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Cayley Graph for the Non-Abelian Tensor Square of Some Finite Groups

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Abstract. Let G be a group and W be a subset of G which is inverse closed and has no identity element. The Cayley graph $\Gamma(G, W)$ is the graph which has the elements of G as its vertices and two vertices $u, v \in G$ are joined by an edge if and only if $v = wu$ for some $w \in W$. Meanwhile, the non-abelian tensor square of a group G, denoted by $G \otimes G$, is a group generated by the symbols $g \otimes h$, $g, h \in G$, subject to the relations $gg' \otimes h = ({}^{g}g' \otimes gh)(g \otimes h)$ and $g \otimes hh' = (g \otimes h)({}^{h}g \otimes {}^{h}h')$ for all $g, g', h, h' \in G$, where ${}^{h}g = hgh^{-1}$. In this research, the Cayley graphs of the non-abelian tensor square are constructed for some groups of small order namely, the symmetric group of order six, S_3, and the dihedral group of order eight, D_8. Seven subsets are obtained from the non-abelian tensor square of S_3, $S_3 \otimes S_3$. Hence, there are seven different Cayley graphs for $S_3 \otimes S_3$. Furthermore, there are 15 subsets with valency one for the non-abelian tensor square of D_8, $D_8 \otimes D_8$. The Cayley graphs for these 15 subsets with valency one of $D_8 \otimes D_8$ are complete graphs.

Keywords: Cayley graph; Finite group; Tensor square.

1. Introduction

Many researches have been done on Cayley graph. Abdollahi and Jazaeri [1] discovered the classification of finite Cayley integral groups from Cayley graphs. Erskine [3] computed the Cayley graphs of dihedral groups that have diameter 2. Then, the formation of Cayley graph by using elements fixing k points for symmetric group has been discovered by Ku et al. [5]. One of the characteristics of Cayley graph for symmetric groups, which is resistance distance has been figured by Vaskouski and Zadorozhnyuk [7]. The basic definitions in group theory and graph theory that are used throughout this paper are non-abelian tensor square, graph and Cayley graph. The nonabelian tensor square for a group G, $G \otimes G$, is generated by the symbols $g \otimes h$, $g, h \in G$, subject to the relations $gg' \otimes h = ({}^{g}g' \otimes gh)(g \otimes h)$ and $g \otimes hh' = (g \otimes h)({}^{h}g \otimes {}^{h}h')$ for all $g, g', h, h' \in G$, where ${}^{h}g = hgh^{-1}$ denotes the conjugate of g by h as defined by Bacon and Kappe [2].

Ramachandran et al. [6] have computed the Cayley table for the non-abelian tensor square of S_3. The non-abelian tensor square of S_3 is stated as in the following:

$$S_3 \otimes S_3 = \{(1)\otimes(1), (123)\otimes(23), (132)\otimes(23), (23)\otimes(23), (23)\otimes(13), (23)\otimes(12)\}.$$

Kappe et al. [4] have generalized the non-abelian tensor square for dihedral group. The non-abelian tensor square for dihedral group with order eight, $D_8 \otimes D_8$ is isomorphic to $\mathbb{Z}_1^2 \times \mathbb{Z}_2^3 \times \mathbb{Z}_4$. The order for $D_8 \otimes D_8$ is 32.

A graph of a group G, Γ_G, is a finite nonempty set of objects called vertices together with a (possibly empty) set of unordered pairs of distinct vertices of Γ_G called edges. The vertex set of Γ_G is denoted by $V(\Gamma_G)$, while the edge set is denoted by $E(\Gamma_G)$. Besides, let G be a finite group and W be an inverse closed subset of G, i.e., $e \notin W$, and $w \in W \rightarrow w^{-1} \in W^{-1}$. The Cayley graph $\Gamma(G, W)$ is the graph which has the elements of G as its vertices and two vertices $u, v \in G$ are joined by an edge if and only if $v = wu$ for some $w \in W$ as defined by Ku et al. [5].

The main results of this paper are presented in eight propositions in the following section.

2. Results and Discussions

In this section, the results of this research are shown in the form of propositions. The results are divided into seven propositions because there are seven subsets of $S_3 \otimes S_3$ which satisfy the condition given for a Cayley graph.

The non-abelian tensor square of S_3, labeled as G, is stated as in the following:

$$\begin{aligned} G &= S_3 \otimes S_3 \\ &= \{(1)\otimes(1), (123)\otimes(23), (132)\otimes(23), (23)\otimes(23), (23)\otimes(13), (23)\otimes(12)\}. \end{aligned}$$

The subsets of G that satisfies the condition in Cayley graph are identified as in the following:

(i) $W_1 = \{(23) \otimes (23)\}$,
(ii) $W_2 = \{(123) \otimes (23), (132) \otimes (23)\}$,
(iii) $W_3 = \{(23) \otimes (13), (23) \otimes (12)\}$,
(iv) $W_4 = \{(123) \otimes (23), (132) \otimes (23), (23) \otimes (23)\}$,
(v) $W_5 = \{(23) \otimes (23), (23) \otimes (13), (23) \otimes (12)\}$,
(vi) $W_6 = \{(123) \otimes (23), (132) \otimes (23), (23) \otimes (13), (23) \otimes (12)\}$,
(vii) $W_7 = \{(123) \otimes (23), (132) \otimes (23), (23) \otimes (23), (23) \otimes (13), (23) \otimes (12)\}$.

All these subsets of G are used to construct the Cayley graphs of G which are presented in the following propositions.

Proposition 2.1. *Let $G = S_3 \otimes S_3 = \{(1) \otimes (1), (123) \otimes (23), (132) \otimes (23), (23) \otimes (23), (23) \otimes (13), (23) \otimes (12)\}$ and $W_1 = \{(23) \otimes (23)\}$ be a subset of G. Then, the Cayley graph of G with subset W_1 is the union of three complete graphs with two vertices, K_2, denoted as $\Gamma(G, W_1) = \cup_{i=1}^{3} K_2$ and the graph is illustrated as in Fig. 1.*

Figure 1: The Cayley graph of G with subset $W_1, \Gamma(G, W_1)$

Proof. Let $W_1 = \{(23) \otimes (23)\}$ be the subset of G. The inverse for the element in W_1 is $(23) \otimes (23)$ since $((23) \otimes (23))((23) \otimes (23)) = (1) \otimes (1)$. Let $x, y \in G$ and $w \in W_1$. Based on the definition of Cayley Graph, x and y are connected if $xy^{-1} = w \in W_1$ which implies $x = wy$. The following calculations for $x = wy$ are shown to observe the relations between the elements in W_1 with all the elements in G.

(i) If $w = (23) \otimes (23)$ and $y = (1) \otimes (1)$, then $x = ((23) \otimes (23))((1) \otimes (1)) = (23) \otimes (23)$ which implies $(23) \otimes (23)$ and $(1) \otimes (1)$ are connected.
(ii) If $w = (23) \otimes (23)$ and $y = (23) \otimes (23)$, then $x = ((23) \otimes (23))((23) \otimes (23)) = (1) \otimes (1)$ which implies $(1) \otimes (1)$ and $(23) \otimes (23)$ are connected.
(iii) If $w = (23) \otimes (23)$ and $y = (123) \otimes (23)$, then $x = ((23) \otimes (23))((123) \otimes (23)) = (23) \otimes (12)$ which implies $(23) \otimes (12)$ and $(123) \otimes (23)$ are connected.
(vi) If $w = (23) \otimes (23)$ and $y = (23) \otimes (12)$, then $x = ((23) \otimes (23))((23) \otimes (12)) = (123) \otimes (23)$ which implies $(123) \otimes (23)$ and $(23) \otimes (12)$ are connected.
(v) If $w = (23) \otimes (23)$ and $y = (132) \otimes (23)$, then $x = ((23) \otimes (23))((132) \otimes (23)) = (23) \otimes (13)$ which implies $(23) \otimes (13)$ and $(132) \otimes (23)$ are connected.

(vi) If $w = (23) \otimes (23)$ and $y = (23) \otimes (13)$, then $x = ((23) \otimes (23))((23) \otimes (13)) = (132) \otimes (23)$ which implies $(132) \otimes (23)$ and $(23) \otimes (13)$ are connected.

Hence, the set of vertices and edges for $\Gamma(G, W_1)$ are listed as in the following:

$$\begin{aligned} V_{\Gamma(G,W_1)} &= \{(1) \otimes (1), (123) \otimes (23), (132) \otimes (23), (23) \otimes (23), (23) \otimes (13), \\ &\quad (23) \otimes (12)\} = S_3 \otimes S_3, \\ E_{\Gamma(G,W_1)} &= \{(1) \otimes (1), (23) \otimes (23)\}, \{(23) \otimes (12), (123) \otimes (23)\}, \{(23) \otimes (13), \\ &\quad (132) \otimes (23)\}. \end{aligned}$$

Thus, the Cayley graph of G with subset W_1 is illustrated as in Fig. 1. ■

By using similar method, the following propositions are obtained for the other subsets of G, namely $W_2, W_3, W_4, W_5, W_6, W_7$.

Proposition 2.2. *Let* $G = S_3 \otimes S_3 = \{(1) \otimes (1), (123) \otimes (23), (132) \otimes (23), (23) \otimes (23), (23) \otimes (13), (23) \otimes (12)\}$ *and* $W_= \{(123) \otimes (23), (132) \otimes (23)\}$ *be a subset of* G*. Then, the Cayley graph of* G *with subset* W_2 *is the union of two complete graphs with three vertices,* K_3*, denoted as* $\Gamma(W, W_2) = \cup_{i=1}^{2} K_3$ *and the graph is illustrated as in Fig. 2.*

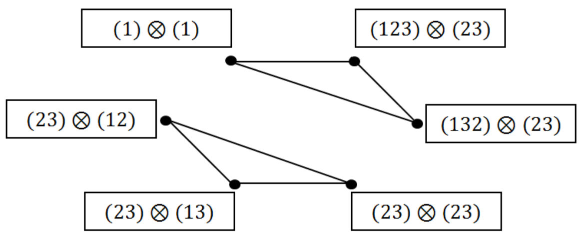

Figure 2: The Cayley graph of G with subset $W_2, \Gamma(G, W_2)$.

Proposition 2.3. *Let* $G = S_3 \otimes S_3 = \{(1) \otimes (1), (123) \otimes (23), (132) \otimes (23), (23) \otimes (23), (23) \otimes (13), (23) \otimes (12)\}$ *and* $W_3 = \{(23) \otimes (13), (23) \otimes (12)\}$ *be a subset of* G*. Then, the Cayley graph of* G *of subset* W_3 *is the incomplete and regular graph with degree two for each vertex. The Cayley graph of* G *of subset* W_3 *is illustrated as in Fig. 3.*

Proposition 2.4. *Let* $G = S_3 \otimes S_3 = \{(1) \otimes (1), (123) \otimes (23), (132) \otimes (23), (23) \otimes (23), (23) \otimes (13), (23) \otimes (12)\}$ *and* $W_4 = \{(123) \otimes (23), (132) \otimes (23), (23) \otimes (23)\}$ *be a subset of* G*. Then, the Cayley graph of* G *of subset* W_4 *is the incomplete graph with six vertices. The Cayley graph of* G *of subset* W_4 *is illustrated as in Fig. 4.*

Proposition 2.5. *Let* $G = S_3 \otimes S_3 = \{(1) \otimes (1), (123) \otimes (23), (132) \otimes (23), (23) \otimes (23), (23) \otimes (13), (23) \otimes (12)\}$ *and* $W_5 = (23) \otimes (13), (23) \otimes (12), (23) \otimes (23)$ *be*

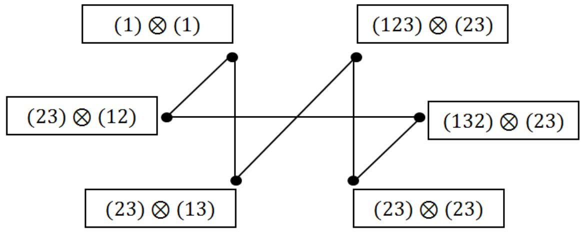

Figure 3: The Cayley graph of G with subset $W_3, \Gamma(G, W_3)$.

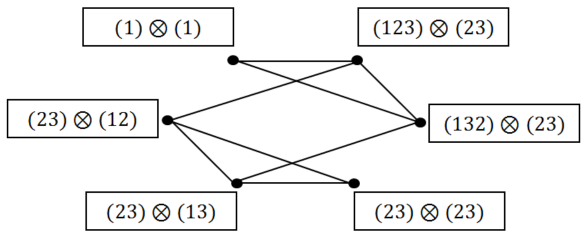

Figure 4: The Cayley graph of G with subset $W_4, \Gamma(G, W_4)$.

a subset of G. Then, the Cayley graph of G of subset W_5 is the incomplete and regular graph with degree three for each vertex. The Cayley graph of G of subset W_5 is illustrated as in Fig. 5.

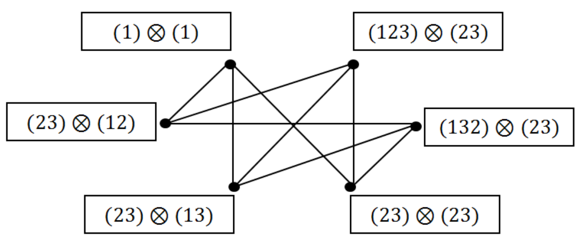

Figure 5: The Cayley graph of G with subset $W_5, \Gamma(G, W_5)$.

Proposition 2.6. *Let $G = S_3 \otimes S_3 = \{(1) \otimes (1), (123) \otimes (23), (132) \otimes (23), (23) \otimes (23), (23) \otimes (13), (23) \otimes (12)\}$ and $W_6 = \{(123) \otimes (23), (132) \otimes (23), (23) \otimes (13), (23) \otimes (12)\}$ be a subset of G. Then, the Cayley graph of G of subset W_6 is the incomplete graph with six vertices. The Cayley graph of G of subset W_6 is illustrated as in Fig. 6.*

Proposition 2.7. *Let $G = S_3 \otimes S_3 = \{(1) \otimes (1), (123) \otimes (23), (132) \otimes (23), (23) \otimes (23), (23) \otimes (13), (23) \otimes (12)\}$ and $W_7 = \{(123) \otimes (23), (132) \otimes (23), (23) \otimes (23), (23) \otimes (13), (23) \otimes (12)\}$ be a subset of G. Then, the Cayley graph of G with*

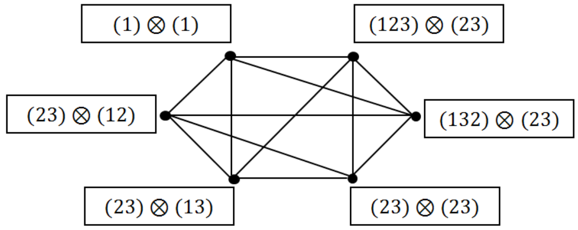

Figure 6: The Cayley graph of G with subset $W_6, \Gamma(G, W_6)$.

subset W_7 is a complete graph with six vertices, K_6, denoted as $\Gamma(W, W_7) = K_6$ as illustrated in Fig. 7.

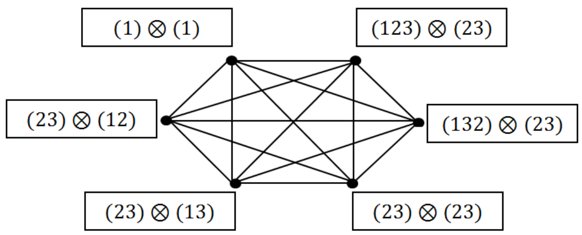

Figure 7: The Cayley graph of G with subset $W_7, \Gamma(G, W_7)$.

Proposition 2.8. *Let $H = D_8 \otimes D_8$ and S is a subset of H with valency one. Then, the Cayley graph of H with subset S is a complete graph with 32 vertices, denoted as $\Gamma(H, S) = K_{32}$ as illustrated in Fig. 8.*

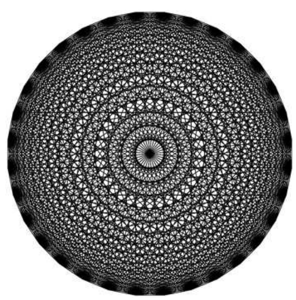

Figure 8: The Cayley graph of H with subsets S of valency one

Proof. Let $H = D_8 \otimes D_8$. Then, there are 15 subsets of H with valency one, namely, {(0, 0, 0, 0, 0, 2)}, {(0, 0, 0, 0, 1, 0)}, {(0, 0, 0, 0, 1, 2)}, {(0, 0, 0, 1, 0, 0)}, {(0, 0, 0, 1, 0, 2)}, {(0, 0, 0, 1, 1, 0)}, {(0, 0, 0, 1, 1, 2)}, {(0, 0, 1, 0, 0, 0)}, {(0, 0, 1, 0, 0, 2)}, {(0, 0, 1, 0, 1, 0)}, {(0, 0, 1, 0, 1, 2)}, {(0, 0, 1, 1, 0, 0)}, {(0, 0, 1, 1, 0, 2)}, {(0, 0, 1, 1, 1, 0)} and {(0, 0, 1, 1, 1, 2)}. Let $W_1 = \{(0,0,0,0,0,2)\}$ be one of the subsets with valency one for H. The set of inverse for W_1 is (0,0,0,0,0,2) since (0,0,0,0,0,2)(0,0,0,0,0,2)=(0,0,0,0,0,0). Let $x, y \in H$ and $w_1 \in W_1$. Then, x and y are connected if $xy^{-1} = w_1 \in W_1$ which implies $x = w_1 y$. By using the same calculations in Proposition 1, the Cayley graph of H with subset W_1 is a complete graph with 32 vertices as illustrated in Figure 8. By using similar method, the other 14 complete graphs with 32 vertices are obtained for each subset in S. ■

3. Conclusion

There are seven Cayley graphs constructed for the non-abelian tensor square of S_3 which are labeled as $\Gamma(G, W_1)$, $\Gamma(G, W_2)$, $\Gamma(G, W_3)$, $\Gamma(G, W_4)$, $\Gamma(G, W_5)$, $\Gamma(G, W_6)$, and $\Gamma(G, W_7)$. There are two graphs, $\Gamma(G, W_1)$ and $\Gamma(G, W_2)$ which are the union of complete graphs and Γ (G, W_7) is a complete graph. The other graphs are incomplete graphs. Besides, the Cayley graphs for the non-abelian tensor square of D_8 with subsets of valency one are obtained as complete graphs with 32 vertices.

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