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Research Article

New generalization of Fuzzy Interior Gamma Ideals of Ordered Gamma Semigroups

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Abstract

Ordered Gamma semigroups as a generalization of ordered semigroup, it is a significance structure in algebra with the associative property and hence has a wide broad of applications in various fields of, coding theory, error correction, computer science, automata theory, and artificial intelligence among others. In this paper, the new form of fuzzy interior ideals in ordered Gamma semigroups which is a new form of generalization of interior ideals is studied and investigated. The characterization and some properties of this new form of generalization in ordered Gamma semigroups is given in this paper.

Keywords : Fuzzy interior Gamma Ideal; Ordered Gamma Semigroups; Regular Ordered Semigroups; Gamma Ideals.

2020 Mathematical Subject Classification Codes : 20M75, 03E72

1 Introduction

Several mathematical problems involving unprecise and uncertainty cannot be solve through classical mathematical methods. The use of fuzzy set theory [1] in such problems has been accomplishing landmark achievements in contemporary mathematics. Zadeh's seminal paper [1] has received much attention and opens a new direction for researchers to tackle problems of uncertainties with a more appropriated mathematical tool. The new findings of fuzzy set theory and other related theories of uncertainties (soft sets, rough sets) are much relevant due to the diverse applications in automata theory, coding theory, decision making, computer sciences, artificial Intelligence and control engineering [2-8] Rosenfeld [9] was the first to apply Zadeh's pioneering idea of fuzzy sets to algebraic structures and introduced fuzzy subgroups. The inception of fuzzy subgroups provides a platform for other researchers to use this icebreaking idea in other algebraic structures along with several applications. Among other algebraic structures, ordered semigroups are having a lot of applications in error correcting codes, control engineering, performance of super computer and information sciences. Mordeson et al. [10] initiated a novel concept i.e., fuzzy subsemigroups along with fuzzy ideals in semigroups while Kehayopulu and Tsingelis [11-13] used fuzzy sets in ordered semigroups to developed fuzzy ideal theory. Khan et al. [14] investigated fuzzy $\Gamma-$ideals and fuzzy generalized bi $\Gamma-$ideals in ordered $\Gamma-$semigroups with characterization of different classes based on the new notion. In this paper, we apply the concept of generalized quasi-coincident with relation in ordered $\Gamma-$semigroup and introduced the new notion of fuzzy interior $\Gamma-$ideals in ordered $\Gamma-$semigroup. Based on this concept of quasi-coincident with relation, various classes of ordered $\Gamma-$semigroups like regular ordered $\Gamma-$semigroups with characterization based on the idea of fuzzy interior $\Gamma-$ideals.

2 Definitions and Preliminaries

Based on the fact that ordered $\Gamma-$semigroups are the basic algebraic structures in some of the advanced fields of computer sciences, error correcting codes, automata theory, robotics, control engineering and formal languages. Therefore, we develop new algebraic structures based on interior $\Gamma-$ideals to tackle the more complicated problems that can be applied in the aforementioned fields of research. In the following, we present some fundamental definitions and previous results that would be used in this paper.

Definition 1. ***Ordered fuzzy point***

Given S an ordered semigroup, and let $a \in S$ with $t \in (0,1]$. An ordered fuzzy point a_t of S is defined by the rule that

$$a_t(x) = \begin{cases} t, & \text{if} \quad x \in (a] \\ 0, & \text{if} \quad x \notin (a]. \end{cases}$$

It is accepted that a_t is a mapping from S in to $[0,1]$, then ordered fuzzy point of S is a fuzzy subset λ of S, denoted as $a_t \subseteq \lambda$ by $a_t \in \lambda$ in sequel.

Definition 2. ***Ordered $\Gamma-$semigroup***

If G and Γ are non-empty sets, then a structure $(G, \Gamma, \leq)$ is called an ordered Γ-semigroup if:

1. $(a\alpha b)\beta c = a\alpha(b\beta c)$ *for all* $a, b, c \in G$ *and* $\alpha, \beta \in \Gamma$,
2. *If* $a \leq b \rightarrow a\alpha x \leq b\alpha x$ *and* $x\alpha a \leq x\alpha b$ *for all* $a, b, x \in G$ *and* $\alpha \in \Gamma$.

Definition 3. ***Fuzzy Subsemigroup***

A non-empty fuzzy subset λ of a $\Gamma-$semigroup S is called a fuzzy subsemigroup of S if $\lambda(a\alpha b) \geq \min\{\lambda(a), \lambda(b)\}$ $\forall a, b \in S$ *and* $\alpha \in \Gamma$.

3 Fuzzy Interior $\Gamma-$Ideal of the Form $(\in, \in \vee q_k)$ in Ordered $\Gamma-$Semigroups

This section, provides a new generalization of interior fuzzy interior $\Gamma-$ideals using the idea of fuzzy point and quasi coincident with relation as another new form of generalization. The fuzzy interior $\Gamma-$ideal of the form $(\in, \in \vee q_k)$ in ordered $\Gamma-$semigroup G is introduced, where $k \in [0,1)$, unless or otherwise stated.

Definition 4. *A fuzzy subset λ of G is called a fuzzy interior $\Gamma-$ideal of G of the form $(\in, \in \vee q_k)$, if it satisfies the following conditions:*

1. *If* $a \leq b \to (b_t \in \lambda \to a_t \in \vee q_k \lambda)$. $\forall a, b \in G$ *with* $t \in (0,1]$.
2. *If* $a_t \in \lambda, b_t \in \lambda \to (a\alpha b)_t \in \vee q_k \lambda$ *(resp.* $(b\alpha a)_t \in \vee q_k \lambda)$, $\forall a, b \in G, \alpha, \beta \in \Gamma$ *and* $t \in (0,1]$.
3. *If* $x_t \in \lambda, \to (a\alpha x\beta c)_t \in \vee q_k \lambda$, $\forall x, a, c \in G, \alpha, \beta \in \Gamma$ *and* $t \in (0,1]$.

A fuzzy subset λ is said to be of the form $(q, \in \vee q_k)$, if given $a, b \in G$ with $a \leq b$ such that $b_t q \lambda$ implies $\lambda(b) + t > 1$ implies that $a \in A$, for any $A \subseteq G$ and $t \in (0,1]$.

The characterization of the new form of interior $\Gamma-$ideal of the form $(\in, \in \vee q_k)$, and $(q, \in \vee q_k)$ are given below.

Theorem 1. *Given a fuzzy subset* λ *of an ordered* $\Gamma-$*semigroup* G *and let* I *be an interior* $\Gamma-$*ideal of* G *and* λ *is defined as:*

$$\lambda(a) = \begin{cases} \frac{1-k}{2}, & \text{if } a \in I, \\ 0, & \text{if } a \notin I. \end{cases}$$

Then,

1. λ *is a fuzzy interior* $\Gamma-$*ideal of* G *of the form* $(\in, \in \vee q_k)$.
2. λ *is a fuzzy interior* $\Gamma-$*ideal of* G *of the form* $(q, \in \vee q_k)$.

Proof:

1. Suppose $a, b \in G$, and $\alpha, \beta \in \Gamma$ with $t, r \in (0,1]$, such that $a_t \in \lambda$ and $b_r \in \lambda$ then $a, b \in I$. Hence, $\lambda(a\alpha b) \geq \frac{1-k}{2}$, then, the following cases are established.

   Case I: If $\min\{t, r\} > \frac{1-k}{2}$, then $\lambda(a\alpha b) + \min\{t, r\} + k > \frac{1-k}{2} + \frac{1-k}{2} + k = 1$. Thus, it shows that $(a\alpha b)_{\min\{t,r\}} \in q_k \lambda$.

   Case II: If $\min\{t, r\} \leq \frac{1-k}{2}$. Then, $\lambda(a\alpha b) \geq \min\{t, r\}$ and hence, $(a\alpha b)_{\min\{t,r\}} \in q_k \lambda$. Let $a, x, c \in G$, and $\alpha, \beta \in \Gamma$ with $t \in (0,1]$, such that $a_t \in \lambda$. Thus, $a \in I$. Therefore, $a\alpha x\beta c \in I$. Hence, $\lambda(a\alpha x\beta c) \geq \frac{1-k}{2}$. If $t > \frac{1-k}{2}$. Since, $\lambda(a\alpha x\beta c) + t + k > \frac{1-k}{2} + \frac{1-k}{2} + \frac{1-k}{2} + k = 1$ and so, $(a\alpha x\beta c)_t \in q_k \lambda$. Similarly, If $t \leq \frac{1-k}{2}$, then $\lambda(a\alpha x\beta c) \geq t$ and so $\lambda(a\alpha x\beta c) \in \lambda$ thus, $(a\alpha x\beta c)_t \in \vee q_k \lambda$. Therefore, by Theorem **??** the remaining part of the theorem follows.

2. Suppose $a, b \in G$, and $\alpha, \beta \in \Gamma$ with $t, r \in (0,1]$, such that $a_t q \lambda$ and $b_r q \lambda$. Thus, $a, b \in I$, so $\lambda(a) + t > 1$. Similarly, $\lambda(a) + r > 1$. Since I is an interior $\Gamma-$ideal by the hypothesis. Then $\lambda(a\alpha b) \geq \frac{1-k}{2}$. Also, if $\min\{t, r\} > \frac{1-k}{2}$, thus, $\lambda(a\alpha b) + \min\{t, r\} + k > \frac{1-k}{2} + \frac{1-k}{2} + k = 1$, and so $(a\alpha b)_{\min\{t,r\}} q_k \lambda$. If $\min\{t, r\} \leq \frac{1-k}{2}$ hence, $\lambda(a\alpha b) \geq \min\{t, r\}$ and therefore, $(a\alpha b)_{\min\{t,r\}} \lambda$ which lead to $(a\alpha b)_{\min\{t,r\}} \in \vee q_k \lambda$. In the same way, let $a, x, c \in G$, and $\alpha, \beta \in \Gamma$ with $t, r \in (0,1]$, such that $a_t q \lambda$. Then $a \in I$, $\lambda(a) + t > 1$. But I is an interior $\Gamma-$ideal of G, then $a\alpha x\beta c \in I$. Thus, $(a\alpha x\beta c) \geq \frac{1-k}{2}$.

If $t > \frac{1-k}{2}$, then
$\lambda(a\alpha x\beta c) + t + k > \frac{1-k}{2} + \frac{1-k}{2} + k = 1$, which implies that $(a\alpha x\beta c)_t q_k t$, thus $(a\alpha x\beta c)_t \in \lambda$. Hence, $(a\alpha x\beta c)_t \vee q_k \lambda$. $\square$

If $k = 0$, with usual binary operation putting in the above theorem, then a corollary below is developed.

Corollary 1. *[**?**] Given a fuzzy subset λ of an ordered $\Gamma-$semigroup G and let I be an interior $\Gamma-$ideal of G and λ is defined as:*

$$\lambda(a) = \begin{cases} \frac{1}{2}, & \text{if } a \in I, \\ 0, & \text{if } a \notin I. \end{cases}$$

Then,

1. *λ is a fuzzy interior $\Gamma-$ideal of G of the form $(\in, \in \vee q)$.*
2. *λ is a fuzzy interior $\Gamma-$ideal of G of the form $(q, \in \vee q)$.*

The necessary and sufficient conditions for a fuzzy subset λ to be fuzzy interior $\Gamma-$ideal of the form $(q, \in \vee q_k)$ are provided in the following.

Theorem 2. *A fuzzy subset λ of an ordered $\Gamma-$semigroup G, is a fuzzy interior $\Gamma-$ideal of G if and only if all of the following are satisfied:*

1. $(\forall a, b \in G)\left(a \leq b \rightarrow \lambda(a) \geq \min\left\{\lambda(b), \frac{1-k}{2}\right\}\right)$,
2. $(\forall a, b \in G)\left(\lambda(a\alpha b) \geq \min\left\{\lambda(a), \lambda(b), \frac{1-k}{2}\right\}\right)$,
3. $(\forall a, c, x \in G)\left(\lambda(a\alpha x\beta c) \geq \min\left\{\lambda(x), \frac{1-k}{2}\right\}\right)$.

Proof: Suppose λ is a fuzzy interior $\Gamma-$ideal of $(\in, \in \vee q_k)$. Suppose that there exist $a, b \in G$, with $a \leq b$, then $\lambda(a) \geq \min\left\{\lambda(b), \frac{1-k}{2}\right\}$ which follows from Theorem **??**. Now, suppose on the contrary there exist $a, b \in G$, $\alpha \in \Gamma$ with $a \leq b$ such that $\lambda(a\alpha b) \leq \min\left\{\lambda(a), \lambda(b), \frac{1-k}{2}\right\}$ at $t \in (0,1]$, by choosing t from the $(0,1]$, such that $\lambda(a\alpha b) < t \leq \min\left\{\lambda(a), \lambda(b), \frac{1-k}{2}\right\}$ then $a_t \in \lambda$, $b_t \in \lambda$, while $\lambda(a\alpha b) < t$ and $\lambda(a\alpha b) + t + k < \frac{1-k}{2} + \frac{1-k}{2} + k = 1$, So $\lambda(a\alpha b)_t\, \overline{q_k}\lambda$. Hence, $\lambda(a\alpha b)_t\, \overline{\in \vee} q_k\lambda$ which is a contradiction. Thus, $\lambda(a\alpha b) \geq \min\left\{\lambda(a), \lambda(b), \frac{1-k}{2}\right\}$ for all $a, b \in G$, $\alpha \in \Gamma$. Suppose there exist $a, x, c \in G$, such that $\lambda(a\alpha x\beta c) < \min\left\{\lambda(x), \frac{1-k}{2}\right\}$. Now, for some $t \in (0,1]$, with $\lambda(a\alpha x\beta c) < t \leq \min\left\{\lambda(x), \frac{1-k}{2}\right\}$, then it shows that $x_t \in \lambda$ but $\lambda(a\alpha x\beta c) < t$ and $\lambda(a\alpha x\beta c) + t + k < \frac{1-k}{2} + \frac{1-k}{2} + k = 1$, so $\lambda(a\alpha x\beta c)_t\, \overline{q_k}\lambda$. Thus, $\lambda(a\alpha x\beta c)_t\, \overline{\in \vee} q_k\lambda$ which is $\lambda(a\alpha x\beta c) \geq \min\left\{\lambda(x), \frac{1-k}{2}\right\}$, for all $a, x, c \in G$.
Conversely, suppose $b_t \in \lambda$ for some $t, r \in (0,1]$. Using Theorem **??**, $a_t \in \vee q_k\lambda$. Let $a_t \in \lambda$,$b_r \in \lambda$, then $\lambda(a) \geq t$ and $\lambda(b) \geq r$. Hence, $\lambda(a\alpha b) \geq \min\left\{\lambda(a), \lambda(b), \frac{1-k}{2}\right\} \geq \min\left\{t, r, \frac{1-k}{2}\right\}$, now if $\min\{t, r\} > \frac{1-k}{2}$, then $\lambda(a\alpha b) \geq \frac{1-k}{2}$ and $\lambda(a\alpha b) + \min\{t, r\} + k > \frac{1-k}{2} + \frac{1-k}{2} + k = 1$. Therefore, $\lambda(a\alpha b)_{\min\{t,r\}}\, q_k\lambda$. Similarly, if $\min\{t, r\} \leq \frac{1-k}{2}$, then $\lambda(a\alpha b) \geq \min\{t, r\}$ which implies that $(a\alpha b)_{\min\{t,r\}} \in \lambda$. Thus, $(a\alpha b)_{\min\{t,r\}} \in \vee q_k\lambda$. Suppose $a_t \in \lambda$, then $\lambda(a) \geq t$, hence, $\lambda(a\alpha x\beta c) \geq \min\left\{\lambda(a), \frac{1-k}{2}\right\} \geq \min\left\{\lambda(a), \frac{1-k}{2}\right\} \geq \min\left\{t, \frac{1-k}{2}\right\}$. Now, if $t > \frac{1-k}{2}$, then $\lambda(a\alpha x\beta c) \geq \frac{1-k}{2}$ and $\lambda(a\alpha x\beta c) + t + k > \frac{1-k}{2} + \frac{1-k}{2} + k = 1$, and so, $\lambda(a\alpha x\beta c)\, q_k\lambda$. Similarly, if $t \leq \frac{1-k}{2}$, then $\lambda(a\alpha x\beta c) \geq t$, thus, $\lambda(a\alpha x\beta c)_t\, \lambda$. Hence, $(a\alpha x\beta c)_t \in \vee q_k\lambda$ which proves that λ is a fuzzy interior $\Gamma-$ideal of G of the form $(\in, \in \vee q_k)$. $\square$

Taking $k = 0$ in the above theorem and considering the Γ operation to be a normal binary operation, the following corollary is obtained which lead to the existing literature result.

Corollary 2. *[**?**] Let λ be a fuzzy subset of an ordered semigroup G. Then λ is a fuzzy interior ideal of G if and only if:*

1. $(\forall a,b\in G)\left(a\leq b\rightarrow\lambda(a)\geq\min\left\{\lambda(b),\frac{1}{2}\right\}\right)$,
2. $(\forall a,b\in G)\left(\lambda(ab)\geq\min\left\{\lambda(a),\lambda(b),\frac{1}{2}\right\}\right)$,
3. $(\forall a,c,x\in G)\left(\lambda(axc)\geq\min\left\{\lambda(x),\frac{1}{2}\right\}\right)$.

Using characteristic function and level subset a link between interior $\Gamma-$ideals of the form $(\in,\in\vee q_k)$.

Theorem 3. *A fuzzy subset λ of an ordered $\Gamma-$semigroup is a fuzzy interior $\Gamma-$ideal of G of the form $(\in,\in\vee q_k)$ if and only if the level subset $U(\lambda,t)\neq\emptyset$ is an interior $\Gamma-$ideal of G for all $t\in\left(0,\frac{1-k}{2}\right]$.*

Proof: Assume that λ is a fuzzy interior $\Gamma-$ideal of G of the form $(\in,\in\vee q_k)$. Let $a,b\in U(\lambda,t)$ for some $t\in\left(0,\frac{1-k}{2}\right]$. Then, $\lambda(a)\geq t$ and $\lambda(b)\geq t$ by the hypothesis. $\lambda(a\alpha b)\geq\min\left\{\lambda(a),\lambda(b),\frac{1-k}{2}\right\}$ $\geq\min\left\{t,t,\frac{1-k}{2}\right\}=t$. Since $t\in\left(0,\frac{1-k}{2}\right]$. Therefore, $a\alpha b\in U(\lambda,t)$. Thus, if $a,x,b\in G$ and $\alpha,\beta\in\Gamma$, with $x\in U(\lambda,t)$ for some $t\in\left(0,\frac{1-k}{2}\right]$, then, $\lambda(x)\geq t$ which shows by the hypothesis, λ is an interior $\Gamma-$ideal which shows that

$$\lambda(a\alpha x\beta c)\geq\min\left\{\lambda(x),\frac{1-k}{2}\right\}\geq\min\left\{t,\frac{1-k}{2}\right\}=t$$

since $t\in\left(0,\frac{1-k}{2}\right]$.
Therefore, $a\alpha x\beta c\in U(\lambda,t)$. Now suppose $a,b\in G$, and $a\leq b$ with $b\in U(\lambda,t)$ for some $t\in\left(0,\frac{1-k}{2}\right]$ which shows that $a\in U(\lambda,t)$ by Theorem **??**.
Conversely, suppose that $U(\lambda;t)(\neq\emptyset)$ is an interior $\Gamma-$ideal of G for all $t\in\left(0,\frac{1-k}{2}\right]$. If there exist $a,b\in G,\alpha\in\Gamma$, with $\lambda(a\alpha b)<\min\left\{\lambda(a),\lambda(b),\frac{1-k}{2}\right\}$ then, there exists a $t\in\left(0,\frac{1-k}{2}\right]$, such that $\lambda(a\alpha b)<t\leq\min\left\{\lambda(a),\lambda(b),\frac{1-k}{2}\right\}$. Hence, $a,b\in U(\lambda;t)$ but $a\alpha b\notin U(\lambda;t)$ which is a contradiction. Thus, $\lambda(a\alpha b)\geq\min\left\{\lambda(a),\lambda(b),\frac{1-k}{2}\right\}$, $\forall a,b\in G,\alpha\in\Gamma$, and $k\in[0,1)$. Now, if there exist $a,x,c\in G$, and $\alpha,\beta\in\Gamma$, with $\lambda(a\alpha x\beta c)<\min\left\{\lambda(x),\frac{1-k}{2}\right\}$, then there exist $t\in\left(0,\frac{1-k}{2}\right]$, such that $\lambda(a\alpha x\beta c)<t<\min\left\{\lambda(x),\frac{1-k}{2}\right\}$. Hence, $a\in U(\lambda;t)$ but $a\alpha x\beta c\notin U(\lambda;t)$ which is a contradiction. Thus, $\lambda(a\alpha x\beta c)\geq\min\left\{\lambda(a),\frac{1-k}{2}\right\}$ $\forall a,b,x\in G,\alpha,\beta\in\Gamma$, and $k\in[0,1)$ (By Theorem **??**). $\lambda(x)\geq\min\left\{\lambda(b),\frac{1-k}{2}\right\}$ $\forall a,b\in G$. Hence, λ is a fuzzy interior $\Gamma-$ideal of G. $\square$

Example 1. *Consider the ordered $\Gamma-$semigroup $G=\{a,b,c,d\}$ and $\Gamma=\{\alpha,\beta\}$ with ordered relation "$\leq$" defined as:*

$$\leq:=\{(a,a),(b,b),(c,c),(d,d),(a,b)\}$$

and a binary operation is defined in the following multiplication table

Table 1: Multiplication table for Example 1

$\boldsymbol{\alpha}$	***a***	***b***	***c***	***d***
a	a	a	a	a
b	a	a	b	a
c	a	a	b	b
d	a	a	a	a

$\boldsymbol{\beta}$	***a***	***b***	***c***	***d***
a	a	a	a	a
b	a	a	a	a
c	a	a	b	a
d	a	a	b	b

Then, $\{a\}, \{a,b\}, \{a,c\}, \{a,d\}, \{a,b,c\}$, $\{a,c,d\}$ *and* $\{a,b,c,d\}$, *are interior* $\Gamma-$*ideals of* G. *Now, define a fuzzy subset* λ *of* G *as follows:*

$$\lambda : G \to [0,1] \, | x \to \lambda(x) = \begin{cases} 0.7, & \text{if } x = a, \\ 0.6, & \text{if } x = b, \\ 0.3, & \text{if } x = c, \\ 0.2, & \text{if } x = d. \end{cases}$$

Then, $U(\lambda; t) = \begin{cases} G, & \text{if } 0 < t \leq 0.2, \\ \{a,b,c\}, & \text{if } 0.2 < t \leq 0.3, \\ \{a,b\}, & \text{if } 0.3 < t \leq 0.6, \\ \{a,c,d\}, & \text{if } 0.6 < t \leq 0.7, \\ \emptyset, & \text{if } 0.7 < t \leq 1. \end{cases}$

Hence, by Theorem 3, the above λ *is a fuzzy interior* $\Gamma-$*ideal of* G *of the form* $(\in, \in \vee q_k)$ *for all* $t \in \left(0, \frac{1-k}{2}\right]$ *and* $k = 0.6$.

The link between interior $\Gamma-$ideal and the fuzzy interior $\Gamma-$ideal of G of the form $(\in, \in \vee q_k)$ is provided in the following proposition.

Proposition 1. *Let* λ *be a nonzero fuzzy interior* $\Gamma-$*ideal of an ordered* $\Gamma-$*semigroup of* G. *Then, the set* $\lambda_0 = \{a \in G \mid \lambda(a) > 0\}$ *is an interior* $\Gamma-$*ideal of* G.

Proof: Suppose λ is a fuzzy interior $\Gamma-$ideal of G. Let $a, b \in G$, with $a \leq b$ and $b \in \lambda_0$, then, $\lambda(b) > 0$. But λ is a fuzzy $\Gamma-$ideal of G, then, from Proposition **??**, it follows that, $\lambda(a) \geq \min\left\{\lambda(b), \frac{1-k}{2}\right\} > 0$. since $\lambda(b) > 0$. Thus, $\lambda(a) > 0$ and so $a \in \lambda_0$. Let $a, b \in \lambda_0$. Similarly, $a \in \lambda_0$ it shows that if $a, c \in \lambda_0$ $\alpha, \beta \in \Gamma$. $\lambda(a\alpha x\beta c) \geq \min\left\{\lambda(a), \frac{1-k}{2}\right\} > 0$, Since $\lambda(a) > 0$. Therefore, $a\alpha x\beta c \in \lambda_0$, which implies that λ_0 is an interior $\Gamma-$ideal of G. $\square$

Interior $\Gamma-$ideal and fuzzy interior $\Gamma-$ideal of the form $(\in, \in \vee q_k)$ is linked in the proposition provided below using the characteristic function.

Proposition 2. *Given a non-empty subset* A *of an ordered* $\Gamma-$*semigroup* G, *is an interior* $\Gamma-$*ideal if and only if the characteristic function* λ_A *of* A *is a fuzzy interior* $\Gamma-$*ideals of* G *of the form* $(\in, \in \vee q_k)$.

Proof: Let A be an interior $\Gamma-$ideal of G. Suppose $a, b \in G$, with $a \leq b$ such that $b_t \in \lambda_A$. Hence, $\lambda_A(b) \geq t$ and $t \in (0,1]$ which shows that $\lambda_A(b) = 1$. Thus, $b \in A$. But A is an interior $\Gamma-$ideal of G and $a \leq b$, hence, $a \in A$ which lead to $\chi_A(a) = 1 \geq t$, thus, $a_t \in \lambda_A$ which implies that $a_t \in \vee q_k \chi_A$. Let $a, b \in G$, $\alpha \in \Gamma$, with $b_t \in \lambda_A$. Then $b \in A$ which follows that $a\alpha b \in A$, which shows that $(a\alpha b)_t \in \vee q_k \chi_A$. Similarly, let $a, x, c \in A$, and $\alpha, \beta \in \Gamma$, then it follows that $(a\alpha x\beta c) \in A$. Since A is interior $\Gamma-$ideal, thus $(a\alpha x\beta c) \in \vee q_k \chi_A$. Therefore, λ_A is a fuzzy interior $\Gamma-$ideal of G of the form $(\in, \in \vee q_k)$. Conversely, Suppose that χ_A is a fuzzy interior $\Gamma-$ideal of G of the form $(\in, \in \vee q_k)$. Let $a, b \in G$, with $b \in A$, then $\chi_A(b) = 1$. Hence, $\chi_A(b) > t \in (0,1]$, thus $b_t \in \chi_A$. But χ_A is a fuzzy interior $\Gamma-$ideal, then $a_t \in \vee q_k \chi_A$. So, if $a_t \in \chi_A$, then $\chi_A(a) \geq t \in (0,1]$. Therefore, $\chi_A(a) = 1$ which shows that $a \in A$. Similarly, if $a_t q_k \chi_A$ then $\chi_A(a) + t + k > 1$ for $k \in [0,1)$. Thus, $\chi_A(a) \neq 0$ which shows that $a \in A$. Now, let $a, b \in G$, $\alpha \in \Gamma$ such that $b \in A$. Then $b_t \in \chi_A$. But χ_A is an interior $\Gamma-$ideal of G of the form $(\in, \in \vee q_k)$ then $(a\alpha b)_t \in \vee q_k \chi_A$ which shows that $a\alpha b \in A$. In a similar way, let $a, b \in G$,$\alpha, \beta \in \Gamma$ with $a \in A$ then $a_t \in \chi_A$ and $(a\alpha x\beta c)_t \in \vee q_k \chi_A$ which implies that $a\alpha x\beta c \in A$. As a result of that A is an interior $\Gamma-$ideal of G. $\square$

Proposition 3. *A non empty subset λ of G is an interior $\Gamma-$ideal if and only if the characteristic function χ_λ of G is the fuzzy interior $\Gamma-$ideal of the form $(\in, \in \vee q_k)$ of G.*

Proof: The proof follows from Proposition 2. □

The fact that every $\Gamma-$ideal is an interior $\Gamma-$ideal, then the following proposition shows that for every $\Gamma-$ideal of the form $(\in, \in \vee q_k)$ is also an interior $\Gamma-$ideal of the form $(\in, \in \vee q_k)$.

Proposition 4. *Every fuzzy $\Gamma-$ideal of G of the form $(\in, \in \vee q_k)$ is a fuzzy interior $\Gamma-$ideal of G of the form $(\in, \in \vee q_k)$.*

Proof: Suppose λ is a fuzzy $\Gamma-$ideal of G of the form $(\in, \in \vee q_k)$. Let $a, b \in G$, with $a \leq b$ but since λ is a fuzzy $\Gamma-$ideal of the form $(\in, \in \vee q_k)$, therefore, $\lambda(a) \geq \min\left\{\lambda(b), \frac{1-k}{2}\right\}$ and if $a, b \in G$, $\alpha \in \Gamma$, then $\lambda(a\alpha b) \geq \min\left\{\lambda(a), \frac{1-k}{2}\right\}$ and $\lambda(a\alpha b) \geq \min\left\{\lambda(b), \frac{1-k}{2}\right\}$ for λ a fuzzy left $\Gamma-$ideal of G and fuzzy right $\Gamma-$ideal of G respectively by condition (1) and (2) of Theorem **??** which implies that $\lambda(a\alpha b) \geq \min\left\{\lambda(a), \lambda(b), \frac{1-k}{2}\right\}$ for all $a, b \in G$, $\alpha \in \Gamma$.
Now, let $a, x, c \in G$, $\alpha, \beta \in \Gamma$. Then,
$\lambda(a\alpha x\beta c) = \lambda(a\alpha(x\beta c)) \geq \min\left\{\lambda(x\beta c), \frac{1-k}{2}\right\}$ since λ a fuzzy left $\Gamma-$ideal. Also, $\lambda(a\alpha x\beta c) \geq \min\left\{\lambda(x), \frac{1-k}{2}\right\}$ since λ a fuzzy right $\Gamma-$ideal. Therefore, using Theorem 2 above it shows that λ a fuzzy interior $\Gamma-$ideal of G of the form $(\in, \in \vee q_k)$. □

The converse of the above stated proposition is not always true as can be seen in the following example.

Example 2. *Consider the ordered $\Gamma-$semigroup $G = \{a, b, c, d\}$ and $\Gamma = \{\alpha, \beta\}$ with ordered relation defined " $\leq$ " as:*

$$\leq := \{(a, a), (b, b), (c, c), (d, d), (a, b)\}$$

with the binary operations define in the following multiplication table

Table 2: Multiplication table for Example 2

$\boldsymbol{\alpha}$	$\boldsymbol{a}$	$\boldsymbol{b}$	$\boldsymbol{c}$	$\boldsymbol{d}$
$\boldsymbol{a}$	a	a	a	a
$\boldsymbol{b}$	a	a	a	a
$\boldsymbol{c}$	a	a	a	b
$\boldsymbol{d}$	a	a	b	c

$\boldsymbol{\beta}$	$\boldsymbol{a}$	$\boldsymbol{b}$	$\boldsymbol{c}$	$\boldsymbol{d}$
$\boldsymbol{a}$	a	a	a	a
$\boldsymbol{b}$	a	a	a	b
$\boldsymbol{c}$	a	a	b	c
$\boldsymbol{d}$	a	a	a	a

Define $\lambda : G \to [0, 1]$, be a fuzzy subset as shown below.

$$\lambda : G \to [0,1] \,|x \to \lambda(x) = \begin{cases} 0.7, & \text{if } x = a, \\ 0.3, & \text{if } x = b, \\ 0.6, & \text{if } x = c, \\ 0, & \text{if } x = d. \end{cases}$$

If $x = a$

$$\lambda(a\alpha x\beta c) = \lambda(a\beta c) = \lambda(a) = 0.7 > 0.4 \geq \min\{0.7, 0.4\}.$$

Now, if $a\alpha b = a$, then $\lambda(a\alpha b) = \lambda(a) = 0.7 > 0.4 \geq \min\left\{\lambda(a), \lambda(b), \frac{1-k}{2}\right\}$

$$= 0.7 \geq \min\{0.7, 0.3, 0.4\}.$$

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$$\lambda(a\alpha b) = 0.7 > 0.3.$$

If $a\alpha b = b$, then

$$\lambda(a\alpha b) = \lambda(b) = 0.3 > 0 \geq \min\left\{\lambda(a), \lambda(b), \frac{1-k}{2}\right\}$$
$$0.3 > 0 = \min\{0.7, 0.3, 0.4\}.$$

If $a\alpha b = c$, then

$$\lambda(a\alpha b) = \lambda(c) = 0.6 > 0 = \min\left\{\lambda(a), \lambda(b), \frac{1-k}{2}\right\}$$

Hence, λ is a fuzzy interior $\Gamma-$ideal of G of the form $(\in, \in \vee q_k)$ for all $k = 0.2$ but not a fuzzy $\Gamma-$ideal of G of the form $(\in, \in \vee q_k)$.

The following result shows the coinciding of fuzzy $\Gamma-$ideal and interior $\Gamma-$ideal all of the form $(\in, \in \vee q_k)$.

Proposition 5. *In a regular ordered $\Gamma-$semigroup G, every fuzzy interior $\Gamma-$ideal is a fuzzy $\Gamma-$ideal of G all of the form $(\in, \in \vee q_k)$.*

Proof: Suppose λ is a fuzzy interior $\Gamma-$ideal of G and let $a, b \in G$, $\alpha, \beta \in \Gamma$. Then, there exist $x \in G$ such that $a \leq a\alpha x\beta a$. Therefore,

$$\begin{aligned}
\lambda(a\alpha b) &\geq \min\left\{\lambda((a\alpha x\beta a)\alpha b), \frac{1-k}{2}\right\} \\
&= \min\left\{\lambda((a\alpha x)\beta a\alpha b), \frac{1-k}{2}\right\} \\
&\geq \min\left\{\left(\lambda(a), \frac{1-k}{2}\right), \frac{1-k}{2}\right\} \\
&= \min\left\{\lambda(a), \frac{1-k}{2}\right\}.
\end{aligned}$$

Since λ is a fuzzy interior $\Gamma-$ideal of G.
In a similar way, it also can be shown that $\lambda(a\alpha b) \geq \min\left\{\lambda(b), \frac{1-k}{2}\right\}$ for every $a, b \in G$,$\alpha, \beta \in \Gamma$. Therefore, λ is a fuzzy interior $\Gamma-$ideal of G of the form $(\in, \in \vee q_k)$. $\square$

Corollary 3. *In ordered $\Gamma-$semigroup, a fuzzy interior $\Gamma-$ideal and the fuzzy $\Gamma-$ideal of G of the form $(\in, \in \vee q_k)$ coincide.*

Proof: The proof follows from the Proposition 5. $\square$
In what follows, the proposition give condition for fuzzy interior $\Gamma-$ideal of G to be fuzzy $\Gamma-$ideal of the form $(\in, \in \vee q_k)$ in ordered $\Gamma-$semigroup.

Proposition 6. *In semisimple ordered $\Gamma-$semigroup G, every fuzzy interior $\Gamma-$ideal I of G, then, I is also a fuzzy $\Gamma-$ideal of G of the form $(\in, \in \vee q_k)$.*

Proof: Let G be a semisimple ordered $\Gamma-$semigroup G, and λ is a fuzzy interior $\Gamma-$ideal of G. Let $a, b \in G$,$\alpha, \beta \in \Gamma$ then there exist $x, y, z \in G$, such that $a \leq x\alpha a\beta y\alpha a\beta z$. Hence,

$$\begin{aligned}\lambda\left(a\alpha b\right) &\geq \min\left\{\lambda\left(x\alpha a\beta y\alpha a\beta z\right)\alpha b, \frac{1-k}{2}\right\}\\ &= \min\left\{\lambda\left(x\alpha a\beta y\right)\alpha a\beta\left(z\alpha b\right), \frac{1-k}{2}\right\}\\ &\geq \min\left\{\left(\lambda\left(a\right), \frac{1-k}{2}\right), \frac{1-k}{2}\right\}\end{aligned}$$

Since λ is a fuzzy interior $\Gamma-$ideal of G.

$$\lambda\left(a\alpha b\right) = \min\left\{\lambda\left(a\right), \frac{1-k}{2}\right\}.$$

In a similar way, it can also be shown that $\lambda\left(a\alpha b\right) \geq \min\left\{\lambda\left(b\right), \frac{1-k}{2}\right\}$. For every $a, b \in G$,$\alpha \in \Gamma$. Therefore, λ is a fuzzy $\Gamma-$ideal of G of the form $(\in, \in \vee q_k)$. □

Theorem 4. *In a semisimple ordered $\Gamma-$semigroup, the fuzzy interior $\Gamma-$ideal with fuzzy $\Gamma-$ideal of the form $(\in, \in \vee q_k)$ coincide.*

Proof: The proof of this proposition follows from Proposition 6. □

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Conclusion

Fuzzy interior $\Gamma-$ideal of ordered $\Gamma-$semigroup and some characterizations play a remarkable role in the study of algebraic structures of ordered $\Gamma-$semigroups. In this Paper, the results on fuzzy interior $\Gamma-$ideals of the form $(\in, \in \vee q_k)$ of an ordered $\Gamma-$semigroup gave more characterizations where simple ordered $\Gamma-$semigroup, $\Gamma-$regular, intra $\Gamma-$regular or semiprime coincide. Some more characterization of fuzzy interior $\Gamma-$ideal of the form $(\in, \in \vee q_k)$, with several properties of an ordered of fuzzy interior $\Gamma-$ideals of the form $(\in, \in \vee q_k)$ have been studied and connections between interior $\Gamma-$ideals and the introduced fuzzy interior $\Gamma-$ideals of the form $(\in, \in \vee q_k)$ using characteristic function are established.

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