3 Conjugate Degree of group

Definition Let G be a group and $x \in G$. Then $x^g := g^{-1}xg$ is called the conjugate of x, where $g \in G$. The set of all conjugates of x is denoted by x^G and is called conjugacy class of x.

$$\text{conjugacy class of } x = x^G = \{x^g \mid g \in G\}$$

Example Let S_3 be symmetric group on 3 symboles. Then we have

$$S_3 = \{e, (12), (13), (23), (123), (132)\}$$

$$e^G = \{e^g \mid g \in S_3\} = \{g^{-1}eg \mid g \in S_3\} = \{e\}$$

$$(12)^G = \{(12)^g \mid g \in S_3\} = \{(12), (13), (23)\}$$

$$(123)^G = \{(123)^g \mid g \in S_3\} = \{(123), (132)\}$$

$$(12)^G = (13)^G = (23)^G, \quad (123)^G = (132)^G$$

It is clear that $S_3 = e^G \cup (12)^G \cup (123)^G$ and so S_3 has 3 distinct conjugacy classes.

Definition Let G be a group. Two elements x and y are called **conjugate** if and only if there is an element $g \in G$ such that $x^g = y$.

Example In S_3, two elements (12) and (13) are conjugate, because we have

$$(12)^{(123)} = (123)^{-1}(12)(123) = (132)(12)(123) = (13)$$

But, two elements (12) and (123) are not conjugate.

Some Facts

(i) If two elements x and y are conjugate then $|x| = |y|$. In other words, they have the same order.

(ii) Every two elements in a conjugacy class of element are conjugate.

(iii) If G is a finite group and $x \in G$.

Then $|x^G| = \frac{|G|}{|C_G(x)|}$.

(iv) If $x \in Z(G)$, then $|x^G| = 1$. In other words, $x^G = \{x\}$

(v) For every two conjugacy classes x^G and y^G we have $x^G \cap y^G = \phi$ or $x^G = y^G$.

Definition Let G be a finite group. Then the conjugate degree of G define as follows:

Conjugate degree of $G = P_{Conj}(G)$

$$= \frac{|\{(x,y) \in G \times G \mid x \text{ and } y \text{ are conjugate}\}|}{|G|^2}$$

Example $S_3 = \{e, (12), (13), (23), (123), (132)\}$

$P_{Conj}(S_3) = |\{(e,e), ((12),(12)), ((12),(13)), ((12),(23)), ((13),(13)),$
$((13),(12)), ((13),(23)), ((23),(23)), ((23),(12)), ((23),(13)),$

$((123),(123)), ((123),(132)), ((132),(132)), ((132),(123))\}| / 36$

$= \frac{14}{36}$.

Lemma Let G be finite group and $x_1^G, x_2^G, \ldots, x_k^G$ are all distinct conjugacy classes of G. Then

$$P_{Conj}(G) = \frac{1}{|G|^2} \sum_{i=1}^{k} |x_i^G|^2.$$

In the previous Example, we can see that S_3 has three distinct conjugacy classes $e^G, (12)^G, (123)^G$ of sizes $1, 3, 2$, respectively. Thus, by the above formula, we will have

$$P_{Conj}(S_3) = \frac{1}{36}(1^2 + 3^2 + 2^2) = \frac{14}{36}$$

Lemma If G is abelian, then

$$P_{Conj}(G) = \frac{1}{|G|}.$$

proof Since G is abelian, so each conjugacy class is a singleton set and the proof follows.

Same problems

1. If G is a finite group with $P_{Conj}(G) = \frac{1}{|G|}$. Is G abelian?

2. If G is a trivial group, then $P_{Conj}(G) = 1$. Is the converse true?

3. What is $P_{Conj}(D_{2n})$ for all even and odd number $n \geq 3$?

4. What is the maximum value of $P_{Conj}(D_{2n})$? Find n?

5. prove that

$$P_{Conj}(G_1 \times G_2) = P_{Conj}(G_1) P_{Conj}(G_2)$$

4 Normality Degree of a subgroup in a group

Definition Let G be group and N be a subgroup of G. Then N is said to be a **normal subgroup**, if for every element $n \in N$ and every element $g \in G$ we have $g^{-1}ng \in N$.

Example $S_3 = \{e, (12), (13), (23), (123), (132)\}$

$H = \{e, (12)\}$; $K = \{e, (123), (132)\}$

(i) H is not a normal subgroup of S_3.

(ii) K is a normal subgroup of S_3.

Because, In H, take $(12) \in H$ and $(123) \in S_3$, then $(12)^{(123)} = (123)^{-1}(12)(123)$

$= (132)(12)(123) = (13) \notin H$

But, for every $x \in K$ and every $g \in S_3$, we can see that $g^{-1}xg \in K$.

Definition Let G be a finite group and H be a subgroup of G. The **normality degree of H in G**, denoted by $P_{normal}(H,G)$ is defined as the following:

$$P_{normal}(H,G) = \frac{|\{(h,g)\in H\times G \mid g^{-1}hg \in H\}|}{|H||G|}$$

* It is obvious that

$$P_{normal}(H,G)=1 \iff H \trianglelefteq G$$

Example $S_3=\{e,(12),(13),(23),(123),(132)\}$

$H=\{e,(12)\}$. clearly, $e^x = x^{-1}ex = e \in H$ for all $x\in S_3$. Now, for $(12)\in H$, we have

$(12)^e = e^{-1}(12)e = (12) \in H$

$(12)^{(12)} = (12)^{-1}(12)(12) = (12) \in H$

$(12)^{(13)} = (13)^{-1}(12)(13) = (13)(12)(13) = (23) \notin H$

$(12)^{(23)} = (23)^{-1}(12)(23) = (23)(12)(23) = (13) \notin H$

$$(12)^{(123)} = (123)^{-1}(12)(123) = (132)(12)(123) = (13) \notin H$$

$$(12)^{(132)} = (132)^{-1}(12)(132) = (132)(12)(132) = (12) \in H$$

Thus,

$$P_{normal}(H, S_3) = |\{(e,e), (e,(12)), (e,(13)), (e,(23)), (e,(123)), (e,(132)), ((12),e), ((12),(12)), ((12),(132))\}| \Big/ |H||S_3|$$

$$= \frac{9}{2 \times 6} = \frac{9}{12} = \frac{3}{4} \implies P_{normal}(H, S_3) = \frac{3}{4}$$

Lemma Let G be a finite group and H be a subgroup of G. Then we define

$$N_H(g) = \{h \in H \mid h^g \in H\}. \text{ So,}$$

$$P_{normal}(H, G) = \frac{1}{|H||G|} \sum_{g \in G} |N_H(g)|$$

Theorem $H_1 \leq G_1$ and $H_2 \leq G$. Then

$$P_{normal}(H_1 \times H_2, G_1 \times G_2) = P_{normal}(H_1, G_1) \cdot P_{normal}(H_2, G_2)$$

Theorem Let G be a finite group and H be a subgroup of G. Assume that p is the smallest prime number dividing $|H|$. Then

$$\frac{1}{[G:H]}+\frac{1}{|H|}-\frac{1}{|G|} \leq P_{normal}(H,G) \leq \frac{1}{p}+\left(1-\frac{1}{p}\right)\frac{1}{[G:N_G(H)]}$$

(note that $N_G(H)=\{g\in G \mid H^g=H^g\}$)

Theorem $H \leq G$. Then

$$d(G) \leq d(H,G) \leq P_{normal}(H,G)$$

Theorem Let H and K be two subgroups of G such that $K \trianglelefteq G$ and $K \leq H$.

Then $P_{normal}(H,G) = P_{normal}\left(\frac{H}{K}, \frac{G}{K}\right)$

Theorem Let G be a finite simple group. Then for every subgroup H of G

$$P_{normal}(H,G) \leq \frac{8}{15}$$

Exercise Let D_{2n} be dihedral group of order $2n$ and H be a non-normal subgroup of D_{2n}. Then prove that

$$P_{normal}(H,G) = \frac{\left\lfloor \frac{[D_{2n}:H]}{2} \right\rfloor + 1}{[D_{2n}:H]}$$

In particular, for $n=4$ and

$$D_8 = \langle a, b \mid a^4 = b^2 = e,\ bab = a^{-1} \rangle$$
$$= \{e, a, a^2, a^3, b, ab, a^2b, a^3b\}$$

$$H = \{e, b\}, \quad [D_{2n}:H] = \frac{2n}{2} = n = 4$$

$$P_{normal}(H,G) = \frac{\left\lfloor \frac{n}{2} \right\rfloor + 1}{n} = \frac{2+1}{4} = \frac{3}{4}.$$

(5) cyclicity Degree of a group

Definition Let G be a group. Then for element $x \in G$, the cyclic subgroup $\langle x \rangle$ is a subgroup that contains all power of x.

In other words,

$$\langle x \rangle = \{x^n \mid n \in \mathbb{Z}\}$$

The group G is called cyclic group if there exists an element $x \in G$ such that $G = \langle x \rangle$.

Example Z_4 is a cyclic group of order 4. We have $Z_4 = \langle x \mid x^4 = e \rangle = \{e, x, x^2, x^3\}$

* Every subgroup of a cyclic group is cyclic.

* If G is a cyclic group of order p, then we have $G = \langle x \mid x^p = e \rangle = \{e, x, x^2, --, x^{p-1}\}$ and one can see that $G = \langle x \rangle = \langle x^2 \rangle = \langle x^3 \rangle = -- = \langle x^{p-1} \rangle$.

Definition Let G be a finite group. Then the cyclicity degree of G, denoted by

$$P_{cyclic}(G) = \frac{|\{(x,y) \in G \times G \mid \langle x,y \rangle \text{ is cyclic}\}|}{|G|^2}$$

That is a probability of a random pair (x,y) of element of G such that the subgroup generated by x and y, $\langle x,y \rangle$ is cyclic.

It is clear that if G is cyclic, then $P_{cyclic}(G) = 1$.

Exercise Is the converse is true?

Example Let $G = \{e, x, y, xy\}$ is a Klein group. We know that G is not cyclic group, $|x| = |y| = |xy| = 2$. Moreover, G can be generated by x and y that is $\langle x,y \rangle = G$. Thus, we have

$\langle e, e\rangle = \langle e\rangle$ is cyclic $\Rightarrow (e,e)$ ✓

$\langle e, x\rangle = \langle x\rangle$ is cyclic $\Rightarrow (e,x)$ ✓

$\langle e, y\rangle = \langle y\rangle$ is cyclic $\Rightarrow (e,y)$ ✓

$\langle e, xy\rangle = \langle xy\rangle$ is cyclic $\Rightarrow (e,xy)$ ✓

$\langle x, e\rangle = \langle x\rangle$ " " $\Rightarrow (x,e)$ ✓

$\langle x, x\rangle = \langle x\rangle$ " " $\Rightarrow (x,x)$ ✓

$\langle x, y\rangle = G$ is not cyclic $\Rightarrow (x,y)$ ✗

$\langle x, xy\rangle = G$ is not cyclic $\Rightarrow (x,xy)$ ✗

$\langle y, e\rangle = \langle y\rangle$ is cyclic $\Rightarrow (y,e)$ ✓

$\langle y, x\rangle = G$ is not cyclic $\Rightarrow (y,x)$ ✗

$\langle y, y\rangle = \langle y\rangle$ is cyclic $\Rightarrow (y,y)$ ✓

$\langle y, xy\rangle = G$ is not cyclic $\Rightarrow (y,xy)$ ✗

$\langle xy, e\rangle = \langle xy\rangle$ is cyclic $\Rightarrow (xy,e)$ ✓

$\langle xy, x\rangle = G$ is not cyclic $\Rightarrow (xy,x)$ ✗

$\langle xy, y\rangle = G$ is not cyclic $\Rightarrow (xy,y)$ ✗

$\langle xy, xy\rangle = \langle xy\rangle$ is cyclic $\Rightarrow (xy,xy)$ ✓

$$P_{cyclic}(G) = \frac{10}{16} = \frac{5}{8}$$

Example $G = S_3 = \{e, (12), (13), (23), (123), (132)\}$

We know that

$$\langle (12),(13)\rangle = \langle (12),(23)\rangle = \langle (12),(123)\rangle = \langle (12),(132)\rangle$$

$$= \langle (13),(23)\rangle = \langle (13),(123)\rangle = \langle (13),(132)\rangle = \cdots = S_3$$

and so they are not cyclic. The only cyclic cases are

e —— all 6 elements 6 cases

(12) —— with $e, (12)$ $\Longrightarrow$ 2 cases

(13) —— with $e, (13)$ $\Longrightarrow$ 2 cases

(23) —— with $e, (23)$ $\Longrightarrow$ 2 cases

(123) —— with $e, (123), (132)$ $\Longrightarrow$ 3 cases

(132) —— with $e, (132), (123)$ $\Longrightarrow$ 3 cases

18 cases

$$P_{cyclic}(S_3) = \frac{18}{36} = \frac{1}{2}$$

Lemma $$\frac{3|G|-2}{|G|^2} \leq P_{cyclic}(G)$$

proof It is obvious that $\langle e, x\rangle$ = cyclic group for all $x \in G$. Moreover if $e \neq x \in G$, then $\langle x, e\rangle = \langle x, x\rangle = \langle x\rangle$ is cyclic. So, the proof follows.

Definition For every element $x \in G$

$$\mathrm{cyc}_G(x) = \{ y \in G \mid \langle x, y \rangle \text{ is cyclic} \}$$

$$\mathrm{cyc}(G) = \bigcap_{x \in G} \mathrm{cyc}_G(x) = \{ y \in G \mid \langle x, y \rangle \text{ is cyclic } \forall y \in G \}$$

Question Is $\mathrm{cyc}_G(x)$ is a subgroup of G? How about $\mathrm{cyc}(G)$?

$$P_{cyclic}(G) = \frac{1}{|G|^2} \sum_{x \in G} |\mathrm{cyc}_G(x)|$$

Suggestion for research

1. Compute suitable lower and upper bounds for $P_{cyclic}(G)$?

2. Determine $P_{cyclic}(D_{2n})$ for $n \geq 3$?

3. Compare $d(G)$ and $P_{cyclic}(G)$?

⑥ Permutability Degree of a group

Definition Let G be a group and H, K be two subgroups of G. Then we say that two subgroups H and K are permutable if and only if $HK = KH$.

Definition Let G be a group. Then the set of all subgroups of G will be denoted by $L(G)$. So, we have

$$L(G) = \{H \leq G \mid H \text{ is a subgroup of } G\}$$

Definition Let G be a finite group. Then the permutability degree of G, denoted by $P(G)$ is defined as

$$P(G) = \frac{|\{(H,K) \in L(G) \times L(G) \mid HK = KH\}|}{|L(G)|^2}$$

Example $S_3 = \{e, (12), (13), (23), (123), (132)\}$

Subgroups of S_3 are:

$H_1 = \{e\}$, $H_2 = \{e, (12)\}$, $H_3 = \{e, (13)\}$

$H_4 = \{e, (23)\}$, $H_5 = \{e, (123), (132)\}$, $H_6 = S_3$

So, $L(S_3) = \{H_1, H_2, H_3, H_4, H_5, H_6\}$

Now, we have:

$H_1 = \{e\} \longrightarrow H_1 H_i = H_i H_1 \quad i = 1, 2, \dots, 6$ (6 cases)

$H_2 = \{e, (12)\} \longrightarrow$
$H_2 H_1 = H_1 H_2 \rightarrow (H_2, H_1)$
$H_2 H_2 = H_2 H_2 \rightarrow (H_2, H_2)$
$H_2 H_5 = H_5 H_2 \rightarrow (H_2, H_5)$
$H_2 H_6 = H_6 H_2 \rightarrow (H_2, H_6)$

$H_3, H_4 \longrightarrow$ Similar to H_2 (4 cases)

$H_5 \longrightarrow H_5 H_i = H_i H_5 \quad 1 \leq i \leq 6$ (6 cases)

$H_6 \longrightarrow H_6 H_i = H_i H_6 \quad 1 \leq i \leq 6$ (6 cases)

$$p(S_3) = \frac{6+4+4+4+6+6}{36} = \frac{30}{36} = \frac{5}{6}$$

Note If $H \trianglelefteq G$, then $HK = KH$ for all $K \in L(G)$.

Definition A group G is called Dedekind group if all subgroups of G are normal.

Some results

1. If G is an abelian group, then $p(G) = 1$.
2. If G is a Dedekind group, then $p(G) = 1$.
3. $$p(G) = \frac{|\{(H,K) \in L(G) \times L(G) \mid HK \in L(G)\}|}{|L(G)|^2}$$
   (note that $HK \leq G \iff HK = KH$)
4. $$p(D_{2p}) = \frac{7p+9}{(p+3)^2}$$, p add prime

problem Determine $p(D_{2n})$, for all $n \geq 3$?