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The Non-Normal Subgroup Graph for Some Generalised Quaternion Groups

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Abstract. In this paper, the non-normal subgroup graphs of some generalised quaternion groups are constructed. The non-normal subgroups of the generalised quaternion groups are determined and furthermore the graph is constructed. The subgroup graph of a group G is defined as a directed simple graph with a vertex set G and two distinct elements x and y are adjacent if $xy \in H$.

Keywords: Subgroup graph; Non-normal subgroup; Generalised quaternion group.

1. Introduction

The algebraic graph theory has become an interesting and active research area in describing the properties of groups. A group can be represented in a graph by its subgroup structure. In group theory, a subgroup H of a group G is a subset of G, where H itself is a group under the same operation as in G. A subgroup H is said to be a normal subgroup if its left and right cosets coincide. The non-normal subgroups can be determined once the normal subgroups of a group G are identified. There were some researches on the graph of a group using its subgroup structure and the graphs obtained were undirected and directed.

The undirected graph of a group that are related to a subgroup were cyclic subgroup graph defined by Devi and Singh [3], a stable subgroup graph defined by Tolue [6] and a normal subgroup-based power graph of a finite group defined by Bhuniya and Bera [2]. Meanwhile, a directed graph related to subgroup was firstly introduced by Anderson [1] and then formally defined by Kakeri and Erfanian [4]. Anderson [1] introduced the subgroup graph as a simple directed graph and the structure of the connected components of this graph were investigated when $|H|$ is either two or three, H is normal subgroup of G and G/H is a finite abelian group. Kakeri and Erfanian [4] defined the subgroup graph formally, studied the complement of subgroup graph and discussed some properties of the graph. Let G be a group and H be a subgroup of a group G. Then the subgroup graph $\Gamma_H(G)$ is defined as a directed simple graph with vertex set G and two distinct elements x and y are adjacent if and only if $xy \in H$. In this paper, the non-normal subgroup graphs are obtained for some generalised quaternion group Q_{4n} of order $4n$. This group can be expressed in a group representation as follows:

$$Q_{4n} = \langle a, b | a^{2n} = b^4 = 1, a^n = b^2, b^{-1}ab = a^{-1} \rangle, n \geq 2, n \in \mathrm{N}.$$

In the following section, the main results are presented on the non-normal subgroup graph of generalised quaternion group.

2. Results and Discussions

In this section, the non-normal subgroup graph are constructed for some generalised quaternion group. The definition of the non-normal subgroup graph is given as follows:

Definition 2.1. [5] *Let G be a group and H be a non-normal subgroup of G. The subgroup graph $\Gamma_H^{NN}(G)$ is a directed graph with vertex set G such that x is the initial vertex and y is the terminal vertex of an edge if and only if $x \neq y$ and $xy \in H$.*

This definition is used to determine the non-normal subgroup graph of certain order in generalised quaternion groups. The results of the graph are shown in the form of propositions. Based on Definition 2.1, the non-normal subgroup graphs are determined for generalised quaternion group of order twelve and sixteen. They are stated in the following propositions.

Proposition 2.2. *Let G be the generalised quaternion group of order twelve, Q_{12} and H be a non-normal subgroup of G. The elements of Q_{12} are $\{1, a, a^2, a^3, a^4, a^5, b, ab, a^2b, a^3b, a^4b, a^5b\}$. Then the non-normal subgroup graphs for H_i where $1 \leq i \leq 3$ are given as follows:*

$$\Gamma_H^{NN}(G) = \begin{cases} \Gamma_{H_1}(Q_{12}) \cup K_4 \ ; & H_1 = \{1, a^3, b, a^3b\} \\ \Gamma_{H_2}(Q_{12}) \cup K_4 \ ; & H_2 = \{1, a^3, ab, a^4b\} \\ \Gamma_{H_3}(Q_{12}) \cup K_4 \ ; & H_3 = \{1, a^3, a^2b, a^5b\} \end{cases}$$

where $\Gamma_{H_i}(Q_{12})$ are illustrated in Figure 1.

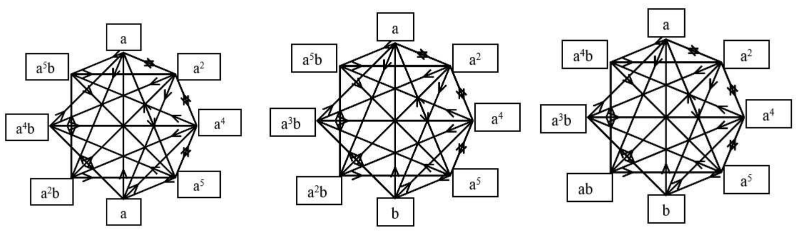

Figure 1: $\Gamma_{H_1}(Q_{12})$, $\Gamma_{H_2}(Q_{12})$ and $\Gamma_{H_3}(Q_{12})$

Proof. The generalised quaternion group of order twelve, Q_{12} has 12 elements. The non-normal subgroups of Q_{12} are $H_1 = \{1, a^3, b, a^3b\}$, $H_2 = \{1, a^3, ab, a^4b\}$ and $H_3 = \{1, a^3, a^2b, a^5b\}$. By Definition 2.1, the set of vertices for the non-normal subgroup graph is the set of elements of Q_{12} where two vertices are adjacent if and only if $x \neq y$ and $xy \in H$. The initial vertex and terminal vertex of an edge are x and y, respectively. For example, let $x = 1$ and $y = a^3$. Since $xy = 1 \cdot a^3 \in H$, there is a direction from $x = 1$ to $y = a^3$. Hence, the non-normal subgroup graph of $H_1 = \{1, a^3, b, a^3b\}$ is a directed graph as illustrated in the Figure 2.

The graph in Figure 2 can be concluded as $\Gamma_{H_1}(Q_{12}) \cup K_4$ where K_4 is a complete digraph with four vertices.

The non-normal subgroup graph of $H_2 = \{1, a^3, ab, a^4b\}$ is shown as in Figure 3 and it can be concluded as $\Gamma_{H_2}(Q_{12}) \cup K_4$.

Next, the non-normal subgroup graph of $H_3 = \{1, a^3, a^2b, a^5b\}$ is shown as in Figure 4 and the graph can be concluded as $\Gamma_{H_3}(Q_{12}) \cup K_4$.

Therefore, for each non-normal subgroup of Q_{12}, the non-normal subgroup graph is a union of a complete graph with four vertices and a directed graph as shown in Figure 1. ■

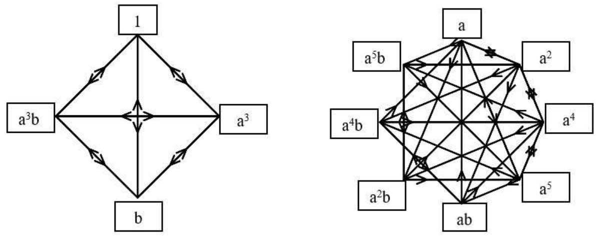

Figure 2: $\Gamma_{H_1}^{NN}(Q_{12})$

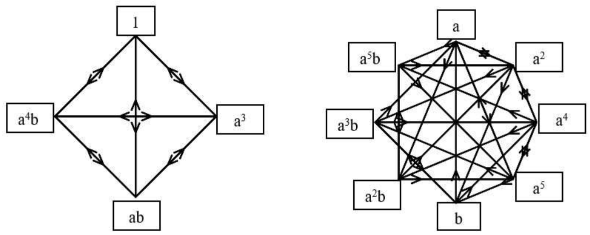

Figure 3: $\Gamma_{H_2}^{NN}(Q_{12})$

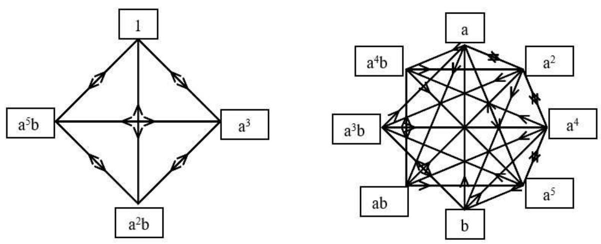

Figure 4: $\Gamma_{H_3}^{NN}(Q_{12})$

Proposition 2.3. *Let G be the generalised quaternion group of order sixteen, Q_{16} and H be a non-normal subgroup of G. The elements of Q_{16} are $\{1, a, a^2, a^3, a^4, a^5, a^6, a^7, b, ab, a^2b, a^3b, a^4b, a^5b, a^6b, a^7b\}$. Then the non-normal subgroup graphs for H_i where $1 \leq i \leq 4$ are given as follows:*

$$\Gamma_H^{NN}(G) = \begin{cases} \Gamma_{H_1}(Q_{16}) \cup K_4 \cup K_4 \ ; & H_1 = \{1, a^4, b, a^4b\} \\ \Gamma_{H_2}(Q_{16}) \cup K_4 \cup K_4 \ ; & H_2 = \{1, a^4, ab, a^5b\} \\ \Gamma_{H_3}(Q_{16}) \cup K_4 \cup K_4 \ ; & H_3 = \{1, a^4, a^2b, a^6b\} \\ \Gamma_{H_4}(Q_{16}) \cup K_4 \cup K_4 \ ; & H_4 = \{1, a^4, a^3b, a^7b\} \end{cases}$$

where $\Gamma_{H_i}(Q_{16})$ are illustrated in Figure 5 and Figure 6.

Proof. The generalised quaternion group of order sixteen, Q_{16} has 16 elements.

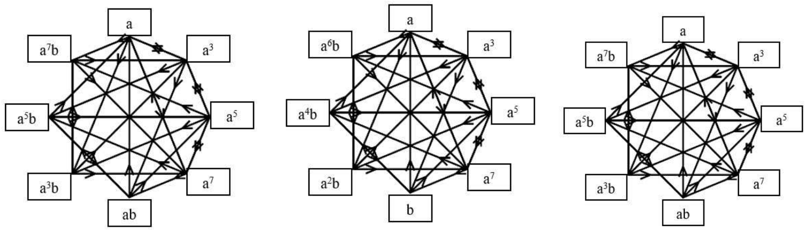

Figure 5: $\Gamma_{H_1}(Q_{16})$, $\Gamma_{H_2}(Q_{16})$ and $\Gamma_{H_3}(Q_{16})$

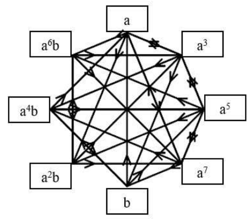

Figure 6: $\Gamma_{H_4}(Q_{16})$

The non-normal subgroups of Q_{16} are $H_1 = \{1, a^4, b, a^4b\}$, $H_2 = \{1, a^4, ab, a^5b\}$, $H_3 = \{1, a^4, a^2b, a^6b\}$ and $H_4 = \{1, a^4, a^3b, a^7b\}$. By Definition 2.1, the set of vertices for the non-normal subgroup graph is the set of elements of Q_{16} where two vertices are adjacent if and only if $x \neq y$ and $xy \in H$. The initial vertex and the terminal vertex of an edge are x and y. For example, let $x = 1$ and $y = a^4$. Since $xy = 1 \cdot a^4 \in H_1$, there is a direction from $x = 1$ to $y = a^4$. Hence, the non-normal subgroup graph of $H_1 = \{1, a^4, b, a^4b\}$ is a directed graph as illustrated in the Figure 7. The graph in Figure 7 can be concluded as $\Gamma_{H_1}(Q_{16}) \cup K_4 \cup K_4$.

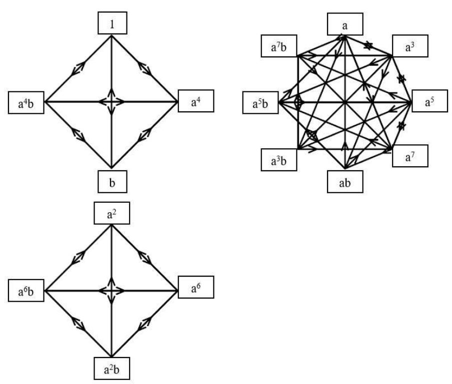

Figure 7: $\Gamma^{NN}_{H_1}(Q_{16})$

The non-normal subgroup graph of $H_2 = \{1, a^4, ab, a^5b\}$ is shown as in Fig. 8 and can be concluded as $\Gamma_{H_2}(Q_{16}) \cup K_4 \cup K_4$.

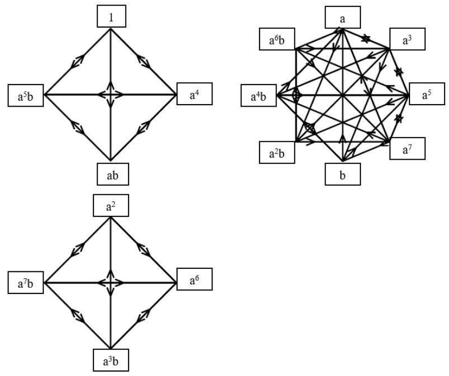

Figure 8: $\Gamma^{NN}_{H_2}(Q_{16})$

Next, the non-normal subgroup graph of $H_3 = \{1, a^4, a^2b, a^6b\}$ is constructed as in Figure 9. This graph can be concluded as $\Gamma_{H_3}(Q_{16}) \cup K_4 \cup K_4$.

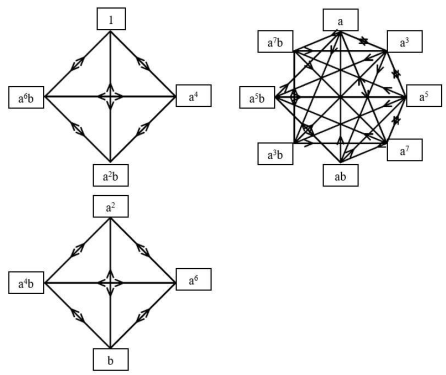

Figure 9: $\Gamma^{NN}_{H_3}(Q_{16})$

Furthermore, the non-normal subgroup graph of $H_4 = \{1, a^4, a^3b, a^7b\}$ is shown as in Figure 10.

The graph in Figure 10 can be concluded as $\Gamma_{H_4}(Q_{16}) \cup K_4 \cup K_4$. Thus, for each non-normal subgroup of Q_{16}, the non-normal subgroup graph is a union of two complete graphs with four vertices and a directed graph as shown in Figs. 5 and 6, respectively. ■

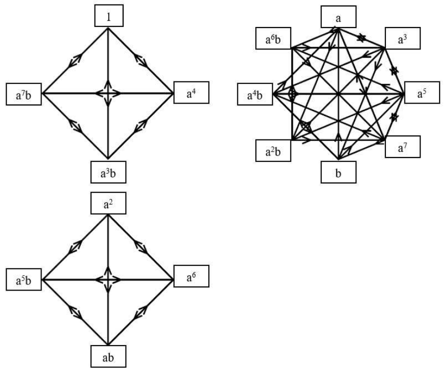

Figure 10: $\Gamma^{NN}_{H_4}(Q_{16})$

3. Conclusion

In this paper, the non-normal subgroup graphs for generalised quaternion of order twelve and sixteen are obtained. It can be seen that the graphs are union of complete graph and directed graph with a similar pattern. The complete graph in the non-normal subgroup graph is directly proportional to the order of the generalised quaternion group. When the order of the group increases, the non-normal subgroup graph will have more complete graph.

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