Applications of Group Theory in Graph Theory

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Part 2 - b

Some Graphs Associated to Groups

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Same graphs associated to a group

① Γ_G non-commuting graph

$V(\Gamma_G) = G \setminus Z(G)$

$x — y \iff xy \neq yx$

$\overline{\Gamma}_G$ commuting graph

$V(\overline{\Gamma}_G) = G \setminus Z(G)$

$x — y \iff xy = yx$

② $\Gamma_{H,G}$ Relative non-commuting graph

$V(\Gamma_{H,G}) = G \setminus C_G(H)$

$x — y \iff (x \text{ or } y \in H) \& (xy \neq yx)$

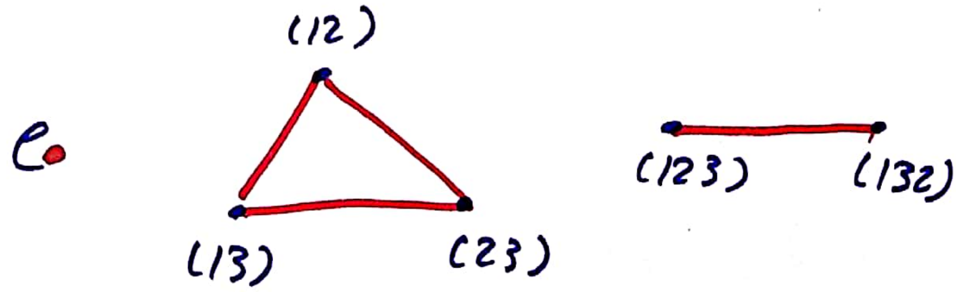
③ Conjugate graph of a group

Definition Let G be a finite group. Then the conjugate graph of G, denoted by $\Gamma^c(G)$ is an undirected simple graph whose vertices are all elements of G and two vertices x and y are adjacent if and only if x and y are conjugate in G.

$$V(\Gamma^c(G)) = G, \quad x — y \iff x^g = y$$
for some $g \in G$

Example $S_3 = \{e, (12), (13), (23), (123), (132)\}$

$\Gamma^c(S_3)$:



* Conjugacy classes of S_3

$e^G = \{e\}$, $(12)^G = \{(12), (13), (23)\}$, $(123)^G = \{(123), (132)\}$

$|e^G| = 1$, $|(12)^G| = 3$, $|(123)^G| = 2$

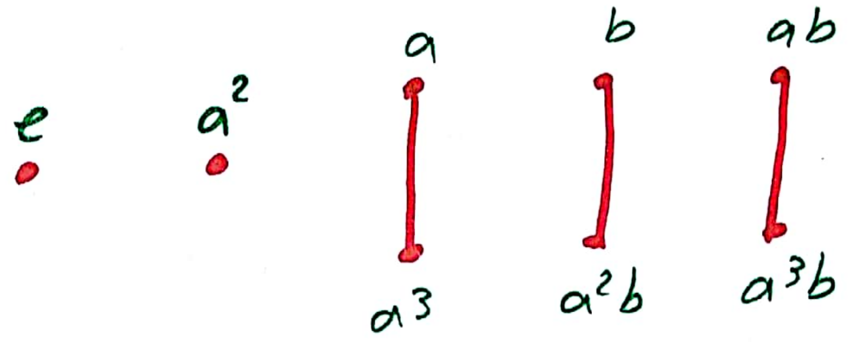
Example $D_8 = \langle a, b \mid a^4 = b^2 = e,\ bab = a^{-1} \rangle$
$= \{e, a, a^2, a^3, b, ab, a^2b, a^3b\}$

Conjugacy classes of D_8:

$e^G = \{e\}$, $a^G = \{a, a^3\}$, $(a^2)^G = \{a^2\}$

$b^G = \{b, a^2b\}$, $(ab)^G = \{ab, a^3b\}$

$\Gamma^c(D_8)$



$\downarrow$

$K_1 \cup K_1 \cup K_2 \cup K_2 \cup K_2 = 2K_1 \cup 3K_2$

Theorem Let G be a finite group with k distinct conjugacy classes $x_1^G, x_2^G, \ldots, x_k^G$ of length $m_1, m_2, \ldots, m_k$, respectively. Then

$$\Gamma^c(G) = K_{m_1} \cup K_{m_2} \cup \cdots \cup K_{m_k}$$

Theorem Let G be a finite group with k distinct conjugacy classes $x_1^G, x_2^G, \ldots, x_k^G$ of length $m_1, m_2, \ldots, m_k$, respectively. Then

(i) $P_{Conj}(G) = \frac{1}{|G|^2} \sum_{i=1}^{k} |x_i^G|^2$

(ii) $|E(\Gamma^c(G))| = \frac{1}{2}\left(\sum_{i=1}^{k} |x_i^G|^2 - |G|\right)$

(iii) $\chi(\Gamma^c(G)) = \max\{m_1, m_2, \ldots, m_k\}$

(iv) $\alpha(\Gamma^c(G)) = \max\{m_1, m_2, \ldots, m_k\}$

(v) $\gamma(\Gamma^c(G)) = k$

(vi) $\Gamma^c(G)$ is connected iff $G = \{e\}$.

(vii) $\Gamma^c(G)$ is planar iff $m_i \leq 4$ for all i, $1 \leq i \leq k$

(viii) $\Gamma^c(G) = K_{m_1} \cup K_{m_2} \cup \cdots \cup K_{m_k}$

(ix) $\overline{\Gamma^c(G)} = K_{m_1, m_2, \ldots, m_k}$

(x) If $m_1, m_2, \ldots, m_k \geq 3$, then $\text{girth}(\Gamma^c(G)) = \min\{m_1, m_2, \ldots, m_k\}$.

Relation between conjugate degree and conjugate graph

$$P_{Conj}(G) = \frac{|\{(x,y) \in G \times G \mid x \text{ conjugate to } y\}|}{|G|^2}$$

We know that x is always conjugate to x.

So, we will have

$$P_{Conj}(G)|G|^2 = |\{(x,y) \in G \times G \mid x \text{ conjugate to } y\}|$$

$$= |\{(x,y) \in G \times G \mid x \text{ conjugate to } y,\ x \neq y\}| + |G|$$

$$= 2|E(\Gamma^c(G))| + |G| \Longrightarrow$$

$$|E(\Gamma^c(G))| = \frac{1}{2}|G|\left(|G|P_{Conj}(G) - 1\right)$$

Example We know that $P_{Conj}(S_3) = \frac{14}{36}$

Thus $|E(\Gamma^c(S_3))| = \frac{1}{2} \times 6\left(6 \times \frac{14}{36} - 1\right)$

$= 3\left(\frac{7}{3} - 1\right) = 7 - 3 = 4$

Lemma If G is abelian group, then $\Gamma^c(G)$ is null graph.

proof As we mentioned before, for abelian group $P_{conj}(G)=\frac{1}{|G|}$. Now, we have

$$|E(\Gamma^c(G))| = \frac{1}{2}|G|(|G|P_{conj}(G)-1) = \frac{1}{2}|G|(|G|\frac{1}{|G|}-1) = \frac{1}{2}|G|(1-1) = 0$$

Hence, $\Gamma^c(G)$ is null graph.

Lemma Let G be a group with n conjugacy classes $x_1^G, x_2^G, \ldots, x_k^G, x_{k+1}^G, \ldots, x_n^G$ such that $x_i \in Z(G)$ $1 \leq i \leq k$, $x_i \notin Z(G)$ $k+1 \leq i \leq n$ and $|x_i^G| = m_i$, for $i=1,2,\ldots,n$. It is clear that $|x_i^G|=1$, for $1 \leq i \leq k$ and so $\Gamma^c(G)$ has the following structure

$$\Gamma^c(G) = K(K_1) \cup K_{m_{k+1}} \cup \cdots \cup K_n$$

Hence, $|E(\Gamma^c(G))| = \sum_{i=k+1}^{n} \frac{m_i(m_i-1)}{2}$.

Now, by formula $|E(\Gamma^c(G))| = \frac{1}{2}|G|(|G|P_{conj}(G)-1)$

we will have

$$\sum_{i=k+1}^{n} \frac{m_i(m_i-1)}{2} = \frac{1}{2}|G|(|G|P_{conj}(G)-1).$$

Thus $$\sum_{i=k+1}^{n} \frac{m_i(m_i-1)}{2} + \frac{1}{2}|G| = \frac{1}{2}|G|^2 P_{conj}(G)$$

Therefore, $$P_{conj}(G) = \frac{1}{|G|^2} \sum_{i=k+1}^{n} m_i(m_i-1) + \frac{1}{|G|}$$

Example Let $D_{12} = \langle a, b \mid a^6 = b^2 = e, bab = a^{-1} \rangle$

$= \{e, a, a^2, \ldots, a^5, b, ab, \ldots, a^5 b\}$. Then we have

the following conjugacy classes

$Z(D_{12}) = \{e, a^3\}$

$e^G = \{e\}$, $(a^3)^G = \{a^3\}$

$m_1 = 1$, $m_2 = 1$

$a^G = \{a, a^5\}$, $(a^2)^G = \{a^2, a^4\}$,

$m_3 = m_4 = 2$

$m_5 = m_6 = 3$

$b^G = \{b, a^2 b, a^4 b\}$

$k = 2$, $n = 6$

$(ab)^G = \{ab, a^3 b, a^5 b\}$

$$P_{Conj}(D_{12}) = \frac{1}{|D_{12}|^2} \sum_{i=k+1}^{6} m_i(m_i-1) + \frac{1}{|D_{12}|}$$

$$= \frac{1}{(12)^2}\left(m_3(m_3-1) + m_4(m_4-1) + m_5(m_5-1) + m_6(m-1)\right)$$

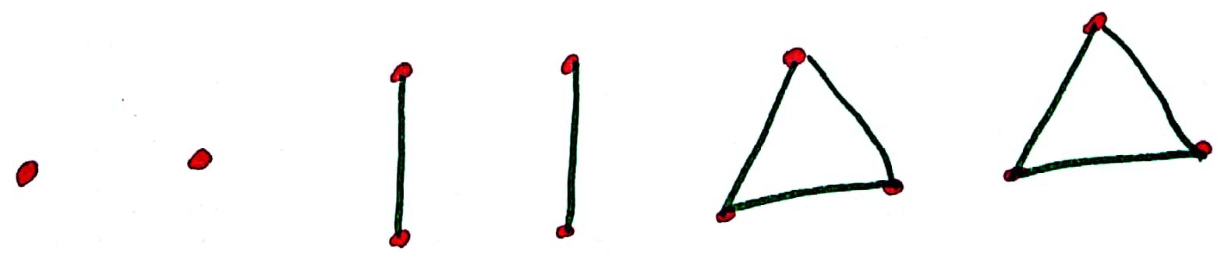
$$+ \frac{1}{12} = \frac{1}{(12)^2}\left(2(2-1)+2(2-1)+3(3-1)+3(3-1)\right)+$$

$$\frac{1}{12} = \frac{1}{(12)^2}(2+2+6+6)+\frac{1}{12} = \frac{16+12}{(12)^2} = \frac{28}{144}$$

$$\Longrightarrow P_{Conj}(D_{12}) = \frac{7}{36}$$

$\Gamma^c(D_{12})$ is the following graph

$$\Gamma^c(D_{12}) = 2K_1 \cup 2K_2 \cup 2K_3$$



$$|E(\Gamma^c(D_{12}))| = \frac{1}{2}|D_{12}|\left(|D_{12}|P_{Conj}(D_{12})-1\right)$$

$$= \left(\frac{1}{2}\right)(12)\left(12\left(\frac{7}{36}\right)-1\right) = 6\left(\frac{4}{3}\right) = 8$$

4 Non-normal graph of a group

Definition Let G be a finite group. Then for a subgroup H of G, we define the subgroups H_G and $N_G(H)$ as the following:

$$H_G = \bigcap_{g \in G} H^g \quad ; \quad N_G(H) = \{g \in G \mid H^g = H\}$$

It is clear that $H_G \subseteq H$ and $N_G(H) \subseteq G$. Now, assume that

$$A = H \setminus H_G \quad , \quad B = G \setminus N_G(H)$$

We define a bipartite graph $\Gamma_{H,G}$ as the following:

$V(\Gamma_{H,G}) = A \dot{\cup} B$, two vertices $h \in A$ and $g \in B$ are adjacent if and only if $h^g \notin H$.

Example

$S_3 = \{e, (12), (13), (23), (123), (132)\}$

$H = \{e, (12)\}$

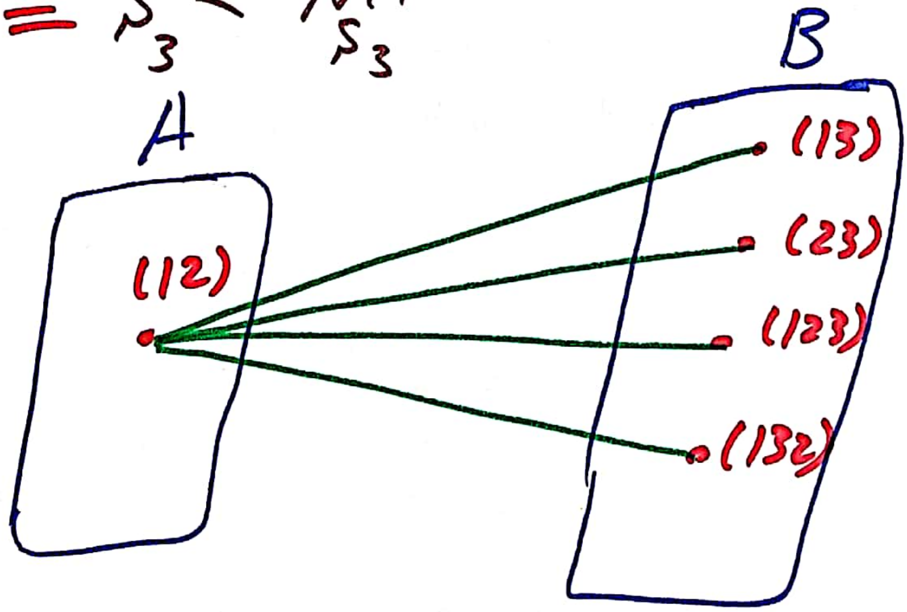
$$H_{S_3} = \bigcap_{g \in S_3} H^g = \{e, (12)\} \cap \{e, (23)\} \cap \{e, (13)\}$$

$$= \{e\}$$

$$N_{S_3}(H) = \{g \in S_3 \mid H^g = H\} = \{e, (12)\}$$

$$A = H - H_{S_3} = \{(12)\}$$

$$B = S_3 - N_{S_3}(H) = \{(13), (23), (123), (132)\}$$



$(12)^{(13)} = (13)^{-1}(12)(13) = (23) \notin H$

$(12)^{(23)} = (23)^{-1}(12)(23) = (13) \notin H$

$(12)^{(123)} = (132)(12)(123) = (13) \notin H$

$(12)^{(132)} = (123)(12)(132) = (23) \notin H$

Relation between normality degree and non-normal graph

$$P_{normal}(H,G) = \frac{|\{(h,g)\in H\times G \mid h^g \in H\}|}{|H||G|}$$

$$H\times G = \{(h,g)\in H\times G \mid h^g \in H\} \cup \{(h,g)\in H\times G \mid h^g \notin H\}$$

$$|H\times G| = |\{(h,g)\in H\times G \mid h^g \in H\}| + |\{(h,g)\in H\times G \mid h^g \notin H\}|$$

$$|H||G| = |H||G| P_{normal}(H,G) + |E(\Gamma_{H,G})|$$

$$\Longrightarrow \boxed{|E(\Gamma_{H,G})| = |H||G|(1 - P_{normal}(H,G))}$$

Example

$S_3 = \{e, (12), (13), (23), (123), (132)\}$

$H = \{e, (12)\}$

$$P_{normal}(H, S_3) = \frac{2}{3}$$

$$|E(\Gamma_{H,S_3})| = |H||S_3|(1 - P_{normal}(H,S_3))$$

$$= (2\times 6)(1 - \frac{2}{3}) = \frac{12}{3} = 4$$

Some results on $\Gamma_{H,G}$

Theorem $\Gamma_{H,G}$ has a pendant vertex if and only if $|H|=2$ and $\Gamma_{H,G}$ is star graph.

Theorem If $|H|>2$, then $\mathrm{girth}(\Gamma_{H,G})=4$

Theorem $\mathrm{diam}(\Gamma_{H,G}) \leq 4$.

Theorem $\mathrm{diam}(\Gamma_{H,G})=2$ if and only if $\Gamma_{H,G}$ is Complete bipartite.

Theorem If H_G is a maximal subgroup of H, then $\Gamma_{H,G}=K_{|H|-|H_G|,\,|G|-|N_G(H)|}$

Theorem $\alpha(\Gamma_{H,G})=|G|-|N_G(H)|$

5 Non-cyclic graph of a group

Definition Let G be a group. As we defined before,

$$cyc_G(x) = \{y \in G \mid \langle x, y \rangle \text{ is cyclic}\}$$

$$cycl(G) = \bigcap_{x \in G} cyc(x)$$

$$= \{y \in G \mid \langle x, y \rangle \text{ is cyclic for all } x \in G\}$$

The non-cyclic graph of G, denoted by Γ_G is a graph with vertex set $V(\Gamma_G) = G \setminus cyc(G)$

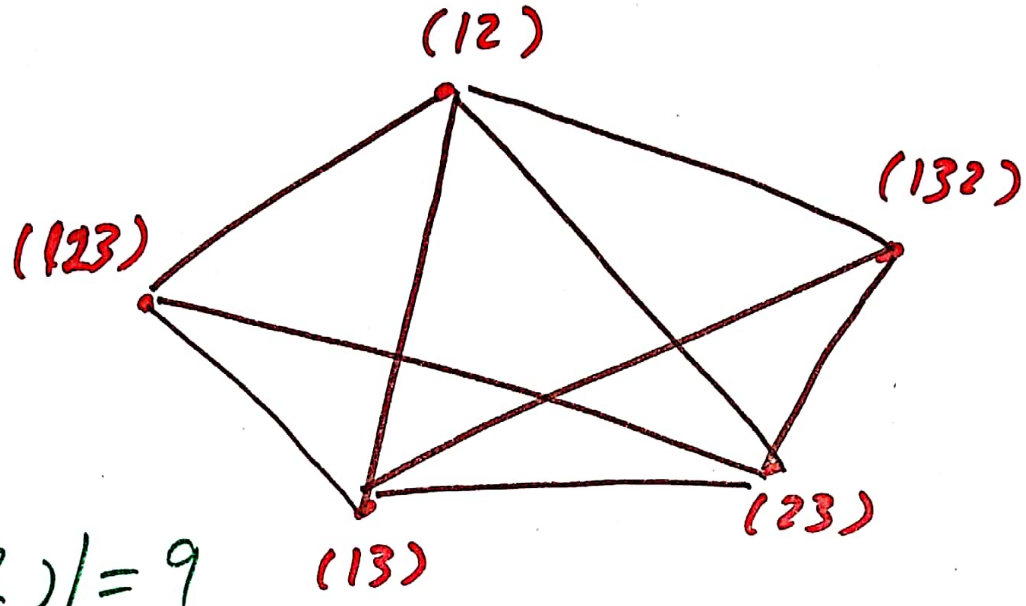
Two vertices x and y are adjacent if and only if $\langle x, y \rangle$ is not cyclic

Example $S_3 = \{e, (12), (13), (23), (123), (132)\}$

$$cyc(S_3) = \{e\}$$

$V(\Gamma_{S_3}) = S_3 \setminus cyc(S_3) = \{(12),(13),(23),(123),(132)\}$

Γ_{S_3}:

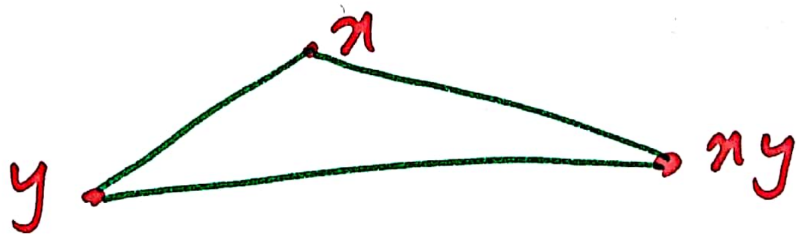


$|E(\Gamma_{S_3})| = 9$

Example Klein group of order 4

$G = \{e, x, xy, y\}$ $|x| = |y| = |xy| = 2$

$cyc(G) = \{e\}$ $V(\Gamma_G) = \{x, y, xy\}$



$|E(\Gamma_G)| = 3$

Relation between cyclicity degree and non-cyclic graph

Let G be a finite group. Then

$$P_{cyclic}(G) = \frac{|\{(x,y) \in G \times G \mid \langle x,y \rangle \text{ is cyclic}\}|}{|G|^2}$$

$$G \times G = \{(x,y) \in G \times G \mid \langle x,y \rangle \text{ is cyclic}\} \cup \{(x,y) \in G \times G \mid \langle x,y \rangle \text{ is not cyclic}\}$$

$$|G|^2 = |G \times G| = |\{(x,y) \in G \times G \mid \langle x,y \rangle \text{ is cyclic}\}| + |\{(x,y) \in G \times G \mid \langle x,y \rangle \text{ is not cyclic}\}| = |G|^2 P_{cyclic}(G) + 2|E(\Gamma_G)|$$

$$\boxed{|E(\Gamma_G)| = \frac{1}{2}|G|^2(1 - P_{cyclic}(G))}$$

Example We have already computed that $P_{cyclic}(S_3) = \frac{1}{2}$. Thus,

$$|E(\Gamma_{S_3})| = \frac{1}{2}|S_3|^2(1 - P_{cyclic}(S_3))$$

$$= \frac{1}{2}(36)(1-\frac{1}{2}) = \frac{1}{2}\times 36 \times \frac{1}{2} = \frac{36}{4} = \boxed{9}$$

Example G = Klein group of order 4

$P_{cyclic}(G) = \frac{5}{8}$, thus

$$|E(\Gamma_G)| = \frac{1}{2}(4^2)(1-\frac{5}{8}) = \frac{16}{2}(\frac{3}{8}) = \boxed{3}$$

Lemma $|E(\Gamma_G)| \leq \frac{1}{2}(|G|^2 - 3|G| + 2)$

proof As we proved before,

$$P_{cyclic}(G) \geq \frac{3|G|-2}{|G|^2}$$

Thus, we have

$$|E(\Gamma_G)| = \frac{1}{2}|G|^2(1-P_{cyclic}(G)) \leq \frac{1}{2}|G|^2(1 - \frac{3|G|-2}{|G|^2}) = \frac{1}{2}(|G|^2 - 3|G| + 2)$$

Example

$$9 = |E(\Gamma_{S_3})| \leq \frac{1}{2}(36-18+2) = 10$$

$$3 = |E(\Gamma_G)| \leq \frac{1}{2}(16-12+2) = 3$$

Definition G is called locally cyclic if every subgroup of G is cyclic. (proper subgroup)

Note If G is locally cyclic, then Γ_G is null graph.

Thus, we always assume that G is not locally cyclic

Theorem G is not locally cyclic. Then $\operatorname{diam}(\Gamma_G) = 1 \Longleftrightarrow G = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2$

Theorem G is not locally cyclic + G is nilpotent $\Longrightarrow \operatorname{diam}(\Gamma_G) \leq 2$.

Theorem G is not locallycyclic and is torsion free $\Longrightarrow \operatorname{diam}(\Gamma_G) = 2$.

6 permutable graph of a group

Definition Let G be a group and $L(G)$ be the set of all subgroups of G. Then the permutable graph of G, denoted by $\Gamma_{perm}(G)$ is a group whose vertices of all subgroups of G and two distinct subgroups H and K are adjacent if and only if $HK = KH$.

$$V(\Gamma_{perm}(G)) = L(G)$$

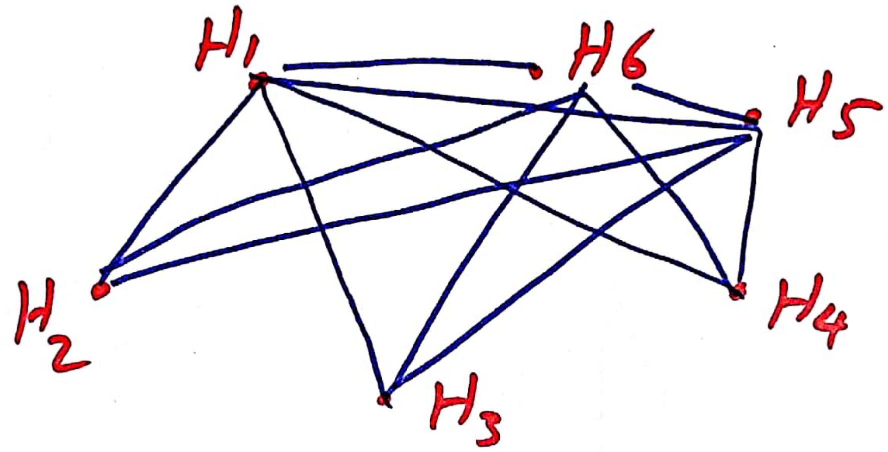
$$H \text{ --- } K \iff HK = KH$$

Example $S_3 = \{e, (12), (13), (23), (123), (132)\}$

$H_1 = \{e\}$, $H_2 = \{e, (12)\}$, $H_3 = \{e, (13)\}$, $H_4 = \{e, (23)\}$

$H_5 = \{e, (123), (132)\}$, $H_6 = S_3$

$V(\Gamma_{perm}(S_3)) = L(S_3) = \{H_1, H_2, H_3, H_4, H_5, H_6\}$



$H_2 H_3 = \{e, (12)\}\{e, (13)\} = \{e, (12), (13), (132)\}$

$H_3 H_2 = \{e, (13)\}\{e, (12)\} = \{e, (12), (13), (123)\}$

$H_2 H_3 \neq H_3 H_2 \implies H_2 \not\sim H_3$

$|E(\Gamma_{perm}(S_3))| = 12$

Some facts

1. Identity subgroup $\{e\}$ is adjacent to all other subgroups of G.
2. G as a subgroup is also adjacent to all subgroups of G.
3. If H is a normal subgroup of G,

then H will adjacent to all subgroups of G.

$$\deg(\{e\}) = \deg(G) = \deg(H) = |L(G)| - 1$$

Example $D_8 = \langle a, b \mid a^4 = b^2 = e,\ bab = a^{-1} \rangle$

$= \{e, a, a^2, a^3, b, ab, a^2b, a^3b\}$

subgroups of D_8 are the following:

$H_1 = \{e\}$

$H_2 = \{e, a^2\}$

$H_3 = \{e, b\}$

$H_4 = \{e, ab\}$

$H_5 = \{e, a^2b\}$

$H_6 = \{e, a^3b\}$

$H_7 = \{e, a, a^2, a^3\}$

$H_8 = \{e, a^2, b, a^2b\}$

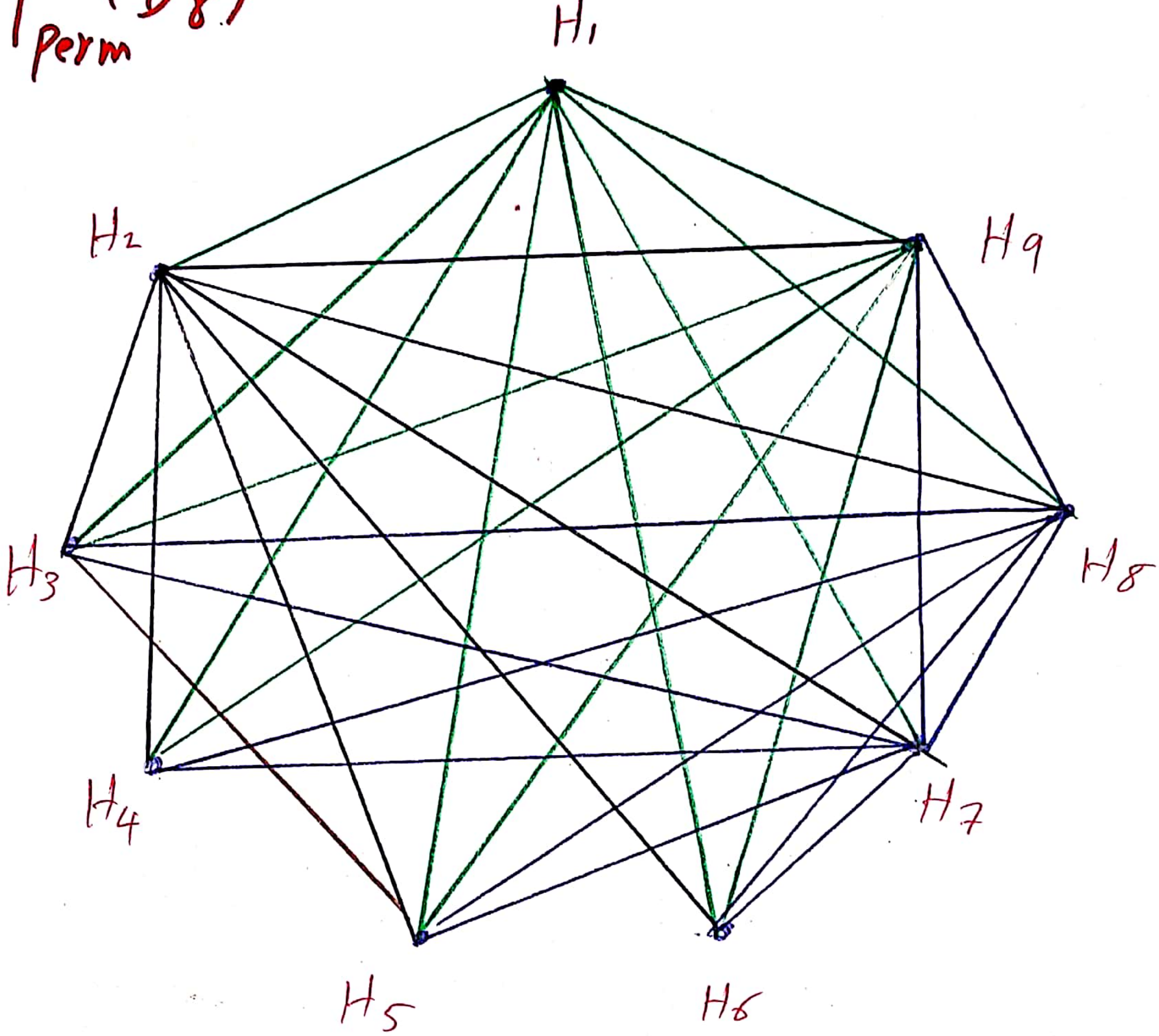
$H_9 = D_8$

Normal subgroups

H_1, H_2, H_7, H_8, H_9

$$V(\Gamma_{perm}(D_8)) = L(D_8)$$
$$= \{H_1, H_2, H_3, H_4, H_5, H_6, H_7, H_8, H_9\}$$

$\Gamma_{perm}(D_8)$



$H_3 H_5 = \{e, b\}\{e, a^2b\} = \{e, b, a^2b, a^2\}$

$H_5 H_3 = \{e, a^2b\}\{e, b\} = \{e, b, a^2b, a^2\}$

$\Longrightarrow H_3 — H_5$

$H_4 H_5 = \{e, ab\}\{e, a^2b\} = \{e, ab, a^2b, a^3\}$

$H_5 H_4 = \{e, a^2b\}\{e, ab\} = \{e, ab, a^2b, a\}$

$\Longrightarrow H_4 \not— H_5$

$|E(\Gamma_{perm}(D_8))| = 31$

Some properties of graph

Theorem Let G be finite group and $L(G)$ be the set of all subgroups of G. Then the graph $\Gamma_{perm}(G)$

(i) Connected

(ii) $diam(\Gamma_{perm}(G)) \leq 2$. Moreover, $diam(\Gamma_{perm}(G)) = 1$ if and only G is a Dedekind group.

(iii) If G is not a cyclic group of order prime number, then $girth(\Gamma_{perm}(G)) = 3$

(iv) $\gamma(\Gamma_{perm}(G)) = 1$

(v) $\chi(\Gamma_{perm}(G)) \geq 2$. If G is not simple, then $\chi(\Gamma_{perm}(G)) \geq 3$.

Relation between permutability degree and permutable graph of a group

$$p(G) = \text{Permutability degree}$$
$$= \frac{|\{(H,K) \in L(G) \times L(G) \mid HK = KH\}|}{|L(G)|^2}$$
$$= \frac{|\{(H,K) \in L(G) \times L(G) \mid HK = KH, \ H \neq K\}| + |L(G)|}{|L(G)|^2}$$
$$= \frac{2|E(\Gamma_{perm}(G))| + |L(G)|}{|L(G)|^2}$$

Thus

$$\boxed{|E(\Gamma_{perm}(G))| = \frac{1}{2}|L(G)|\left(|L(G)|p(G) - 1\right)}$$

Example As we computed before,

$p(S_3) = \frac{5}{6}$. So, we have

$$|E(\Gamma_{perm}(S_3))| = \frac{1}{2} \times 6 \left(6 \times \frac{5}{6} - 1\right) = \boxed{12}$$

Example We found that $|E(\Gamma_{perm}(D_8))| = 31$

So, we can find $P(D_8)$ as the following

$$|E(\Gamma_{perm}(D_8))| = \frac{1}{2}|L(D_8)|\left(|L(D_8)|\,P(D_8) - 1\right)$$

$$\Rightarrow 31 = \frac{1}{2} \times 9\,(9P(D_8) - 1) \Rightarrow$$

$$62 = 81\,P(D_8) - 9 \Rightarrow$$

$$71 = 81\,P(D_8) \Rightarrow P(D_8) = \frac{71}{81}$$

Example $D_{2p} = \langle a, b \mid a^p = b^2 = e,\ bab = a^{-1} \rangle$

$= \{e, a, a^2, \dots, a^{p-1}, b, ab, \dots, a^{p-1}b\}$, $p \begin{cases} \text{pime} \\ \text{odd} \end{cases}$

$|L(D_{2p})| = p+3$, $P(D_{2p}) = \dfrac{7p+9}{(p+3)^2}$

So,

$$|E(\Gamma_{perm}(D_{2p}))| = \frac{1}{2} \times (p+3)\left[(p+3)\frac{7p+9}{(p+3)^2} - 1\right]$$

$$= \frac{1}{2}(p+3)\left(\frac{7p+9}{p+3} - 1\right) = \frac{1}{2}(7p+9-p-3)$$

$$= \frac{1}{2}(6p+6) = \boxed{3p+3}$$

Exercises

1. Compute $|E(\Gamma_{perm}(D_{2n}))|$ for all $n \geq 3$.

2. Compute $|E(\Gamma_{perm}(A_n))|$ and $|E(\Gamma_{perm}(S_n))|$ for all $n \geq 3$.

3. Find values of each $\chi(\Gamma_{perm}(G))$, $\alpha(\Gamma_{perm}(G))$, $\omega(\Gamma_{perm}(G))$?

4. When $\Gamma_{perm}(G)$ is complete, complete bipartite, tree ?

THANK YOU