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# The Independence and Clique Polynomials of the Center Graphs of Some Finite Groups*

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**Abstract.** The independence polynomial and the clique polynomial are the graph polynomials that are used to describe the combinatorial information of graphs, including the graphs related to group theory. An independence polynomial of a graph is the polynomial in which its coefficients are the number of independent sets in the graph. The independent set of a graph is a set of vertices that are not adjacent. A clique polynomial of a graph is the polynomial containing coefficients that represent the number of cliques in the graph. The clique of a graph is a set of vertices that are adjacent to each other in the graph. Meanwhile, the center graph of a group $G$ is a graph in which the vertices are all the elements of $G$ and two distinct vertices $a, b$ are adjacent if an only if $ab$ is in the center of $G$. In this research, the independence polynomial and the clique polynomial are established for the center graphs of three finite non-abelian groups, namely the dihedral group, the generalized quarternion group and the quasidihedral group.

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**Keywords:** Independence polynomial; Clique polynomial; Center graph; Finite group.

## 1. Introduction

A simple graph, $\Gamma = (V, E)$ consists of a set of vertices, $V$ and a set of edges, $E$. For vertices $u, v \in V$, $u$ and $v$ are adjacent if they are connected by an edge, $e = (u, v) \in E$. A complete graph, $K_n$, is a common type of graph in which it contains $n$ vertices and each pair of distinct vertices is connected by an edge [14]. Meanwhile, a compete bipartite graph, $K_{m,n}$ is a graph that has its vertex set partitioned into two subsets of $m$ and $n$ vertices, respectively, in which there is an edge between two vertices if and only if the first vertex is in the first subset and the second vertex is in the second subset [14]. Let $\Gamma_1 = (V_1, E_1)$ and $\Gamma_2 = (V_2, E_2)$, then the union of $\Gamma_1$ and $\Gamma_2$, denoted by $\Gamma_1 \cup \Gamma_2$, is the graph $\Gamma = (V, E)$, where $V = V_1 \cup V_2$ and $E = E_1 \cup E_2$ [2].

Graph polynomials, such as the independence polynomials and the clique polynomials of graphs have been studied by many researchers to analyze the properties of graphs. An independence polynomial of a graph is a polynomial containing coefficients that represent the number of independent sets in the graph [9]. Meanwhile, a clique polynomial of a graph is a polynomial containing coefficients that represent the number of cliques in the graph [9].

Many researches have been extensively done to associate the graphs in graph theory to the groups in group theory. Some examples of the graphs associated to groups include the conjugate graph [5], the commuting graph [13] and the noncommuting graph [1]. Another type of graph associated to group is the center graph that is being considered in this research. A center graph of a group $G$, $\Gamma_G^z$ is defined as the graph whose vertex set, $V(\Gamma_G^z)$ contains all elements of $G$, that is $|V(\Gamma_G^z)| = |G|$ in which two distinct vertices $a$ and $b$ are adjacent if and only if $ab$ is in $Z(G)$ [3]. In group theory, $Z(G)$ is the center of a group $G$, in which it is the subset of the elements of $G$ that commute with every element in $G$ and can be denoted as $\{a \in G | ag = ga \text{ for all } g \in G\}$ [8].

Graph polynomials are commonly found for the graphs in graph theory, such as the cycle graph, the complete graph and the complete bipartite graph. In this research, we focus in establishing two types of graph polynomials for the graph of groups. The independence polynomial and the clique polynomial are determined for the center graphs of three types of finite nonabelian groups. The groups involved are the dihedral group of order $2n$, the generalized quaternion group of order $4n$ and the quasidiheral group of order $2^n$, that can be expressed, respectively, in the group presentations as follows:

(i) $D_{2n} = \langle a, b : a^n = b^2 = 1, bab = a^{-1} \rangle$, $n \geq 3$, $n \in \mathbb{N}$ [15].
(ii) $Q_{4n} = \langle a, b : a^{2n} = b^4 = 1, a^n = b^2, b^{-1}ab = a^{-1} \rangle$, $n \geq 2$, $n \in \mathbb{N}$ [10].
(iii) $QD_{2^n} = \langle a, b : a^{2^{(n-1)}} = b^2 = 1, bab^{-1} = a^{2^{(n-2)}-1} \rangle$, $n \geq 4$, $n \in \mathbb{N}$ [4].

## 2. Preliminaries

This section presents some preliminaries that are used throughout this research. First, some basic concepts related to the independence polynomial are given. An independent set of a graph $\Gamma$ is the set of vertices in $\Gamma$ such that no two distinct vertices are adjacent. The independence number of a graph $\Gamma$, denoted by $\alpha(\Gamma)$ is the maximum number of vertices in an independent set of the graph [14]. Hoedi and Li [9] have defined that the independence polynomial of a graph $\Gamma$ is the polynomial whose coefficient on $x^k$ is given by the number of independent sets of size $k$ in $\Gamma$. This is denoted by $I(\Gamma; x) = \sum_{k=0}^{\alpha(\Gamma)} a_k x^k$, where $a_k$ is the number of independent sets of size $k$ in $\Gamma$ and $\alpha(\Gamma)$ is the independence number of $\Gamma$. Previous researchers have obtained some properties related to the independence polynomial such as stated in the following propositions.

**Proposition 2.1.** [9] *Let $\Gamma_1$ and $\Gamma_2$ be two vertex-disjoint graphs. Then, $I(\Gamma_1 \cup \Gamma_2; x) = I(\Gamma_1; x) \cdot I(\Gamma_2; x)$.*

**Proposition 2.2.** [6] *The independence polynomial of the union of $m$ complete graphs each of $n_i$ vertices is $I\left(\bigcup_{i=1}^{m} K_{n_i}; x\right) = \prod_{i=1}^{m}(1 + n_i x)$.*

**Proposition 2.3.** [6] *The independence polynomial of a complete bipartite graph of $n_1 + n_2$ vertices is $I\left(K_{n_1,n_2}; x\right) = (1 + x)^{n_1} + (1 + x)^{n_2} - 1$.*

Next, some basic concepts related to the clique polynomials are stated. A clique of a graph $\Gamma$ is the set of vertices in which every vertex is adjacent to every other vertices [2]. Pardalos and Xue [12] have stated that $S$ is a clique of a graph if and only if $S$ is an independent set of the complement of the graph. The clique of the graph is called a maximal clique if it is not a subset of a larger clique in the graph [2, 12]. Eventhough there are some researchers referred the maximal clique as clique, however, in this research, clique is defined only as the complete subgraph. Furthermore, the maximal clique is different from the maximum clique of a graph, that is the clique with the biggest cardinality in the graph [12]. The clique number of a graph $\Gamma$, denoted by $\omega(\Gamma)$ is the size of the maximum clique [2].

The clique polynomial of a graph $\Gamma$ has been defined by Hoede and Li [9] in 1994 in which it is the polynomial whose coefficient on $x^k$ is given by the number of cliques of size $k$ in $\Gamma$. This is denoted by $C(\Gamma; x) = \sum_{k=0}^{\omega(\Gamma)} c_k x^k$, where $c_k$ is the number of cliques of size $k$ in $\Gamma$ and $\omega(\Gamma)$ is the clique number of $\Gamma$. Some properties of the clique polynomials are given in the following propositions.

**Proposition 2.4.** [9] *Let $\Gamma_1$ and $\Gamma_2$ be two vertex-disjoint graphs. Then, $C(\Gamma_1 \cup \Gamma_2; x) = C(\Gamma_1; x) + C(\Gamma_2; x) - 1$.*

**Proposition 2.5.** [7] *The clique polynomial of the union of $m$ complete graphs each of $n$ vertices is* $C\left(\bigcup_{i}^{m} K_n; x\right) = m(1+x)^n - (m-1)$.

**Proposition 2.6.** [7] *The clique polynomial of a complete bipartite graph of $n_1+n_2$ vertices is* $C\left(K_{n_1,n_2}; x\right) = (1+n_1x)(1+n_2x)$.

Finally, the following concepts on the center graph of groups are stated. The element $a$ in a group $G$ can be of two types, either $a^2 \in Z(G)$ or $a^2 \notin Z(G)$. Therefore, $G = X \cup Y$ in which $X = \{g \in G : g^2 \in Z(G)\}$ and $Y = \{g \in G : g^2 \notin Z(G)\}$ [3]. Balakrishnan et al. [3] have obtained the center graph of the nonabelian groups as stated in the following proposition.

**Proposition 2.7.** [3] *Let $G$ be a nonabelian group. Then the center graph of $G$ is*

$$\Gamma_G^z = \bigcup_{i=1}^{\frac{|X|}{|Z(G)|}} K_{|Z(G)|} \cup \bigcup_{i=1}^{\frac{|Y|}{2|Z(G)|}} K_{|Z(G)|,|Z(G)|}.$$

For the generalized quaternion group and the quasidihedral group, the general form of the center graphs are not yet constructed by other researchers and therefore will be presented in the next section. However, Karimi et al. have established the general form of the center graph for the dihedral group as stated in the following lemma.

**Lemma 2.8.** [11] *Let $D_{2n}$ be a dihedral group of order $2n$, where $n \geq 3$, $n \in \mathbb{N}$. The center graph of $D_{2n}$ is*

$$\Gamma_{D_{2n}}^z = \begin{cases} \left(\bigcup_{i=1}^{n+1} K_1\right) \cup \left(\bigcup_{i=1}^{\frac{n-1}{2}} K_{1,1}\right) & \text{if } n \text{ is odd,} \\ \left(\bigcup_{i=1}^{\frac{n}{2}+2} K_2\right) \cup \left(\bigcup_{i=1}^{\frac{n}{4}-1} K_{2,2}\right) & \text{if } \frac{n}{2} \text{ is even,} \\ \left(\bigcup_{i=1}^{\frac{n}{2}+1} K_2\right) \cup \left(\bigcup_{i=1}^{\frac{n-2}{4}} K_{2,2}\right) & \text{if } \frac{n}{2} \text{ is odd.} \end{cases}$$

## 3. Main Results

This section contains the results of this research. The general form of the center graphs are first being determined for generalized quaternion and quasidihedral groups. Then, the independence polynomial and the clique polynomial are established for the dihedral group, the generalized quaternion group and the quasidihedral group.

**Lemma 3.1.** *Let $Q_{4n}$ be the generalized quaternion group of order $4n$, $n \geq 2$, $n \in \mathbb{N}$. The center graph of $Q_{4n}$ can be presented as follows:*

$$\Gamma^z_{Q_{4n}} = \begin{cases} \left(\bigcup_{i=1}^{n+1} K_2\right) \cup \left(\bigcup_{i=1}^{\frac{n-1}{2}} K_{2,2}\right) & \text{if } n \text{ is odd,} \\ \left(\bigcup_{i=1}^{n+2} K_2\right) \cup \left(\bigcup_{i=1}^{\frac{n-2}{2}} K_{2,2}\right) & \text{if } n \text{ is even.} \end{cases}$$

*Proof.* Suppose that $Q_{4n}$ is a generalized quaternion group of order $4n$, $n \geq 2$, $n \in \mathbb{N}$, with $\Gamma^z_{Q_{4n}}$ as its center graph. The center of $Q_{4n}$ is $Z(Q_{4n}) = \{1, a^n\}$. When $n$ is odd, set $X$ of $Q_{4n}$ can be expressed as $X = \{1, a^n, a^i b\}, 0 \leq i \leq 2n-1$. It follows that $|X| = 2 + 2n$. Since $|Q_{4n}| = 4n$ and $Y = G \setminus X$, therefore $|Y| = |G| - |X| = 4n - (2 + 2n) = 2n - 2$. And by Proposition 2.8,

$$\Gamma^z_{Q_{4n}} = \bigcup_{i=1}^{\frac{2+2n}{2}} K_2 \cup \bigcup_{i=1}^{\frac{2n-2}{2(2)}} K_{2,2} = \bigcup_{i=1}^{n+1} K_2 \cup \bigcup_{i=1}^{\frac{n-1}{2}} K_{2,2}.$$

Next, when $n$ is even, set $X$ of $Q_{4n}$ can be expressed as $X = \{1, a^{\frac{n}{2}}, a^n, a^{\frac{3n}{2}}, a^i b\}$, $0 \leq i \leq 2n - 1$. It follows that $|X| = 4 + 2n$. Since $|Q_{4n}| = 4n$ and $Y = G \setminus X$, therefore $|Y| = |G| - |X| = 4n - (4 + 2n) = 2n - 4$. And by Proposition 2.8,

$$\Gamma^z_{Q_{4n}} = \bigcup_{i=1}^{\frac{4+2n}{2}} K_2 \cup \bigcup_{i=1}^{\frac{2n-4}{2(2)}} K_{2,2} = \bigcup_{i=1}^{n+2} K_2 \cup \bigcup_{i=1}^{\frac{n-2}{2}} K_{2,2}.$$

■

**Lemma 3.2.** *Let $QD_{2^n}$ be the quasidihedral group of order $2^n$, $n \geq 4$, $n \in \mathbb{N}$. The center graph of $QD_{2^n}$ can be presented as follows:*

$$\Gamma^z_{QD_{2^n}} = \left(\bigcup_{i=1}^{2^{(n-2)}+2} K_2\right) \cup \left(\bigcup_{i=1}^{2^{(n-3)}-1} K_{2,2}\right).$$

*Proof.* Suppose that $QD_{2^n}$ is a quasidihedral group of order $2^n$, $n \geq 4$, $n \in \mathbb{N}$, with center $Z(QD_{2^n}) = \{1, a^{2^{(n-2)}}\}$ and $\Gamma^z_{QD_{2^n}}$ as its center graph. The set $X$ of $QD_{2^n}$ is $X = \{1, a^{2^{(n-3)}}, a^{2^{(n-2)}}, a^{-2^{(n-3)}}, a^i b\}, 0 \leq i \leq 2^{(n-1)} - 1$. It follows that $|X| = 4 + 2^{(n-1)}$. Since $|QD_{2^n}| = 2^n$ and $Y = G \setminus X$, therefore $|Y| = |G| - |X| = 2^n - (4 + 2^{(n-1)}) = 2^{(n-1)} - 4$. And by Proposition 2.8,

$$\Gamma^z_{QD_{2^n}} = \bigcup_{i=1}^{\frac{4+2^{(n-1)}}{2}} K_2 \cup \bigcup_{i=1}^{\frac{2^{(n-1)}-4}{2(2)}} K_{2,2} = \bigcup_{i=1}^{2^{(n-2)}+2} K_2 \cup \bigcup_{i=1}^{2^{(n-3)}-1} K_{2,2}.$$

■

Next, the general forms for the independence polynomial and the clique polynomial of the center graph for the dihedral group are stated in the following theorems.

**Theorem 3.3.** *Let $D_{2n}$ be a dihedral group of order $2n$, where $n \geq 3$, $n \in \mathbb{N}$. The independence polynomial of the center graph of $D_{2n}$ is*

$$I(\Gamma^z_{D_{2n}};x) = \begin{cases} (1+x)^{n+1}(1+2x)^{\frac{n-1}{2}} & \text{if } n \text{ is odd,} \\ (1+2x)^{\frac{n}{2}+2}(1+4x+2x^2)^{\frac{n}{4}-1} & \text{if } \frac{n}{2} \text{ is even,} \\ (1+2x)^{\frac{n}{2}+1}(1+4x+2x^2)^{\frac{n-2}{4}} & \text{if } \frac{n}{2} \text{ is odd.} \end{cases}$$

*Proof.* Suppose that $D_{2n}$ is the dihedral group of order $2n$, where $n \geq 3$, $n \in \mathbb{N}$, and $\Gamma^z_{D_{2n}}$ is its center graph. When $n$ is odd, from Lemma 2.9, $\Gamma^z_{D_{2n}} = \left(\bigcup_{i=1}^{n+1} K_1\right) \cup \left(\bigcup_{i=1}^{\frac{n-1}{2}} K_{1,1}\right)$. Using Propositions 2.1, 2.2 and 2.3, the independence polynomial of $\Gamma^z_{D_{2n}}$ is computed:

$$I(\Gamma^z_{D_{2n}};x) = (1+x)^{n+1}\left[2(1+x)-1\right]^{\frac{n-1}{2}} = (1+x)^{n+1}(1+2x)^{\frac{n-1}{2}}.$$

When $\frac{n}{2}$ is even, $\Gamma^z_{D_{2n}} = \left(\bigcup_{i=1}^{\frac{n}{2}+2} K_2\right) \cup \left(\bigcup_{i=1}^{\frac{n}{4}-1} K_{2,2}\right)$. Using Props. 2.1, 2.2 and 2.3, the independence polynomial of $\Gamma^z_{D_{2n}}$ is computed as follows:

$$\begin{aligned} I(\Gamma^z_{D_{2n}};x) &= (1+2x)^{\frac{n}{2}+2}\left[2(1+x)^2-1\right]^{\frac{n}{4}-1} \\ &= (1+2x)^{\frac{n}{2}+2}\left[2(1+2x+x^2)-1\right]^{\frac{n}{4}-1} \\ &= (1+2x)^{\frac{n}{2}+2}\left(1+4x+x^2]\right)^{\frac{n}{4}-1} \end{aligned}$$

When $\frac{n}{2}$ is odd, $\Gamma^z_{D_{2n}} = \left(\bigcup_{i=1}^{\frac{n}{2}+1} K_2\right) \cup \left(\bigcup_{i=1}^{\frac{n-2}{4}} K_{2,2}\right)$.Using Propositions 2.1, 2.2 and 2.3, the independence polynomial of $\Gamma^z_{D_{2n}}$ is computed as follows:

$$\begin{aligned} I(\Gamma^z_{D_{2n}};x) &= (1+2x)^{\frac{n}{2}+1}\left[2(1+x)^2-1\right]^{\frac{n-2}{4}} \\ &= (1+2x)^{\frac{n}{2}+1}\left[2(1+2x+x^2)-1\right]^{\frac{n-2}{4}} \\ &= (1+2x)^{\frac{n}{2}+1}\left(1+4x+x^2]\right)^{\frac{n-2}{4}}. \end{aligned}$$

■

**Theorem 3.4.** *Let $D_{2n}$ be a dihedral group of order $2n$, where $n \geq 3$, $n \in \mathbb{N}$. The clique polynomial of the center graph of $D_{2n}$ is*

$$C(\Gamma^z_{D_{2n}};x) = \begin{cases} 1+2nx+\left(\frac{n-1}{2}\right)x^2 & \text{if } n \text{ is odd,} \\ 1+2nx+\left(\frac{3n}{2}-2\right)x^2 & \text{if } \frac{n}{2} \text{ is even,} \\ 1+2nx+\left(\frac{3n-2}{2}\right)x^2 & \text{if } \frac{n}{2} \text{ is odd.} \end{cases}$$

*Proof.* Suppose that $D_{2n}$ is the dihedral group of order $2n$, where $n \geq 3$, $n \in \mathbb{N}$, and $\Gamma^z_{D_{2n}}$ is its center graph. When $n$ is odd, from Lemma 2.9, $\Gamma^z_{D_{2n}} = \left(\bigcup_{i=1}^{n+1} K_1\right) \cup \left(\bigcup_{i=1}^{\frac{n-1}{2}} K_{1,1}\right)$. Using Props. 2.4, 2.5 and 2.6, the clique polynomial of $\Gamma^z_{D_{2n}}$ is computed as follows:

$$\begin{aligned} C(\Gamma^z_{D_{2n}};x) &= (n+1)+(n+1)x+\left(\tfrac{n-1}{2}\right)+(n-1)x+\left(\tfrac{n-1}{2}\right)x^2-\tfrac{3n-1}{2} \\ &= \left(n+1+\tfrac{n-1}{2}-\tfrac{3n-1}{2}\right)+\big((n+1)+(n-1)\big)x+\left(\tfrac{n-1}{2}\right)x^2 \\ &= \tfrac{2n+2+n-1-3n+1}{2}+2nx+\left(\tfrac{n-1}{2}\right)x^2 \\ &= 1+2nx+\left(\tfrac{n-1}{2}\right)x^2. \end{aligned}$$

When $\frac{n}{2}$ is even, $\Gamma^z_{D_{2n}}=\left(\bigcup_{i=1}^{\frac{n}{2}+2}K_2\right)\cup\left(\bigcup_{i=1}^{\frac{n}{4}-1}K_{2,2}\right)$. Using Props. 2.4, 2.5 and 2.6, the clique polynomial of $\Gamma^z_{D_{2n}}$ is computed as follows:

$$\begin{aligned} C(\Gamma^z_{D_{2n}};x) &= \left[\left(\tfrac{n}{2}+2\right)(1+x)^2-\tfrac{n}{2}-1\right]+\left[\left(\tfrac{n}{4}-1\right)(1+2x)^2-\tfrac{n}{4}+2\right]-1 \\ &= \left(\tfrac{n}{2}+\tfrac{n}{4}-\tfrac{3n}{4}+1\right)+\left(\tfrac{2n}{2}+\tfrac{4n}{4}\right)x+\left(\tfrac{n}{2}+\tfrac{4n}{4}-2\right)x^2 \\ &= 1+2nx+\left(\tfrac{3n}{2}-2\right)x^2. \end{aligned}$$

When $\frac{n}{2}$ is odd, $\Gamma^z_{D_{2n}}=\left(\bigcup_{i=1}^{\frac{n}{2}+1}K_2\right)\cup\left(\bigcup_{i=1}^{\frac{n-2}{4}}K_{2,2}\right)$. Thus, the clique polynomial of $\Gamma^z_{D_{2n}}$ is computed as follows:

$$\begin{aligned} C(\Gamma^z_{D_{2n}};x) &= \left[\left(\tfrac{n}{2}+1\right)(1+x)^2-\tfrac{n}{2}\right]+\left[\left(\tfrac{n-2}{4}\right)(1+2x)^2-\tfrac{n-2}{4}+1\right]-1 \\ &= \left(\tfrac{n}{2}+1+\tfrac{n-2}{4}-\tfrac{3n-2}{4}\right)+(n+2+n-2)x+\left(\tfrac{n}{2}+1+n-2\right)x^2 \\ &= 1+2nx+\left(\tfrac{3n-2}{2}\right)x^2. \end{aligned}$$

■

The following theorems present the general forms for the independence polynomial and the clique polynomial of the center graph for the generalized quaternion group.

**Theorem 3.5.** *Let $Q_{4n}$ be a generalized quaternion group of order $4n$, where $n \geq 2$, $n \in \mathbb{N}$. The independence polynomial of the center graph of $Q_{4n}$ is*

$$I(\Gamma^z_{Q_{4n}};x)=\begin{cases}(1+2x)^{n+1}(1+4x+2x^2)^{\frac{n-1}{2}} & \text{if } n \text{ is odd}\\ (1+2x)^{n+2}(1+4x+2x^2)^{\frac{n-2}{2}} & \text{if } n \text{ is even.}\end{cases}$$

*Proof.* Suppose that $Q_{4n}$ is the generalized quaternion group of order $4n$, where $n \geq 2$, $n \in \mathbb{N}$, and $\Gamma^z_{Q_{4n}}$ is its center graph. When $n$ is odd, $\Gamma^z_{Q_{4n}}=\left(\bigcup_{i=1}^{n+1}K_2\right)\cup\left(\bigcup_{i=1}^{\frac{n-1}{2}}K_{2,2}\right)$. Using Propositions 2.1, 2.2 and 2.3, the independence polynomial of $\Gamma^z_{Q_{4n}}$ is computed:

$$\begin{aligned} I(\Gamma^z_{Q_{4n}};x) &= (1+2x)^{n+1}\left[(1+x)^2+(1+x)^2-1\right]^{\frac{n-1}{2}} \\ &= (1+2x)^{n+1}\left(1+2x+x^2+1+2x+x^2-1\right)^{\frac{n-1}{2}} \\ &= (1+2x)^{n+1}\left(1+4x+2x^2\right)^{\frac{n-1}{2}}. \end{aligned}$$

When $n$ is even, $\Gamma^z_{Q_{4n}} = \left(\bigcup_{i=1}^{n+2} K_2\right) \cup \left(\bigcup_{i=1}^{\frac{n-2}{2}} K_{2,2}\right)$. The independence polynomial of $\Gamma^z_{Q_{4n}}$ is computed as follows:

$$\begin{aligned} I(\Gamma^z_{Q_{4n}};x) &= (1+2x)^{n+2}\left[2(1+x)^2-1\right]^{\frac{n-2}{2}} \\ &= (1+2x)^{n+2}\left[2(1+2x+x^2)-1\right]^{\frac{n-2}{2}} \\ &= (1+2x)^{n+2}\left(1+4x+x^2\right)^{\frac{n-2}{2}}. \end{aligned}$$ ∎

**Theorem 3.6.** *Let $Q_{4n}$ be a generalized quaternion group of order $4n$, where $n \geq 2$, $n \in \mathbb{N}$. The clique polynomial of the center graph of $Q_{4n}$ is*

$$C(\Gamma^z_{Q_{4n}};x) = \begin{cases} 1+4nx+(3n-1)x^2 & \text{if } n \text{ is odd,} \\ 1+4nx+(3n-2)x^2 & \text{if } n \text{ is even.} \end{cases}$$

*Proof.* Suppose that $Q_{4n}$ is the generalized quaternion group of order $4n$, where $n \geq 2$, $n \in \mathbb{N}$, and $\Gamma^z_{Q_{4n}}$ is its center graph. When $n$ is odd, $\Gamma^z_{Q_{4n}} = \left(\bigcup_{i=1}^{n+1} K_2\right) \cup \left(\bigcup_{i=1}^{\frac{n-1}{2}} K_{2,2}\right)$. Using Propositions 2.4, 2.5 and 2.6, the clique polynomial of $\Gamma^z_{Q_{4n}}$ is computed as follows:

$$\begin{aligned} C(\Gamma^z_{Q_{4n}};x) &= \left[(n+1)(1+x)^2-n\right] + \left[\left(\tfrac{n-1}{2}\right)(1+2x)^2 - \tfrac{n-1}{2} + 1\right] - 1 \\ &= (n+1)(1+2x+x^2) + \left(\tfrac{n-1}{2}\right)\left(1+4x+4x^2\right) - n - \tfrac{n-1}{2} \\ &= \left(n+1+\tfrac{n-1}{2}-\tfrac{3n-1}{2}\right) + (2n+2n)x + (n+2n-1)x^2 \\ &= 1+4nx+(3n-1)x^2. \end{aligned}$$

When $n$ is even, $\Gamma^z_{Q_{4n}} = \left(\bigcup_{i=1}^{n+2} K_2\right) \cup \left(\bigcup_{i=1}^{\frac{n-2}{2}} K_{2,2}\right)$. Thus, the clique polynomial of $\Gamma^z_{Q_{4n}}$ is computed as follows:

$$\begin{aligned} C(\Gamma^z_{Q_{4n}};x) &= \left[(n+2)(1+x)^2-(n+1)\right] + \left[\left(\tfrac{n-2}{2}\right)(1+2x)^2 - \left(\tfrac{n-2}{2}\right)\right] \\ &= (n+2)(1+2x+x^2) + \left(\tfrac{n-2}{2}\right)(1+4x+4x^2) - \tfrac{3n}{2} \\ &= \left(n+2+\tfrac{n-2}{2}-\tfrac{3n}{2}\right) + (2n+4+2n-4)x + (n+2+2n-4)x^2 \\ &= 1+4nx+(3n-2)x^2. \end{aligned}$$ ∎

Lastly, the independence polynomial and the clique polynomial of the center graph for the quasidihedral group are presented.

**Theorem 3.7.** *Let $QD_{2^n}$ be a quasidihedral group of order $2^n$, where $n \geq 4$, $n \in \mathbb{N}$. The independence polynomial of the center graph of $QD_{2^n}$ is:*

$$I(\Gamma^z_{QD_{2^n}};x) = (1+2x)^{2^{(n-2)}+2}(1+4x+2x^2)^{2^{(n-3)}-1}.$$

*Proof.* Suppose that $QD_{2^n}$ is the quasidihedral group of order $2^n$, where $n \geq 4$, $n \in \mathbb{N}$, and $\Gamma^z_{QD_{2^n}}$ is its center graph. By using Lemma 3.2 and Props. 2.1, 2.2 and 2.3, the independence polynomial of $\Gamma^z_{QD_{2^n}}$ is computed as follows:

$$\begin{aligned} I(\Gamma^z_{QD_{2^n}};x) &= (1+2x)^{2^{(n-2)}+2}\left[(1+x)^2+(1+x)^2-1\right]^{2^{(n-3)}-1} \\ &= (1+2x)^{2^{(n-2)}+2}\left(1+4x+2x^2\right)^{2^{(n-3)}-1}. \end{aligned}$$

■

**Theorem 3.8.** *Let $QD_{2^n}$ be a quasidihedral group of order $2^n$, where $n \geq 4$, $n \in \mathbb{N}$. The clique polynomial of the center graph of $QD_{2^n}$ can be expressed as follows:*

$$C(\Gamma^z_{QD_{2^n}};x) = 1 + 2^n x + \left(3(2^{(n-2)}) - 2\right)x^2.$$

*Proof.* Suppose that $QD_{2^n}$ is the quasidihedral group of order $2^n$, where $n \geq 4$, $n \in \mathbb{N}$, and $\Gamma^z_{QD_{2^n}}$ is its center graph. By using Lemma 3.2 and Props. 2.4, 2.5 and 2.6, the clique polynomial of $\Gamma^z_{QD_{2^n}}$ is computed as follows:

$$\begin{aligned} C(\Gamma^z_{QD_{2^n}};x) &= \left[\left(2^{(n-2)}+2\right)(1+x)^2 - 2^{(n-2)} - 1\right] + \left[\left(2^{(n-3)}-1\right)\right. \\ &\quad \left.(1+2x)^2 - 2^{(n-3)} + 2\right] - 1 \\ &= 1 + \left[\left(2^n(2)\right) + \left(2^{n-3}(4)\right)\right]x + \left[\left(2^{n-2}\right) + \left(2^{n-3}(4)\right) - 2\right]x^2 \\ &= 1 + 2^n x + \left(3(2^{n-2}-2)\right)x^2. \end{aligned}$$

■

## 4. Conclusion

In this paper, the independence polynomial and the clique polynomial are established for the center graphs associated to the dihedral group, the generalized quaternion group and the quasidihedral group. The independence polynomials obtained for all the three groups have degrees depending on the order of the group. Meanwhile, the clique polynomials that are obtained for the three groups are all of degree two.

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