4}\{(1)^1, (-1)^1\}$ which also can be written as $\lambda = \pm 1$ with multiplicity $\frac{n!}{4}$. Therefore, the energy of the Cayley graph, denoted as $\varepsilon(Cay(A_4, S)) = \frac{n!}{4}(1) + \frac{n!}{4}(|-1|) = 2(\frac{n!}{4}) = \frac{n!}{2}$. ■

Theorem 3.2. *Let A_n be the alternating group and $n \geq 4$. Let $S \subset A_n$, such that $|S| = 2$, $S = \{a, a^{-1}\}$ where $o(a) = m$ and $m \neq 2$. Then the energy of the Cayley graph of A_n with respect to a subset S, denoted as $\varepsilon(Cay(A_n, S))$ is $\sum_{j=0}^{m-1} \frac{n!}{m} \left|\cos\left(\frac{2j\pi}{m}\right)\right|$.*

Proof. Consider the alternating group of order $\frac{n!}{2}$, A_n. By Theorem 2.2, $Cay(A_n, S) = (\frac{n!}{2m})C_m$ where $S = \{a, a^{-1}\}$ such that $o(a) = m$ and $m \neq 2$. Since the adjacency spectrum of a cycle graph, C_n is $\{2cos(\frac{2\pi j}{n})\}$ for $j = \{0, 1, \dots, n-1\}$. Then the eigenvalues of $(\frac{n!}{2m})C_m$ are $\lambda_i = \frac{n!}{2m}\{2cos(\frac{2\pi j}{m})\}$ for $j = \{0, 1, \dots, m-1\}$ which also can be written as $\{2, 2cos(\frac{2\pi}{m}), 2cos(\frac{4\pi}{m}), \cdots, 2cos(\frac{2(m-1)\pi}{m})\}$ with multiplicity $\frac{n!}{2m}$. Therefore, the energy of the Cayley graph, denoted as $\varepsilon(Cay(A_n, S)) = \sum_{j=0}^{m-1} \frac{n!}{m} \left|\cos\left(\frac{2j\pi}{m}\right)\right|$. ■

Theorem 3.3. *Let A_n be the alternating group and $n \geq 4$. Let $S = \{a, b\} \subset A_n$, such that $a \neq b$, $o(a) = o(b) = 2$. Then the energy of the Cayley graph of A_n with respect to a subset S, denoted as $Cay(A_n, S)$ is $\sum_{j=0}^{2m-1} \frac{n!}{2m} \left|\cos\left(\frac{j\pi}{m}\right)\right|$.*

Proof. Consider the alternating group of order $\frac{n!}{2}$, A_n. By Theorem 2.3, $Cay(A_n, S) = (\frac{n!}{4m})C_{2m}$ where $S = \{a, b\} \subset A_n$, such that $a \neq b$, $o(a) = o(b) = 2$. Since the adjacency spectrum of a cycle graph, C_n is $\{2cos(\frac{2\pi j}{n})\}$ for $j = \{0, 1, \dots, n-1\}$. Then the eigenvalues of $(\frac{n!}{4m})C_{2m}$

are $\lambda_i = \frac{n!}{4m}\{2cos(\frac{2\pi j}{2m})\}$ for $j = \{0, 1, \ldots, 2m-1\}$ which also can be written as $\{2, 2cos(\frac{\pi}{m}), 2cos(\frac{2\pi}{m}), 2cos(\frac{3\pi}{m}), \cdots, 2cos(\frac{(2m-1)\pi}{m})\}$ with multiplicity $\frac{n!}{4m}$. Therefore, the energy of the Cayley graph, denoted as $\varepsilon(Cay(A_n, S)) = \sum_{j=0}^{2\,m-1} \frac{n!}{2m} \left|\cos\left(\frac{j\pi}{m}\right)\right|$. ■

4. Conclusion

In this paper, the energy of the Cayley graphs with respect to subsets S of valency one and two of alternating groups are found. The results of the respected Cayley graphs are as presented in the following main theorems.

Theorem 4.1. *Let A_n be the alternating group and S be a subset A_n, such that $e \notin S$ and $S^{-1} = S$. Then*

(i) *if $n \geq 3$ and $|S| = 1$, then $Cay(A_n, S) = (\frac{n!}{4})K_2$.*

(ii) *if $n \geq 4$, and $|S| = 2$, where $S = \{a, b\}$,*

  (a) *if $b = a^{-1}$, then $Cay(A_n, S) = (\frac{n!}{2m})C_m$, for which $o(a) = m \neq 2$.*

  (b) *if $b \neq a^{-1}$ and o(a)=o(b)=2, then $Cay(A_n, S) = (\frac{n!}{4m})C_{2m}$, where $m = o(ab)$.*

Theorem 4.2. *Let A_n be the alternating group and S be a subset A_n, such that $e \notin S$ and $S^{-1} = S$. Then*

(i) *if $n \geq 3$ and $|S| = 1$, then $\varepsilon(Cay(A_n, S)) = \frac{n!}{2}$.*

(ii) *if $n \geq 4$, and $|S| = 2$, where $S = \{a, b\}$,*

  (a) *if $b = a^{-1}$, then $\varepsilon(Cay(A_n, S)) = \sum_{j=0}^{m-1} \frac{n!}{m} \left|\cos\left(\frac{2j\pi}{m}\right)\right|$, for which $o(a) = m \neq 2$.*

  (b) *if $b \neq a^{-1}$ and o(a)=o(b)=2, then $\varepsilon(Cay(A_n, S)) = \sum_{j=0}^{2\,m-1} \frac{n!}{2m} \left|\cos\left(\frac{j\pi}{m}\right)\right|$, where $m = o(ab)$.*

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