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Probabilistic Characterizations of Some Finite Ring of Matrices and Its Zero Divisor Graph*

Nurhidayah Zaid and Nor Haniza Sarmin
Department of Mathematical Sciences, Faculty of Science, University Teknologi Malaysia, 81310 Johor Bahru, Johor, Malaysia
Email: nurhidayah57@graduate.utm.my; nhs@utm.my

Sanhan Muhammad Salih Khasraw
Department of Mathematics, College of Basic Education, Salahaddin University-Erbil, Kurdistan Region, Iraq
Email: sanhan.khasraw@su.edu.krd

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Abstract. Let R be a finite ring. In this study, the probability that two random elements chosen from a finite ring have product zero is determined for some finite ring of matrices over $\mathbb{Z}_n$. Then, the results are used to construct the zero divisor graph which is defined as a graph whose vertices are the nonzero zero divisors of R and two distinct vertices x and y are adjacent if and only if $xy = 0$.

Keywords: Commutativity degree; Zero divisor graph; Noncommutative rings.

1. Introduction

A ring is an algebraic structure in which addition and multiplication operations are defined. The study of finite rings has been an interest of various researchers

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in the field of pure mathematics, especially in algebra. Meanwhile, also in the field of algebra, probability theory has been widely applied, especially in finite groups. This paper focuses on the usage of probability theory in finite rings.

Furthermore, graph is a mathematical structure which contains vertices and edges. In both groups and rings, graph is often used to visualize the features of the group or ring itself. For example, a commuting graph of a group shows the commuting relations between the elements of a group. With all non-central elements of a group as its vertices, two vertices are connected if the elements commute [8]. In this paper, the zero divisor graph of finite rings is also studied.

2. Probabilistic Characterizations of Finite Rings

In this section, past researches and recent updates on the applications of probability theory in rings are provided. One of the initial ideas of studies in probability of groups is given by Erdos and Turan [5] in 1968. In the study, the authors who are mainly interested in symmetric groups, S_n found some statistical relations of the groups that involve its order. This idea has then inspired Gustafson [7] in 1973 to introduce one type of probability in groups, which is the probability that two group elements commute or it is usually known as the commutativity degree of a group. The definition is given in the following:

Definition 2.1. *Let G be a group of finite order n. The commutativity degree or the probability, $Pr(G)$ that two elements selected at random (with replacement) from G are commutative is $\frac{|C|}{n^2}$ where $C = \{(x, y) \in G \times G | xy = yx\}$.*

In the study, Gustafson [7] has shown that the commutativity degree can also be calculated as $\frac{k(G)}{|G|}$, where $k(G)$ is the number of conjugacy classes in G. This concept of commutativity in groups has led to many extensions, such as the relative commutativity degree [6] and the conjugation degree [14].

In rings, the study on commutativity degree is started by MacHale [11] in 1976. The author has studied the probability that a pair of elements in a finite ring R commute. The probability is written as $P(R) = \frac{\sum_{r \in R} |C_R(r)|}{|R|^2}$, where $C_R(r)$ is the subring $\{x \in R | xr = rx\}$. Besides that, MacHale has also studied the probability in subrings of finite rings and found that $P(R) \leq P(S)$ whenever S is a subring of R. This idea has welcomed a number of researches on the commutativity in rings.

Recently in 2014, Buckley and MacHale [3] have defined the commuting probability in R as in the following mathematical formula.

Definition 2.2. *The commuting probability is defined as*

$$Pr(R) = \frac{|\{(x,y) \in R \times R | xy = yx\}|}{|R|^2},$$

where R is a finite ring, and $|R|$ denotes cardinality of a set R.

In 2016, Buckley and MacHale [3] have studied the results in [11] and later found that $P(R) = P(S)$ if and only if the commutator, $[x, S] = [x, R]$ for all $x \in R$. The authors then have continued the study on the commuting probability in finite rings and determined that if A and B are finite subsets of a possibly nonassociative ring R, then its commuting probability is defined by

$$\mathrm{Pr_R}(A, B) = \frac{|\{(x,y) \in A \times B | xy = yx\}|}{|A| \cdot |B|}.$$

Another recent study in commuting probability in rings has been done by Dutta et al. [4]. In the study, $P(R)$ is generalized as the relative commuting probability of the subring S in a finite ring R and the mathematical formula is given as:

$$Pr(S, R) = \frac{|\{(s,r) \in S \times R | sr = rs\}|}{|S \times R|}.$$

Moving on to another type of probability that involves rings, which is the probability that two elements of a finite ring have product zero, which has been introduced by Khasraw [9] in 2018. This idea is inspired by the zero divisor graph, which will be discussed in the next section. Khasraw [9] has defined this probability as in the following:

Definition 2.3. *The probability $P(R)$ that two elements chosen at random (with replacement) from a ring R have product zero is*

$$P(R) = \frac{|Ann|}{|R \times R|},$$

in which $Ann = \{Ann(x) | x \in R\}$, where $Ann(x) = \{y \in R | xy = 0\}$.

Some bounds of the probability have been obtained in the study, including

$$P(R) \geq \frac{2l + |Z(R)| - 1}{l^2},$$

where $|Ann(0)| = l = |R|$ and $Z(R)$ is the set of zero divisors of R. In this paper, the probability that two elements of a finite ring have product zero is determined for some ring of matrices.

3. Zero Divisor Graph of Finite Rings

In this section, some basic concepts on the zero divisor graph of finite rings are discussed. The idea on the zero divisor graph has been introduced by Beck [2] in 1988. Mainly focusing on the coloring of commutative rings, Beck has considered a simple graph whose vertices are the elements of a finite commutative ring R, such that two different elements x and y are adjacent if $xy = 0$.

Later in 1999, Anderson and Livingston [1] have formally introduced the zero divisor graph of a finite commutative ring R. The definition is given as in the following:

Definition 3.1. *Let R be a commutative ring (with 1) and let $Z(R)$ be its set of zero divisors. Then, the zero divisor graph, $\Gamma(R)$ is a simple graph of R with vertices $Z(R)^* = Z(R) - \{0\}$, the set of nonzero zero divisors of R, and for distinct $x, y \in Z(R)^*$, the vertices x and y are adjacent if and only if $xy = 0$.*

In 2002, Redmond [13] extended the definition of the zero divisor graph, where a study has been done on finite non-commutative rings. Redmond has defined that the zero divisor graph, $\Gamma(R)$ of a finite non-commutative ring is a directed graph. In the paper, Redmond has shown that in finite non-commutative rings, $\Gamma(R)$ need not to be a connected graph.

The study on zero divisor graph has also been extended to some other graphs, including the ideal based zero divisor graph [12] and the compressed zero divisor graph [10]. In this paper, the main focus is to obtain the zero divisor graph of some finite ring of matrices.

4. Results and Discussions

In this section, the results of this study are given in the form of propositions. The results are divided into two parts, which are the probability that two elements of a finite ring have product zero and the zero divisor graph. The finite ring that is considered in this study is the ring of 2×2 matrices over set of integers modulo two, $M_2(\mathbb{Z}_2)$.

Definitions 2.3 and 3.1 are mainly used in proving the propositions. Besides that, two new definitions are introduced in this section, which are stated as follows.

Definition 4.1. *The probability $\widetilde{P}(R)$ that two elements chosen at random (with replacement) from a ring R have product zero is*

$$\widetilde{P}(R) = \frac{\{(x, y) \in R \times R | xy = yx = 0\}}{|R \times R|}.$$

Definition 4.2. *The zero divisor graph, $\widetilde{\Gamma}(R)$ is a simple graph of R with vertices $Z(R)$, the set of nonzero zero divisors of R, and for distinct $x, y \in Z(R)$, the vertices x and y are adjacent if and only if $xy = yx = 0$.*

Proposition 4.3. *Let $R = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \middle| a, b, c, d \in \mathbb{Z}_2 \right\}$. Then, the probability that two elements of R have product zero, $P(R) = \frac{29}{128}$ and $\widetilde{P}(R) = \frac{11}{128}$.*

Proof. Let $R = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \middle| a, b, c, d \in \mathbb{Z}_2 \right\}$, where its elements are:

$$R = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \right.$$
$$\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix},$$
$$\left. \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \right\}.$$

Therefore, it is clear that $|R| = 16$. Recall from Definition 2.3, $Ann(x) = \{y \in R | xy = 0\}$. For example, $\left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} \right)$ is in $Ann\left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \right)$ because the product of these elements is zero. After some calculations, it is found that the size of Ann, $|Ann| = 58$. Hence, by definition, the probability that two elements of R have product zero is $P(R) = \frac{58}{256} = \frac{29}{128}$.

Now, considering the second case where the annihilator is defined as $Ann(x) = \{y \in R | xy = yx = 0\}$. For example, $\left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right)$ is in $Ann\left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \right)$ because $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$. It is found that in this case, the size of Ann, $|Ann| = 22$. Hence, by definition, the probability that two elements of R have product zero is $\widetilde{P}(R) = \frac{22}{256} = \frac{11}{128}$. ∎

Proposition 4.4. *Let $R = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \middle| a, b, c, d \in \mathbb{Z}_2 \right\}$. Then, the zero divisor graph of R, $\Gamma(R)$ is a directed graph of nine vertices and 24 arcs and $\widetilde{\Gamma}(R)$ is a directed graph of nine vertices and six arcs.*

Proof. Let $R = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \middle| a, b, c, d \in \mathbb{Z}_2 \right\}$. From the elements of R that have been obtained in Proposition 4.3, it is found that R has nine nonzero zero divisors which are stated in the following:

$$Z(R) = \left\{ \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, \right.$$

$$\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\Bigg\}.$$

Hence, the zero divisor graph of R, $\Gamma(R)$ has nine vertices. Since two vertices are adjacent when the product is zero, $\Gamma(R)$ has 24 arcs. The graph is shown as follows.

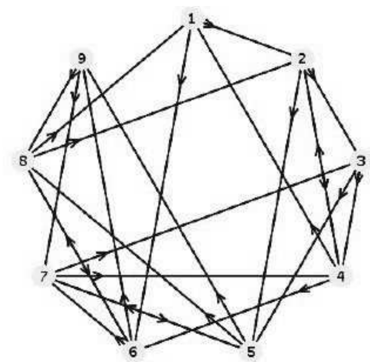

Figure 1: The zero divisor graph of R, $\Gamma(R)$

Now, considering the case where two vertices, say A and B are adjacent if and only if $AB = BA = 0$, it is found that $\widetilde{\Gamma}(R)$ has nine vertices and six arcs. The graph is shown below.

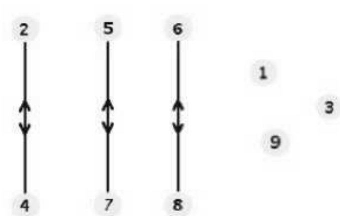

Figure 2: The zero divisor graph of R, $\widetilde{\Gamma}(R)$

■

The results from Propositions 4.3 and 4.4 lead to the following two propositions that show the relations between the probability and arcs of the zero divisor graph.

Proposition 4.5. *Let* $R = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \middle| a, b, c, d \in \mathbb{Z}_2 \right\}$ *and* $|R| = 16$. *Then,*

$$P(R) = \frac{|A(\Gamma(R))| + 2(|R| - 1) + c}{|R|^2},$$

where $|A(\Gamma(R))|$ *is the number of arcs in* $\Gamma(R)$ *and* $c = |\{(x,x) \in R \times R | xx = 0\}|$.

Proof. Let $R = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \middle| a,b,c,d \in \mathbb{Z}_2 \right\}$. Clearly, $|Ann(0)| = |R| = 16$. Since the vertices of $\Gamma(R)$ are nonzero elements of R, there is $2 \cdot |R|$ as all elements in R have product zero when multiplied with zero itself. From Prop. 4.4, the arcs of the zero divisor graph, $A(\Gamma(R))$ represents $Ann(x)$. Hence, $|A(\Gamma(R))| = |Ann(x)|$. In addition, since $\Gamma(R)$ is a simple graph, the elements of R that multiply itself to zero is not shown as arcs in $\Gamma(R)$. Thus,

$$|Ann| = |A(\Gamma)| + 2(|R| - 1) + c,$$

where $c = |\{(x,x) \in R \times R | xx = 0\}|$. Therefore,

$$P(R) = \frac{|A(\Gamma(R))| + 2(|R| - 1) + c}{|R|^2}.$$

■

Using similar method as in Proposition 4.5, the following proposition is obtained for $\widetilde{P}(R)$.

Proposition 4.6. *Let* $R = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \middle| a,b,c,d \in \mathbb{Z}_2 \right\}$ *and* $|R| = 16$. *Then,*

$$\widetilde{P}(R) = \frac{|A(\widetilde{\Gamma}(R))| + |R| + c - 2}{|R|^2},$$

where $|A(\widetilde{\Gamma}(R))|$ *is the number of arcs in* $\widetilde{\Gamma}(R)$ *and* $c = |\{(x,x) \in R \times R | xx = 0\}|$.

5. Conclusions

In this study, a type of probabilistic characterizations of finite ring, namely the probability that two elements of a finite ring have product zero is determined for the ring of 2×2 matrices over integers modulo two. It is found that the probability, $P(R) = \frac{29}{128}$ and $\widetilde{P}(R) = \frac{11}{128}$.

Next, the zero divisor graphs of the same ring are constructed and the graphs are presented in the form of directed graphs. The zero divisor graph, $\Gamma(R)$ is a graph with nine vertices and 24 arcs, while $\widetilde{\Gamma}(R)$ has nine vertices and six arcs.

Lastly, it is also found that there is a relation between the arcs of $\Gamma(R)$ and $\widetilde{\Gamma}(R)$ with the probabilities $P(R)$ and $\widetilde{P}(R)$, where the number of arcs represents the number of annihilators of x in R, $Ann(x)$.

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