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Consistency Relations of an Extension Polycyclic Free Abelian Lattice Group by Quaternion Point Group

Consistency Relations of an Extension Polycyclic Free Abelian Lattice Group by Quaternion Point Group

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Abstract. An extension of a free abelian lattice group by finite group is a torsion free crystallographic group. It expounds its symmetrical properties or known as homological invariants. One of the methods to compute its homological invariants is by determining the polycyclic presentation of the group. These polycyclic presentations are first shown to satisfy its consistency relations. Therefore, our focus is to show that this extension polycyclic free abelian lattice group by quaternion point group satisfy its consistency relations.

1. Introduction

The computation of the homological invariants of a group has been started long time ago [1]. However, researches on the computation of the homological invariants for an extension polycyclic of a free abelian lattice group by finite group are only carried out starting in 2009 [2]. The research focused on the computations of the nonabelian tensor squares for polycyclic groups with cyclic point group of order two, C_2 and elementary abelian 2-group point group, $C_2 \times C_2$ as the abelian extension. Later in 2014, Mat Hassim *et al.* [3], extended the research by finding the other homological invariant of a group which is the exterior square. In the same year, Mat Hassim *et al.* [4] computed the abelianizations of all extension of polycyclic group with cyclic point group of order three, C_3 in order to find its homological invariants. Later, Mat Hassim [5] also computed the other homological invariants for groups extended with cyclic point group of order two, three and five. Abdul Ladi *et al.* [6, 7] and Masri *et al.* [8], continue to compute the other homological invariants for the extended polycyclic group with point group, $C_2 \times C_2$.

In 2011 the nonabelian extension has been taken into consideration. Mohd Idrus [9] started to work on the dihedral point group of order eight, D_4, followed by Wan Mohd Fauzi *et al.* [10] in 2015 who worked on the same extension but with different dimension of the group. A year later, Tan *et al.* [11] used the symmetric group of order six, S_3 as the extension. All of these groups are isomorphic to one of the groups that were designed by Opgenorth *et al.* [12] to enable the user to construct and recognize space groups. Eick and Nickel [13] showed that the nonabelian tensor square of a polycyclic group given by a polycyclic presentation can be computed. Therefore, the technique on computing the nonabelian tensor squares of polycyclic groups developed by Blyth and Morse [14] is used throughout this research. All of these groups have been transformed into polycyclic presentations and proved to satisfy its consistency relations.

Based on [12], there are four groups of extension of a free abelian lattice group by quaternion point group of order eight, Q_8 found and all of them are of dimension six. In 2015, Mohammad *et al.*[15] computed the polycyclic presentation of the fourth extension of a free abelian lattice group with point group Q_8. However, there are some problems on the calculations of its homological invariants by using this polycyclic presentation. The new generator c was developed among the relationship between generators in G_4 but the generator c is not well defined in here. The polycylic presentation is stated as in the following:

$$\begin{aligned}Q_4(6) = \langle a, b, c, l_1, l_2, l_3, l_4, l_5, l_6 | & a^2 = cl_6, b^2 = cl_5 l_6^{-1}, b^a = bcl_5^{-2} l_6^2, c^2 = l_5 l_6^{-1}, c^a = cl_5^{-1} l_6, c^b = c, \\ & l_1^a = l_4^{-1}, l_1^b = l_3^{-1}, l_1^c = l_1^{-1}, l_2^a = l_3, l_2^b = l_4^{-1}, l_2^c = l_2^{-1}, l_3^a = l_2^{-1}, l_3^b = l_1, l_3^c = l_3^{-1}, l_4^a = l_1, \\ & l_4^b = l_2, l_4^c = l_4^{-1}, l_5^a = l_6, l_5^b = l_5, l_5^c = l_5, l_6^a = l_5, l_6^b = l_6, l_6^c = l_6, l_j^{l_i} = l_j, l_j^{l_i^{-1}} = l_j \\ & \text{for } j > i,\ 1 \leqslant i,\ j \leqslant 6\rangle.\end{aligned}$$

This presentation cannot be used in computing the homological invariants. Problem arise as the calculations taking over. Therefore, at the end of the calculations, the polycylic presentation is concluded to be not consistent. Hence, the main motivation of this research is to show the new polycyclic presentation that satisfy its consistency relations for the fourth extension of a free abelian lattice group with point group Q_8. Let G_4 be the fourth extension of a free abelian lattice group with point group Q_8, then

$$G_4 = \langle a_0, a_1, l_1, l_2, l_3, l_4, l_5, l_6 \rangle, \text{where} \tag{1}$$

$$a_0 = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & -\frac{1}{4} \\ 0 & 0 & 0 & 0 & 0 & 1 & \frac{1}{4} \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad a_1 = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

$$l_1 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad l_2 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

$$l_3 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad l_4 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

$$l_5 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \text{ and} l_6 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

This group will later be shown to be isomorphic to the new $Q_4(6) = \langle a, b, c, l_1, l_2, l_3, l_4, l_5, l_6 \rangle$ with the new generator c where Q is the group (with point group Q_8) while $_4$ indicates the number 4^{th} group and (6) here indicates the dimension of the group.

2. Preliminary

As mentioned in Section 1, the technique developed by Blyth and Morse [14] will be used throughout this research. The polycyclic presentation of this extension of a free abelian lattice group by finite group is shown to be consistent. The following definitions are used throughout this research.

Definition 1. *[13] Let F_n be a free group on generators $g_i, \ldots, g_n$ and R be a set of relations of group F_n. The relations of a polycyclic presentation have the form:*

$$\begin{aligned} g_i^{e_i} &= g_{i+1}^{x_{i,i+1}} \cdots g_n^{x_{i,n}} && \text{for } i \in I, \\ g_j^{-1} g_i g_j &= g_{j+1}^{y_{i,j,j+1}} \cdots g_n^{y_{i,j,n}} && \text{for } j < i, \\ g_j g_i g_j^{-1} &= g_{j+1}^{z_{i,j,j+1}} \cdots g_n^{z_{i,j,n}} && \text{for } j < i \text{ and } j \notin I \end{aligned}$$

for some $I \subseteq \{1, \ldots n\}$, certain exponents $e_i \in \mathbb{N}$ for $i \in I$ and $x_{i,j}, y_{i,j,k}, z_{i,j,k} \in \mathbb{Z}$ for all i, j and k.

Blyth and Morse [14] proved that if G is polycyclic, then $G \otimes G$ is polycyclic. Hence, $G \otimes G$ has a consistent polycyclic presentation. By using the above definition, the presentation is then checked to satisfy all the consistency relations. It is crucial to check the polycyclic presentation is consistent in order to compute its homological invariants later.

Definition 2. *[13] Let G be a group generated by $g_1, \ldots, g_n$ and the consistency relations in G can be evaluated in the polycyclic presentation of G using the collection from the left as in the following:*

$$\begin{aligned} g_k(g_j g_i) &= (g_k g_j) g_i && \text{for } k > j > i, \\ (g_j^{e_j}) g_i &= g_j^{e_j - 1}(g_j g_i) && \text{for } j > i, j \in I, \\ g_j(g_i^{e_i}) &= (g_j g_i) g_i^{e_i - 1} && \text{for } j > i, i \in I, \\ (g_i^{e_i}) g_i &= g_i(g_i^{e_i}) && \text{for } i \in I, \\ g_j &= (g_j g_i^{-1}) g_i && \text{for } j > i, i \notin I \end{aligned}$$

for some $I \subseteq \{1, \ldots, n\}$, $e^i \in \mathbb{N}$. Then, G is said to be given by a consistent polycyclic presentation.

3. Results and Discussion

In this section, the transformation of a group in (1) into a polycyclic group is shown. By using matrix form in (1), $G_4 = \langle a_0, a_1, l_1, l_2, l_3, l_4, l_5, l_6 \rangle$ where l_1, l_2, l_3, l_4, l_5 and l_6 are its lattices in which its basis matrix is the identity matrix, this group is shown to be isomorphic to a new group which is polycyclic, namely $Q_4(6)$. By taking the same generators of l_1, l_2, l_3, l_4, l_5 and l_6 in G_4, whereas a_0 is written as a and a_1 is as b, meanwhile a new generator c is developed based on the relationship among the generators in the group G_4 therefore, $G_4 = \langle a_0, a_1, l_1, l_2, l_3, l_4, l_5, l_6 \rangle$ now is transformed into $Q_4(6) = \langle a, b, c, l_1, l_2, l_3, l_4, l_5, l_6 \rangle$. The polycyclic presentation can now vary depending on what generator c is. This is to make sure that the new group $Q_4(6)$ satisfies its consistency relations aligning with Definition 2 later. By Definition 1, the polycyclic presentation of $Q_4(6)$ is presented as in the following.

Take $\gamma : G_4 \to Q_4(6)$ such that $\gamma(a_0) = a$, $\gamma(a_1) = b$. Let $c = a_0^2 l_6^{-1}$. The mapping γ, is well defined since γ maps the generators of G_4 to generators of $Q_4(6)$. Now, under this mapping γ, all relations hold in $Q_4(6)$ are constructed.

Now, $c = a_0^2 l_6^{-1}$ implies,

$$
\begin{aligned}
c &= \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & -\frac{1}{4} \\ 0 & 0 & 0 & 0 & 0 & 1 & \frac{1}{4} \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}
\begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & -\frac{1}{4} \\ 0 & 0 & 0 & 0 & 0 & 1 & \frac{1}{4} \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}
\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}^{-1} \\
&= \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & -\frac{1}{4} \\ 0 & 0 & 0 & 0 & 0 & 1 & \frac{1}{4} \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}^2
\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & -\frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.
\end{aligned}
$$

Thus, by mapping $\gamma(a_0) = a$, $c = a_0^2 l_6^{-1} = a^2 l_6^{-1}$.

Next, a_0^2 can be shown to be equal to cl_6.

$$
\begin{aligned}
a_0^2 &= \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & -\frac{1}{4} \\ 0 & 0 & 0 & 0 & 0 & 1 & \frac{1}{4} \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}
\begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & -\frac{1}{4} \\ 0 & 0 & 0 & 0 & 0 & 1 & \frac{1}{4} \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & -\frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 & 1 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.
\end{aligned}
$$

$$
\begin{aligned}
cl_6 &= \begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & -\frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}
\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & -\frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 & 1 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.
\end{aligned}
$$

Thus, by mapping $\gamma(a_0) = a$, $a_0^2 = a^2 = cl_6$.

Next, a_1^2 gives,

$$\begin{aligned}
a_1^2 &= \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & -\frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \\
&= c.
\end{aligned}$$

Thus, by mapping $\gamma(a_1) = b$, $a_1^2 = b^2 = c$.

Furthermore, c^2 can be written as $l_5^{-1}l_6^{-1}$.

$$\begin{aligned}
c^2 &= \begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & -\frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & -\frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.
\end{aligned}$$

$$\begin{aligned}
l_5^{-1}l_6^{-1} &= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}^{-1} \\
&= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.
\end{aligned}$$

Thus, $c^2 = l_5^{-1}l_6^{-1}$.

For the conjugation action of each generator, the first relation is shown below:

$$
\begin{aligned}
a_1^{a_0} &= a_0^{-1} a_1 a_0 \\
&= \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & -\frac{1}{4} \\ 0 & 0 & 0 & 0 & 0 & 1 & \frac{1}{4} \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}^{-1}
\begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}
\begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & -\frac{1}{4} \\ 0 & 0 & 0 & 0 & 0 & 1 & \frac{1}{4} \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & -\frac{1}{4} \\ 0 & 0 & 0 & 0 & 0 & 1 & \frac{1}{4} \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}^{-1}
\begin{bmatrix} 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -\frac{1}{4} \\ 0 & 0 & 0 & 0 & 1 & 0 & -\frac{1}{4} \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & -\frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.
\end{aligned}
$$

Meanwhile, $a_1 c^{-1} l_5^{-1}$

$$
\begin{aligned}
&= \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}
\begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & -\frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}^{-1}
\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}^{-1} \\
&= \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}
\begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & -\frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 & 1 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & -\frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.
\end{aligned}
$$

This shows that, $a_1^{a_0} = a_1 c^{-1} l_5^{-1}$. By mapping $\gamma(a_0) = a$, $\gamma(a_1) = b$, $b^a = bc^{-1} l_5^{-1}$.

Next, to show $c^{a_0} = c$,

$$
\begin{aligned}
c^{a_0} &= a_0^{-1} c a_0 \\
&= \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & -\frac{1}{4} \\ 0 & 0 & 0 & 0 & 0 & 1 & \frac{1}{4} \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & -\frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & -\frac{1}{4} \\ 0 & 0 & 0 & 0 & 0 & 1 & \frac{1}{4} \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & \frac{1}{4} \\ 0 & 0 & 0 & 0 & 0 & 1 & -\frac{1}{4} \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & -\frac{3}{4} \\ 0 & 0 & 0 & 0 & 0 & 1 & -\frac{1}{4} \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & -\frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \\
&= c.
\end{aligned}
$$

Hence, $c^{a_0} = a_0^{-1} c a_0 = c$. By mapping $\gamma(a_0) = a$, $c^{a_0} = c^a = c$.

The next calculation shows that $c^{a_1} = c$.

$$
\begin{aligned}
c^{a_1} &= a_1^{-1} c a_1 \\
&= \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & -\frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 & -\frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & -\frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \\
&= c.
\end{aligned}
$$

Thus, $c^{a_1} = a_1^{-1} c a_1 = c$. By mapping $\gamma(a_1) = b$, $c^{a_1} = c^b = c$.

Therefore, all possible relations which are formed by conjugation between each generator and power of certain exponent have been constructed. Thus, $G_4 = \langle a_0, a_1, l_1, l_2, l_3, l_4, l_5, l_6 \rangle$ has been shown to be isomorphic to the new $Q_4(6) = \langle a, b, c, l_1, l_2, l_3, l_4, l_5, l_6 \rangle$ with $c = a_0^2 l_6^{-1}$. Hence, by collecting all possible relations that have been constructed, the new polycyclic presentation of

$Q_4(6)$ is established as:

$$Q_4(6) = \langle a, b, c, l_1, l_2, l_3, l_4, l_5, l_6 | a^2 = cl_6, b^2 = c, c^2 = l_5^{-1} l_6^{-1}, b^a = bc^{-1} l_5^{-1}, c^a = c, c^b = c, l_1^a = l_3, l_1^b = l_4, l_1^c = l_1^{-1}, l_2^a = l_4, l_2^b = l_3^{-1}, l_2^c = l_2^{-1}, l_3^a = l_1^{-1}, l_3^b = l_2, l_3^c = l_3^{-1}, l_4^a = l_2^{-1}, l_4^b = l_1^{-1}, l_4^c = l_4^{-1}, l_5^a = l_5, l_5^b = l_6, l_5^c = l_5, l_6^a = l_6, l_6^b = l_5, l_6^c = l_6, l_2^{l_1} = l_2, l_3^{l_1} = l_3, l_4^{l_1} = l_4, l_5^{l_1} = l_5, l_6^{l_1} = l_6, l_3^{l_2} = l_3, l_4^{l_2} = l_4, l_5^{l_2} = l_5, l_6^{l_2} = l_6, l_4^{l_3} = l_4, l_5^{l_3} = l_5, l_6^{l_3} = l_6, l_5^{l_4} = l_5, l_6^{l_4} = l_6, l_6^{l_5} = l_6, l_2^{l_1^{-1}} = l_2, l_3^{l_1^{-1}} = l_3, l_4^{l_1^{-1}} = l_4, l_5^{l_1^{-1}} = l_5, l_6^{l_1^{-1}} = l_6, l_3^{l_2^{-1}} = l_3, l_4^{l_2^{-1}} = l_4, l_5^{l_2^{-1}} = l_5, l_6^{l_2^{-1}} = l_6, l_4^{l_3^{-1}} = l_4, l_5^{l_3^{-1}} = l_5, l_6^{l_3^{-1}} = l_6, l_5^{l_4^{-1}} = l_5, l_6^{l_4^{-1}} = l_6, l_6^{l_5^{-1}} = l_6 \rangle. \tag{2}$$

To show that the group is polycyclic, the polycyclic presentation has to be consistent. In this section, all relations as given in (2) are shown to satisfy the five consistency relations as given in Definition 2. Hence, some calculations on checking of the consistency polycyclic presentation of the group, are presented in the following theorem.

Theorem 1. *Let $Q_4(6)$ be an extension of a free abelian lattice group of dimension six with point group Q_8 and has presentation as in (2), its polycyclic presentation is found to be*

$$\begin{aligned} Q_4(6) = \langle a, b, c, l_1, l_2, l_3, l_4, l_5, l_6 | &a^2 = cl_6, b^2 = c, c^2 = l_5^{-1} l_6^{-1}, \\ &b^a = bc^{-1} l_5^{-1}, c^a = c, c^b = c, \\ &l_1^a = l_3, l_1^b = l_4, l_1^c = l_1^{-1}, \\ &l_2^a = l_4, l_2^b = l_3^{-1}, l_2^c = l_2^{-1}, \\ &l_3^a = l_1^{-1}, l_3^b = l_2, l_3^c = l_3^{-1}, \\ &l_4^a = l_2^{-1}, l_4^b = l_1 - 1, l_4^c = l_4^{-1}, \\ &l_5^a = l_5, l_5^b = l_6, l_5^c = l_5, \\ &l_6^a = l_6, l_6^b = l_5, l_6^c = l_6, \\ &l_j^{l_i} = l_j, l_j^{l_i^{-1}} = l_j \text{ for } j > i,\ 1 \leqslant i,\ j \leqslant 6 \rangle. \end{aligned}$$

Then, $Q_4(6)$ is consistent.

Proof. $Q_4(6)$ is generated by a, b, c, l_1, l_2, l_3, l_4, l_5 and l_6. Referring to Definition 2, let $g_1 = a$, $g_2 = b$, $g_3 = c$, $g_4 = l_1$, $g_5 = l_2$, $g_6 = l_3$, $g_7 = l_4$, $g_8 = l_5$ and $g_9 = l_6$. For the first consistency relation, i.e. $g_k(g_j g_i) = (g_k g_j) g_i$ for $k > j > i$, the following relations hold:

(i) $c(ba) = (cb)a$,
(ii) $l_1(cb) = (l_1c)b$,
(iii) $l_1(ca) = (l_1c)a$,
(iv) $l_1(ba) = (l_1b)a$,
(v) $l_2(l_1c) = (l_2l_1)c$,
(vi) $l_2(l_1b) = (l_2l_1)b$,
(vii) $l_2(l_1a) = (l_2l_1)a$,
(viii) $l_2(cb) = (l_2c)b$,
(ix) $l_2(ca) = (l_2c)a$,
(x) $l_2(ba) = (l_2b)a$,
(xi) $l_3(l_2l_1) = (l_3l_2)l_1$,
(xii) $l_3(l_2c) = (l_3l_2)c$,
(xiii) $l_3(l_2b) = (l_3l_2)b$,
(xiv) $l_3(l_2a) = (l_3l_2)a$,
(xv) $l_3(l_1c) = (l_3l_1)c$,
(xvi) $l_3(l_1b) = (l_3l_1)b$,
(xvii) $l_3(l_1a) = (l_3l_1)a$,
(xviii) $l_3(cb) = (l_3c)b$,
(xix) $l_3(ca) = (l_3c)a$,
(xx) $l_3(ba) = (l_3b)a$,
(xxi) $l_4(l_3l_2) = (l_4l_3)l_2$,
(xxii) $l_4(l_3l_1) = (l_4l_3)l_1$,
(xxiii) $l_4(l_3c) = (l_4l_3)c$,
(xxiv) $l_4(l_3b) = (l_4l_3)b$,
(xxv) $l_4(l_3a) = (l_4l_3)a$,
(xxvi) $l_4(l_2l_1) = (l_4l_2)l_1$,
(xxvii) $l_4(l_2c) = (l_4l_2)c$,
(xxviii) $l_4(l_2b) = (l_4l_2)b$,
(xxix) $l_4(l_2a) = (l_4l_2)a$,
(xxx) $l_4(l_1c) = (l_4l_1)c$,

(xxxi) $l_4(l_1b) = (l_4l_1)b$,
(xxxii) $l_4(l_1a) = (l_4l_1)a$,
(xxxiii) $l_4(cb) = (l_4c)b$,
(xxxiv) $l_4(ca) = (l_4c)a$,
(xxxv) $l_4(ba) = (l_4b)a$,
(xxxvi) $l_5(l_4l_3) = (l_5l_4)l_3$,
(xxxvii) $l_5(l_4l_2) = (l_5l_4)l_2$,
(xxxviii) $l_5(l_4l_1) = (l_5l_4)l_1$,
(xxxix) $l_5(l_4c) = (l_5l_4)c$,
(xl) $l_5(l_4b) = (l_5l_4)b$,
(xli) $l_5(l_4a) = (l_5l_4)a$,
(xlii) $l_5(l_3l_2) = (l_5l_3)l_2$,
(xliii) $l_5(l_3l_1) = (l_5l_3)l_1$,
(xliv) $l_5(l_3c) = (l_5l_3)c$,
(xlv) $l_5(l_3b) = (l_5l_3)b$,
(xlvi) $l_5(l_3a) = (l_5l_3)a$,
(xlvii) $l_5(l_2l_1) = (l_5l_2)l_1$,
(xlviii) $l_5(l_2c) = (l_5l_2)c$,
(xlix) $l_5(l_2b) = (l_5l_2)b$,
(l) $l_5(l_2a) = (l_5l_2)a$,
(li) $l_5(l_1c) = (l_5l_1)c$,
(lii) $l_5(l_1b) = (l_5l_1)b$,
(liii) $l_5(l_1a) = (l_5l_1)a$,
(liv) $l_5(cb) = (l_5c)b$,
(lv) $l_5(ca) = (l_5c)a$,
(lvi) $l_5(ba) = (l_5b)a$,
(lvii) $l_6(l_5l_4) = (l_6l_5)l_4$,
(lviii) $l_6(l_5l_3) = (l_6l_5)l_3$,
(lix) $l_6(l_5l_2) = (l_6l_5)l_2$,
(lx) $l_6(l_5l_1) = (l_6l_5)l_1$,
(lxi) $l_6(l_5c) = (l_6l_5)c$,
(lxii) $l_6(l_5b) = (l_6l_5)b$,
(lxiii) $l_6(l_5a) = (l_6l_5)a$,
(lxiv) $l_6(l_4l_3) = (l_6l_4)l_3$,
(lxv) $l_6(l_4l_2) = (l_6l_4)l_2$,
(lxvi) $l_6(l_4l_1) = (l_6l_4)l_1$,
(lxvii) $l_6(l_4c) = (l_6l_4)c$,
(lxviii) $l_6(l_4b) = (l_6l_4)b$,
(lxix) $l_6(l_4a) = (l_6l_4)a$,
(lxx) $l_6(l_3l_2) = (l_6l_3)l_2$,
(lxxi) $l_6(l_3l_1) = (l_6l_3)l_1$,
(lxxii) $l_6(l_3c) = (l_6l_3)c$,
(lxxiii) $l_6(l_3b) = (l_6l_3)b$,
(lxxiv) $l_6(l_3a) = (l_6l_3)a$,
(lxxv) $l_6(l_2l_1) = (l_6l_2)l_1$,
(lxxvi) $l_6(l_2c) = (l_6l_2)c$,
(lxxvii) $l_6(l_2b) = (l_6l_2)b$,
(lxxviii) $l_6(l_2a) = (l_6l_2)a$,
(lxxix) $l_6(l_1c) = (l_6l_1)c$,
(lxxx) $l_6(l_1b) = (l_6l_1)b$,
(lxxxi) $l_6(l_1a) = (l_6l_1)a$,
(lxxxii) $l_6(cb) = (l_6c)b$,
(lxxxiii) $l_6(ca) = (l_6c)a$,
(lxxxiv) $l_6(ba) = (l_6b)a$.

Hence, by the relations of $Q_4(6)$;
For (i);

$$c(ba) = cabc^{-1}l_5^{-1} = acbc^{-1}l_5^{-1} = abcc^{-1}l_5^{-1} = abl_5^{-1}.$$
$$(cb)a = bca = bac = abc^{-1}l_5^{-1}c = abl_5^{-1}c^{-1}c = abl_5^{-1}.$$

For the second consistency relation, i.e. $(g_j^{e_j})g_i = g_j^{e_j-1}(g_jg_i)$ for $j > i, j \in I$, the following relations hold:

(i) $b^2a = b(ba)$,
(ii) $c^2a = c(ca)b$,
(iii) $c^2b = c(cb)$.

By the relations of $Q_4(6)$;
For (i);

$$\begin{aligned} b^2a &= ca = ac. \\ b(ba) &= babc^{-1}l_5^{-1} = abc^{-1}l_5^{-1}bc^{-1}l_5^{-1} = abc^{-1}bl_6^{-1}c^{-1}l_5^{-1} = abbc^{-1}l_6^{-1}c^{-1}l_5^{-1} \\ &= ab^2c^{-1}c^{-1}l_6^{-1}l_5^{-1} = ab^2c^{-2}l_6^{-1}l_5^{-1} = acc^{-2}l_6^{-1}l_5^{-1} = ac^{-1}l_6^{-1}l_5^{-1} \\ &= ac^{-1}l_5^{-1}l_6^{-1} = ac^{-1}c^2 = ac. \end{aligned}$$

The third consistency relation i.e. $g_j(g_i^{e_i}) = (g_jg_i)g_i^{e_i-1}$ for $j > i, i \in I$, is satisfied given that the following relations hold:

(i) $ba^2 = (ba)a$,
(ii) $ca^2 = (ca)a$,
(iii) $l_1a^2 = (l_1a)a$,
(iv) $l_2a^2 = (l_2a)a$,
(v) $l_3a^2 = (l_3a)a$,
(vi) $l_4a^2 = (l_4a)a$,
(vii) $l_5a^2 = (l_5a)a$,
(viii) $l_6a^2 = (l_6a)a$,
(ix) $cb^2 = (cb)b$,

(x) $l_1b^2 = (l_1b)b$,
(xi) $l_2b^2 = (l_2b)b$,
(xii) $l_3b^2 = (l_3b)b$,
(xiii) $l_4b^2 = (l_4b)b$,
(xiv) $l_5b^2 = (l_5b)b$,
(xv) $l_6b^2 = (l_6b)b$,
(xvi) $l_1c^2 = (l_1c)c$,
(xvii) $l_2c^2 = (l_2c)c$,
(xviii) $l_3c^2 = (l_3c)c$,
(xix) $l_4c^2 = (l_4c)c$,
(xx) $l_5c^2 = (l_5c)c$,
(xxi) $l_6c^2 = (l_6c)c$.

Based on the relations of $Q_4(6)$;
For (i);

$$\begin{aligned}
ba^2 &= bcl_6.\\
(ba)a &= abc^{-1}l_5^{-1}a = abc^{-1}al_5^{-1} = abac^{-1}l_5^{-1} = abac^{-1}l_5^{-1} = aabc^{-1}l_5^{-1}c^{-1}l_5^{-1}\\
&= a^2bc^{-2}l_5^{-1}l_5^{-1} = a^2bl_6l_5l_5^{-1}l_5^{-1} = a^2bl_6l_5^{-1} = cl_6bl_6l_5^{-1} = cbl_5l_6l_5^{-1}\\
&= cbl_6 = bcl_6.
\end{aligned}$$

Next, the relations of $Q_4(6)$ are shown to satisfy the fourth consistency relation by proving that $(g_i^{e_i})g_i = g_i(g_i^{e_i})$ for $i \in I$. Therefore,

(i) $(a^2)a = aa^2$,
(ii) $(b^2)b = bb^2$,
(iii) $(c^2)c = cc^2$.

By the relations of $Q_4(6)$,
For (i);

$$\begin{aligned}
(a^2)a &= cl_6a = cal_6 = acl_6.\\
aa^2 &= acl_6.
\end{aligned}$$

For (ii);

$$\begin{aligned}
(b^2)b &= cb = bc.\\
bb^2 &= bc.
\end{aligned}$$

For (iii);

$$\begin{aligned}
(c^2)c &= l_5^{-1}l_6^{-1}c = l_5^{-1}cl_6^{-1} = cl_5^{-1}l_6^{-1}.\\
cc^2 &= cl_5^{-1}l_6^{-1}.
\end{aligned}$$

Finally, for the fifth consistency relations i.e. $g_j = (g_jg_i^{-1})g_i$for $j > i, i \notin I$, the following relations are shown to be true.

(i) $l_2 = (l_2l_1^{-1})l_1$,
(ii) $l_3 = (l_3l_1^{-1})l_1$,
(iii) $l_4 = (l_4l_1^{-1})l_1$,
(iv) $l_5 = (l_6l_1^{-1})l_1$,
(v) $l_6 = (l_6l_1^{-1})l_1$,
(vi) $l_3 = (l_3l_2^{-1})l_2$,
(vii) $l_4 = (l_4l_2^{-1})l_2$,
(viii) $l_5 = (l_5l_2^{-1})l_2$,
(ix) $l_6 = (l_6l_2^{-1})l_2$,
(x) $l_4 = (l_4l_3^{-1})l_3$,
(xi) $l_5 = (l_5l_3^{-1})l_3$,
(xii) $l_6 = (l_6l_3^{-1})l_3$,
(xiii) $l_5 = (l_5l_4^{-1})l_4$,
(xiv) $l_6 = (l_6l_4^{-1})l_4$,
(xv) $l_6 = (l_6l_5^{-1})l_5$.

By applying the relations of $Q_4(6)$, it is found that:

(i) $(l_2 l_1^{-1}) l_1 = l_1^{-1} l_2 l_1 = l_2$.
(ii) $(l_3 l_1^{-1}) l_1 = l_1^{-1} l_3 l_1 = l_3$.
(iii) $(l_4 l_1^{-1}) l_1 = l_1^{-1} l_4 l_1 = l_4$.
(iv) $(l_5 l_1^{-1}) l_1 = l_1^{-1} l_5 l_1 = l_5$.
(v) $(l_6 l_1^{-1}) l_1 = l_1^{-1} l_6 l_1 = l_6$.
(vi) $(l_3 l_2^{-1}) l_2 = l_2^{-1} l_3 l_2 = l_3$.
(vii) $(l_4 l_2^{-1}) l_2 = l_2^{-1} l_4 l_2 = l_4$.
(viii) $(l_5 l_2^{-1}) l_2 = l_2^{-1} l_5 l_2 = l_5$.
(ix) $(l_6 l_2^{-1}) l_2 = l_2^{-1} l_6 l_2 = l_6$.
(x) $(l_4 l_3^{-1}) l_3 = l_3^{-1} l_4 l_3 = l_4$.
(xi) $(l_5 l_3^{-1}) l_3 = l_3^{-1} l_5 l_3 = l_5$.
(xii) $(l_6 l_3^{-1}) l_3 = l_3^{-1} l_6 l_3 = l_6$.
(xiii) $(l_5 l_4^{-1}) l_4 = l_4^{-1} l_5 l_4 = l_5$.
(xiv) $(l_6 l_4^{-1}) l_4 = l_6^{-1} l_5 l_4 = l_6$.
(xv) $(l_6 l_5^{-1}) l_5 = l_5^{-1} l_6 l_5 = l_6$.

Since the presentation of $Q_4(6)$ satisfies the consistency relations given in Definition 2, therefore $Q_4(6)$ has a consistent polycyclic presentation.

4. Conclusion

In this research, the new polycyclic presentation that satisfy its consistency relations for one of the four groups of extension of a free abelian lattice group by quaternion point group is shown. This polycyclic presentation which is consistent is later will be used in finding the homological invariants of the group.

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