Applications of Group Theory in Graph Theory

By:

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Part 2 :

Some Graphs Associated to Groups

Monday	25 January, 2021	2.00 p.m. - 4.00 p.m.
Wednesday	27 January, 2021	3.00 p.m. - 5.00 p.m.

Some probabilities in group theory

1. Commutativity degree

$$d(G) = \frac{1}{|G|^2} |\{(x,y) \in G \times G \mid xy = yx\}|$$

2. Relative commutativity degree

$$d(H,G) = \frac{1}{|H||G|} |\{(h,g) \in H \times G \mid hg = gh\}|$$

3. Conjugate degree

$$P_{conj}(G) = \frac{1}{|G|^2} |\{(x,y) \in G \times G \mid x \text{ and } y \text{ are conjugate}\}|$$

4. Normality degree

$$P_{normal}(H,G) = \frac{1}{|H||G|} |\{(h,g) \in H \times G \mid h^g \in H\}|$$

5. cyclicity degree

$$P_{cyclic}(G) = \frac{1}{|G|^2} |\{(x,y) \in G \times G \mid \langle x,y \rangle \text{ is cyclic}\}|$$

6. permutability degree

$$p(G) = \frac{1}{|L(G)|^2} |\{(H,K) \in L(G) \times L(G) \mid HK = KH\}|$$

Some basic concepts in graph theory

Graph (V, E) V = vertex set, E = edge set

Loop

multiple edges

Simple graph = no loop + no multiple edges

directed graph $x \longrightarrow y$

undirected graph $x \text{———} y$

Connected graph

$x - t_1 - t_2 - \cdots - t_k - y$

distance $d(x,y)$ = the length of shortest path between x and y

diameter $\mathrm{diam}(\Gamma) = \max\{d(x,y) \mid x, y \in V(\Gamma)\}$

girth $\mathrm{girth}(\Gamma)$ = length of shortest cycle

1 Commuting and Non-Commuting graph

Definition Let G be a finite group and $Z(G)$ be the centre of G. Then the Non-commuting graph of G, denoted by Γ_G is an undirected simple graph whose vertices are non-central elements (elements in $G \setminus Z(G)$) and two vertices x and y are adjacent if and only if $xy \neq yx$.

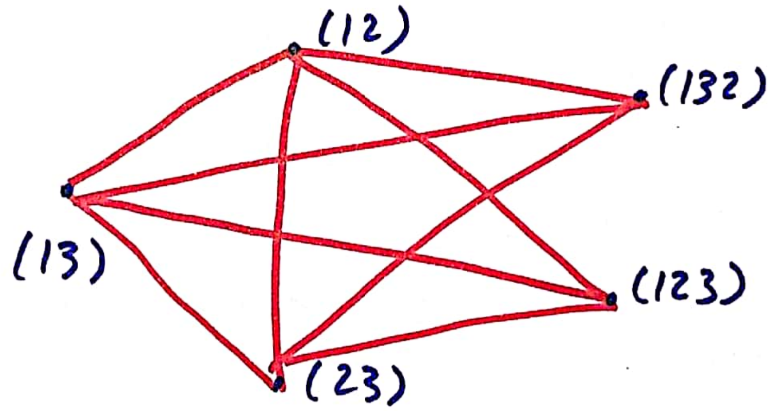
Definition Commuting graph of G is the complement of non-commuting graph. In other words, vertex set is $G \setminus Z(G)$, two vertices x and y are adjacent if and only if $xy = yx$.

Example $S_3 = \{e, (12), (13), (23), (123), (132)\}$

$Z(S_3) = \{e\}$.

Γ_{S_3} : Non-Commuting graph of S_3

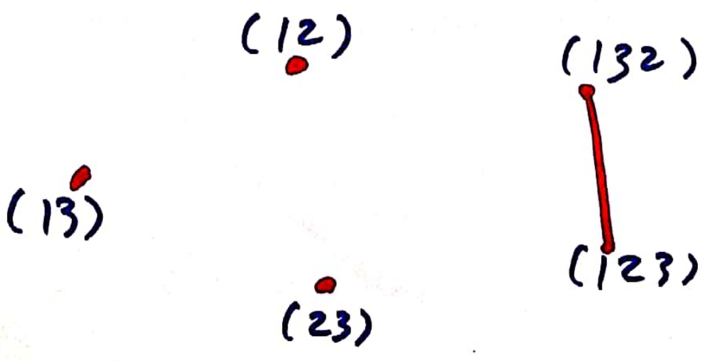
$V(\Gamma_{S_3}) = \{(12), (13), (23), (123), (132)\}$



$|E(\Gamma_{S_3})| = 9$

note that $(123)(132) = (132)(123)$

$\overline{\Gamma}_{S_3}$: Commuting graph of S_3



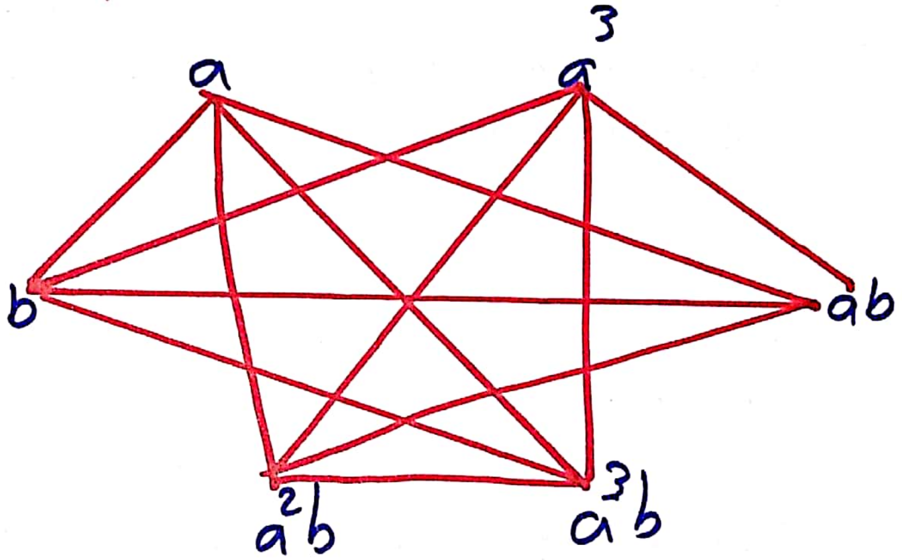
$|E(\overline{\Gamma}_{S_3})| = 1$

Example

$$D_8 = \langle a, b \mid a^4 = b^2 = e,\ bab = a^{-1} \rangle$$
$$= \{e, a, a^2, a^3, b, ab, a^2b, a^3b\}$$

$$Z(D_8) = \{e, a^2\}$$

Γ_{D_8}:



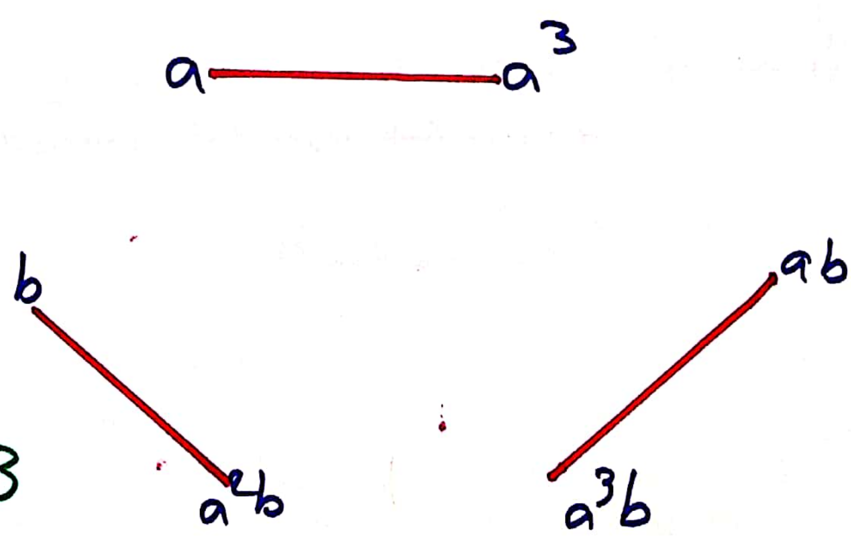
$|E(\Gamma_{D_8})| = 12$

For example $b \nsim a^2b$ because, we have

$b a^2 b = a^2 b b = a^2$, $a^2 b b = a^2 \Longrightarrow b(a^2b) = (a^2b)b$

$\downarrow$
$\in Z(D_8)$

$\overline{\Gamma}_{D_8}$:



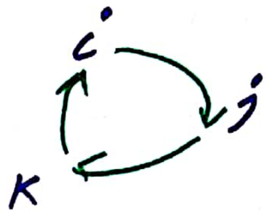
$|E(\overline{\Gamma}_{D_8})| = 3$

Example

$Q_8=\{1,-1,i,-i,j,-j,k,-k\}$

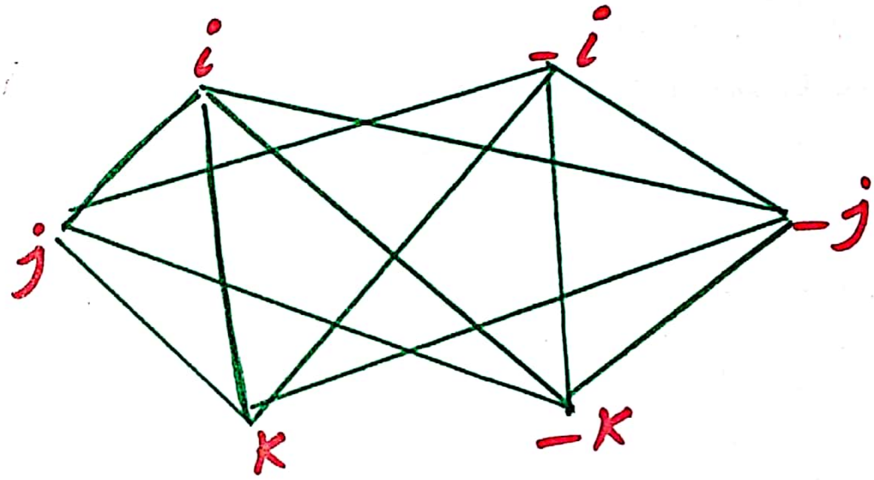
$i^2=-1$
$j^2=-1$
$k^2=-1$

$ij=k$
$jk=i$
$ki=j$



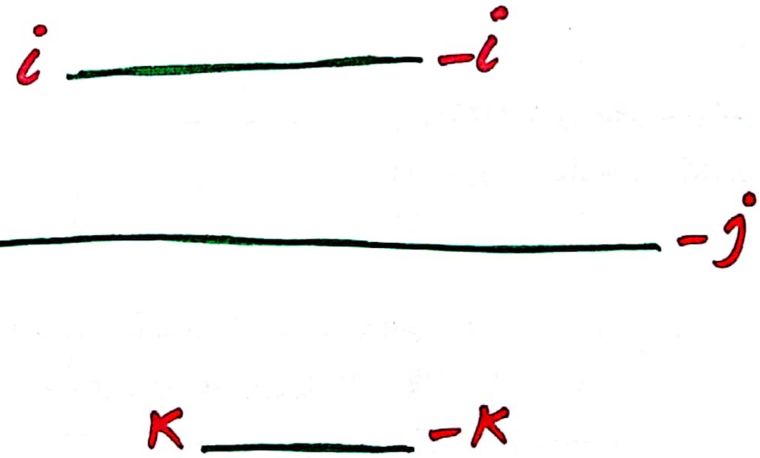
$Z(Q_8)=\{1,-1\}$

Γ_{Q_8} :



$|E(\Gamma_{Q_8})|=12$

$\overline{\Gamma_{Q_8}}$:



$|E(\overline{\Gamma_{Q_8}})|=3$

Relation between commutativity degree and commuting & non-commuting graphs

As I mentioned before,

$$d(G) = \frac{|\{(x,y) \in G \times G \mid xy = yx\}|}{|G|^2}$$

suppose that $A = \{(x,y) \in G \times G \mid xy = yx\}$
$B = \{(x,y) \in G \times G \mid xy \neq yx\}$

Then $G \times G = \{(x,y) \mid x, y \in G\} =$

$\{(x,y) \in G \times G \mid xy = yx\} \cup \{(x,y) \in G \times G \mid xy \neq yx\}$

$\Longrightarrow \quad G \times G = A \cup B \quad \Longrightarrow |G|^2 = |A| + |B|$

$d(G) = \frac{|A|}{|G|^2} \Longrightarrow \quad |A| = |G|^2 d(G) \quad (1)$

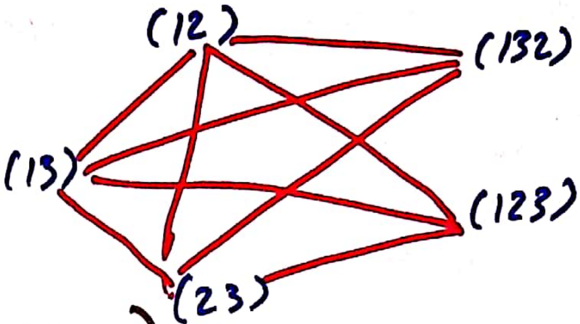
$|B| = 2|E(\Gamma_G)| \quad (2)$

$|G|^2 = |A| + |B| = |G|^2 d(G) + 2|E(\Gamma_G)|$

$\Longrightarrow \boxed{|E(\Gamma_G)| = \frac{1}{2}|G|^2(1 - d(G))}$

Similarly, we can find $|E(\overline{\Gamma_G})|$ in terms of $d(G)$.

Example Γ_{S_3}

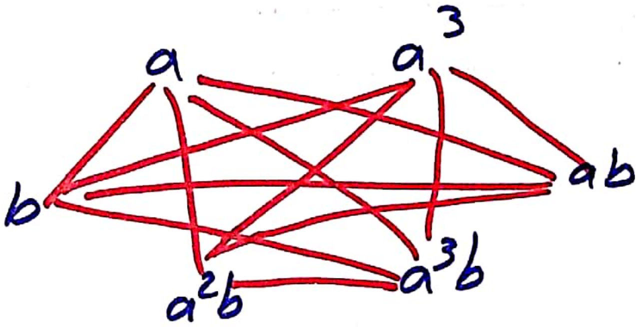


$d(S_3) = \frac{1}{2}$

$|E(\Gamma_{S_3})| = \frac{1}{2}|S_3|^2(1-d(S_3))$

$= \frac{1}{2}(36)(1-\frac{1}{2}) = 9$

Example Γ_{D_8}



$|E(\Gamma_{D_8})| = 12$

$12 = |E(\Gamma_{D_8})| = \frac{1}{2}|D_8|^2(1-d(D_8))$

$= \frac{1}{2}(64)(1-d(D_8)) = 32(1-d(D_8))$

$= 32 - 32\,d(D_8) \implies 32\,d(D_8) = 20$

$\implies d(D_8) = \frac{20}{32} = \frac{5}{8}$

Similarly, $d(Q_8) = \frac{5}{8}$

Proposition We can see that knowing $d(G)$ will determine $|E(\Gamma_G)|$ and conversely. Moreover, lower bound (upper bound) for $d(G)$ will deduce upper bound (lower bound) for $|E(\Gamma_G)|$.

Theorem Let G be a finite non-abelian group. Then $|E(\Gamma_G)| \geq \frac{3}{16}|G|^2$.

Proof We know that for non-abelian finite group G, we have $d(G) \leq \frac{5}{8}$.

Thus, we have

$$|E(\Gamma_G)| = \frac{1}{2}|G|^2(1-d(G)) \geq \frac{1}{2}|G|^2(1-\frac{5}{8})$$
$$= \frac{1}{2}|G|^2(\frac{3}{8}) = \frac{3}{16}|G|^2$$

Ig For Γ_{D_8}, we can see that

$$12 = |E(\Gamma_{D_8})| \geq \frac{3}{16}|D_8|^2 = \frac{3 \times 8^2}{16} = 12$$

Theorem Let G be a finite non-abelian group. Then

$$d(G) \geq 2\frac{|Z(G)|}{|G|} + \frac{1}{|G|} - \frac{|Z(G)|^2}{|G|^2} - \frac{|Z(G)|}{|G|^2}$$

proof We know that if we have a graph with n vertices, then the number of edges is at most $\frac{n(n-1)}{2}$. Now, consider Γ_G. We have $|V(\Gamma_G)| = |G| - |Z(G)|$, so by using the above upper bound for $|E(\Gamma_G)|$ will deduce that

$$\frac{1}{2}|G|^2(1-d(G)) = |E(\Gamma_G)| \leq \frac{(|G|-|Z(G)|)(|G|-|Z(G)|-1)}{2}$$

and therfore $|G|^2 - |G|^2 d(G) \leq (|G|-|Z(G)|)(|G|-|Z(G)|-1)$

$$\Longrightarrow d(G) \geq \frac{2|Z(G)|}{|G|} + \frac{1}{|G|} - \frac{|Z(G)|^2}{|G|^2} - \frac{|Z(G)|}{|G|^2}$$

For S_3: $\frac{1}{2} = d(S_3) \geq \frac{2}{6} + \frac{1}{6} - \frac{1}{36} - \frac{1}{36} = \frac{4}{9}$

$|Z(S_3)| = 1$, $|S_3| = 6$

$\frac{1}{2} > \frac{4}{9}$ ✓

Theorem Let G be a finite non-abelian group. Then

(i) Γ_G has no isolated vertex.

(ii) Γ_G is connected.

(iii) $diam(\Gamma_G) = 2$

(iv) $girth(\Gamma_G) = 3$

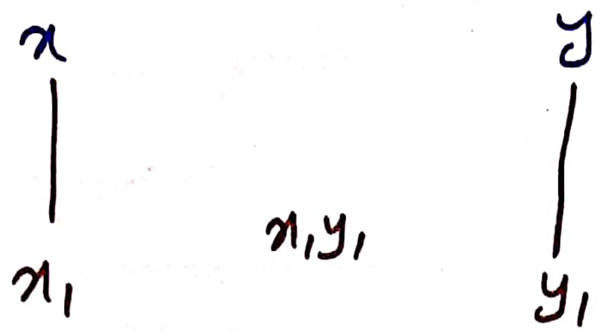
(v) Γ_G is Hamiltonian

(vi) Γ_G is planar $\Longleftrightarrow G \cong S_3$ or D_8 or Q_8

proof

(i) $x \in V(\Gamma_G) = G - Z(G) \implies x \notin Z(G)$
$\implies \exists y \in G$ such that $xy \neq yx$. Since $xy \neq yx \implies y \notin Z(G) \implies y \in V(\Gamma_G)$. Moreover $xy \neq yx \implies x$ is adjacent to $y \implies$ $deg(x) \geq 1$. Hence x is not isolated vertex.

(ii) Take two arbitrary vertices x and y. If x is adjacent to y, then we have a path. otherwise we will have the path *



By (i), $\deg(x) \geq 1$, $\deg(y) \geq 1$. So, there are vertices x_1, y_1 such that $x - x_1$ and $y - y_1$. If $x - y_1$, then $x - y_1 - y$ is a path. If $y - x_1$, then $y - x_1 - x$ is a path. If $x \not\!\!- y_1$ and $y \not\!\!- x_1$ then we will have the path

$$x \text{ —— } x_1y_1 \text{ —— } y$$

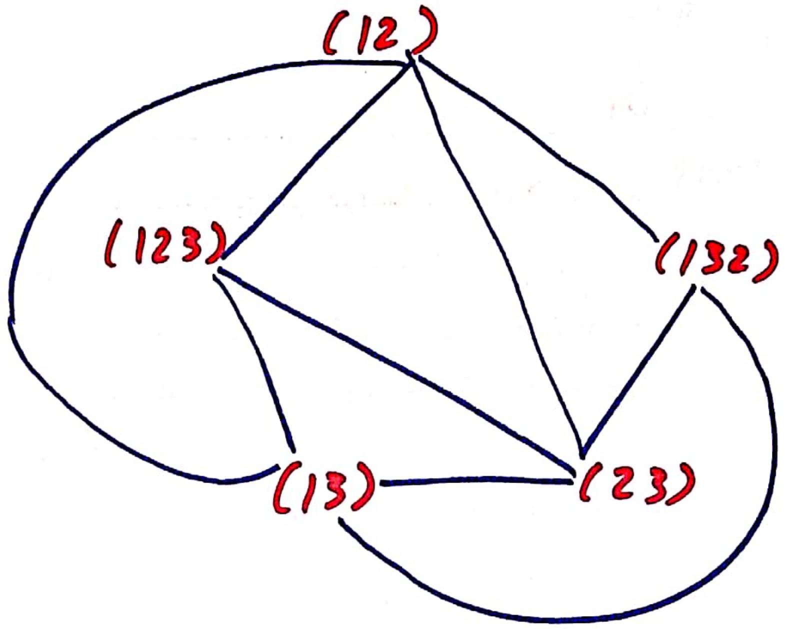
Thus we always have a path between x and y. Hence Γ_G is connected.

(iii), (iv) follows from (ii).

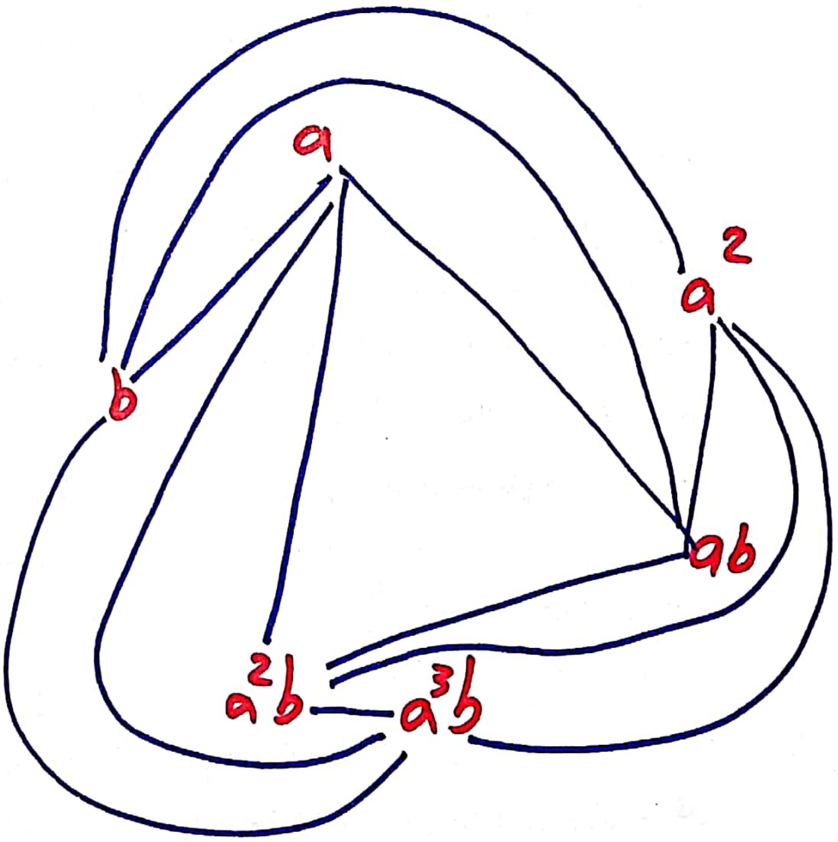
(v) It follows from $\deg(x) > \frac{|V(\Gamma_G)|}{2}$ for every vertex x.

$\Leftarrow$

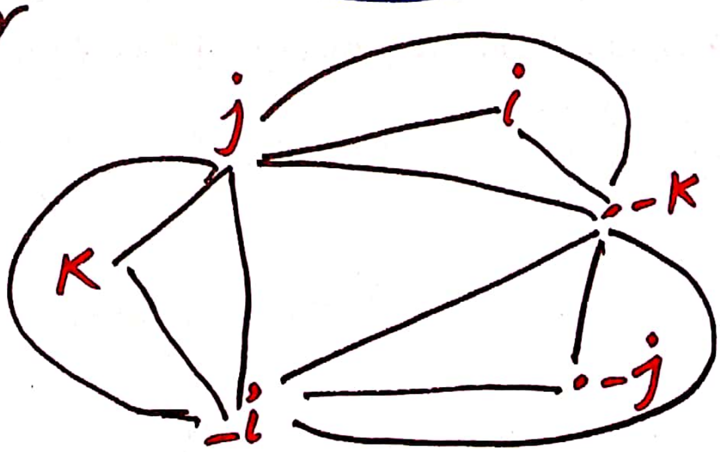
(vi) Γ_{S_3} is planar



Γ_{D_8} is planar



Γ_{Q_8} is planar



$\Longrightarrow$?

Some problems

1. Let G_1 and G_2 be two non-abelian finite group. Let $\Gamma_{G_1} \cong \Gamma_{G_2}$. IS $|G_1| = |G_2|$?

2. Let G be finite non-abelian simple group and $\Gamma_G \cong \Gamma_H$ for some group H. IS H non-abelian simple ?

3. Determine the graph structure of $\Gamma_{D_{2n}}$, Γ_{S_n}, Γ_{A_n} for $n \geq 3$?

4. IS there any group G such that
(i) Γ_G is complete graph
(ii) Γ_G is complete bipartite graph
(iii) Γ_G is regular graph.

2 Relative Non-commuting graph

Definition Let G be a finite group and H be a subgroup of G. Then the relative non-commuting graph of subgroup H of G, denoted by $\Gamma_{H,G}$ is an undirected simple graph where vertex set is $G \setminus C_G(H)$ and two vertices x and y are adjacent if and only if $xy \neq yx$ and at least one of x or y is in H.

In other words,

$V(\Gamma_{H,G}) = G - C_G(H)$

$C_G(H) = \{g \in G \mid gh = hg \ \forall h \in H\}$

$x \text{ — } y \iff (x \in H \text{ or } y \in H)$ & $xy \neq yx$

Example $S_3=\{e,(12),(13),(23),(123),(132)\}$

$H=\{e,(12)\}$

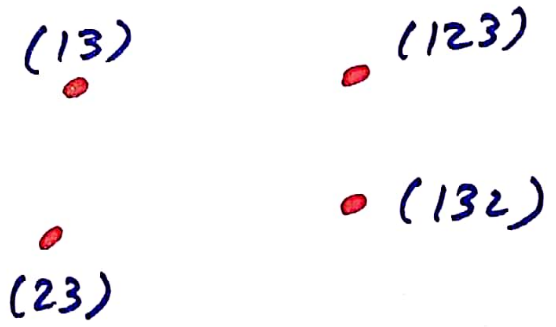
$K=\{e,(123),(132)\}$

$$C_{S_3}(H)=\{x\in S_3 \mid xh=hx \ \forall h\in H\} = \{e,(12)\}=H$$

$$C_{S_3}(K)=\{e,(123),(132)\}=K$$

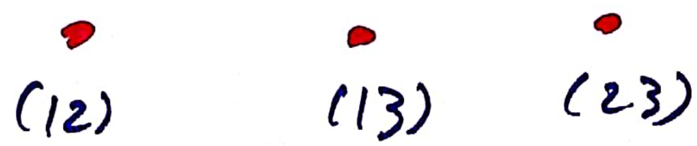
$$V(\Gamma_{H,G}) = S_3 \setminus C_{S_3}(H) = S_3\setminus H = \{(13),(23),(123),(132)\}$$

Γ_{H,S_3}:



$$V(\Gamma_{K,S_3}) = S_3\setminus K = \{(12),(13),(23)\}$$

Γ_{K,S_3}:



Note If H is an abelian subgroup of G, then $H \subseteq C_G(H)$ and so $V(\Gamma_{H,G})$ has no element of H. Hence $\Gamma_{H,G}$ has no edge and contains some isolated vertices.

Relation between relative commutativity degree and number of edges of $\Gamma_{H,G}$

Theorem Let G be a finite group and H be a subgroup of G. Then

$$|E(\Gamma_{H,G})| = |H||G|(1-d(H,G)) - \frac{|H|^2}{2}(1-d(H))$$

if H is non-abelian subgroup. If H is abelian, then $|E(\Gamma_{H,G})| = 0$

Theorem Let $\Gamma_{H,G}$ be a relative non-commuting graph of G. Then

$$|E(\Gamma_{H,G})| \geq \frac{1}{2}|H||G| - \frac{1}{4}|H|^2 - \frac{1}{4}|G||Z(H)| - \frac{1}{4}|H||C_G(H)| + \frac{1}{4}|Z(H)||H|$$

Theorem Let H be a non-abelian subgroup of G and p be the smallest prime number divides $|G|$. Then

$$|E(\Gamma_{H,G})| \leq |H|\left(|G| - \frac{3}{16}|H| - p\right) - |Z(G) \cap H|\,(|G| - p)$$

Theorem Let H be a non-abelian subgroup of G. If $\frac{H}{H \cap Z(G)} \cong Z_2 \times Z_2$, then

$$|E(\Gamma_{H,G})| \geq \frac{-3}{16}|H|^2 + \frac{3}{8}|H||G|$$

Theorem There is no non-abelian group G and subgroup H such that $\Gamma_{H,G}$ is complete graph.

Similarly, $\Gamma_{H,G}$ is not possible to be complete bipartite graph.

Theorem Let H be a non-abelian subgroup of G such that $\Gamma_{H,G} \cong \Gamma_{S_n}$. If $|C_G(H)| \leq n$, then $|G| = |S_n|$. In particular, $G \cong S_n$ for $n = 3, 4, 5$.

Theorem For non-abelian group G and its subgroup H with trivial centre, then

(i) $\operatorname{diam}(\Gamma_{H,G}) = 2$

(ii) $\operatorname{girth}(\Gamma_{H,G}) = 3$

Exercises

1. Draw graph Γ_{A_4,S_4} ?

Hint: we know that $d(A_4)=\frac{1}{3}$, $d(A_4,S_4)=\frac{1}{4}$. So, we have

$$|E(\Gamma_{A_4,S_4})| = |A_4||S_4|(1-d(A_4,S_4)) - \frac{|A_4|^2}{2}(1-d(A_4))$$

$$= (12)(\cancel{24})(1-\tfrac{1}{4}) - \frac{(12)^2}{2}(1-\tfrac{1}{3}) = 20$$

So, Γ_{A_4,S_4} has 20 edges.

2. Is $\Gamma_{H,G}$ Hamiltonian ?

3. When $\Gamma_{H,G}$ is planar ?

4. Suppose that $\Gamma_{H_1,G_1} \cong \Gamma_{H_2,G_2}$ and $|H_1 \setminus Z(H_1)| = |H_2 \setminus Z(H_2)|$. Is $|H_1| = |H_2|$?