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The perfect codes of commuting zero divisor graph of some matrices of dimension two

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Abstract. The study of graph properties has gathered many attentions in the past years. The graph properties that are commonly studied include the chromatic number, the clique number and the domination number of a finite graph. In this study, a type of graph properties, which is the perfect code is studied. The perfect code is originally used in coding theory, then extended to other fields including graph theory. Hence, in this paper, the perfect code is determined for the commuting zero divisor graphs of some finite rings of matrices. First, the commuting zero divisor graph of the finite rings of matrices is constructed where its vertices are all zero divisors of the ring and two distinct vertices, say x and y, are adjacent if and only if $xy = yx = 0$. Then, from the vertex set of the graph, the neighborhood elements of the vertices are determined in order to compute the perfect codes of the graph.

1. Introduction

A graph is defined as a structure which consist of vertices and edges [1]. It has been widely used in various fields to show relations between objects. For instance, graphs are used to show the covalently bonded compound in chemistry [2]. In addition, graphs are used to describe tight-binding models [3] in physics.

In mathematics, particularly in group theory, graphs have been frequently used to show relations between the algebraic structures of groups. For example, a co-prime graph of a finite group G is a graph where its vertices are the elements of G and two vertices are adjacent if and only if the greatest common divisor of the order of the elements is equal to one [4]. Hence, the co-prime characteristics between the elements are described by the co-prime graph. There are many more graphs introduced related to finite groups, which includes the commuting graph [5], noncommuting graph [6], conjugacy class graph [7] and orbit graph [8].

The idea of graphs has also gained the attention of ring theorists, where it is studied on some finite rings. This includes the zero divisor graph [9], inclusion ideal graph [10] and Jacobson graph [11]. In this article, a variant of the zero divisor graph, which is the commuting zero divisor graph is studied.

Meanwhile, properties of graph have been a popular topic in graph theory. The properties of graph are usually determined to assist in understanding the characteristics of the graph. Some

graph properties which are commonly studied by the researchers are the chromatic number, clique number, dominating number and independent number of finite graphs. In this article, one of the graph properties, which is the perfect codes will be discussed for the commuting zero divisor graph.

This article is divided into five sections. The first section is the introduction, followed by some preliminaries on zero divisor graph and perfect codes in the second and third section, respectively. The fourth section presents the results and discussion. Lastly, the summary of this study will be provided in the conclusion as the fifth section.

2. Zero Divisor Graph of Finite Rings

In this section, past researches and recent updates on a graph related to finite rings, which is the zero divisor graph is provided. To begin, the definition of zero divisor of a finite ring is firstly given as follows.

Definition 1. [12] *Let x and y be two nonzero elements of a finite ring R. If $xy = 0$, then x and y are the zero divisors of R. The set of all zero divisors in R is denoted as $Z(R)$.*

The idea of zero divisor graph was originated from Beck [13] where the focus of the study was on the coloring of graphs. Beck had discussed on a graph where the vertices are the elements of a finite commutative ring R and two vertices are adjacent if and only if the product is zero.

Then, few years later, Anderson and Livingston [9] had formally defined the graph by the following definition.

Definition 2. [9] *Let R be a commutative ring (with 1) and $Z(R)$ be its set of zero divisors. Then, the zero divisor graph, $\Gamma(R)$ is a simple graph with vertices $Z(R)^* = Z(R) - \{0\}$, the set of nonzero zero divisors of R, and for distinct $x, y \in Z(R)^*$, the vertices x and y are adjacent if and only if $xy = 0$.*

Afterwards, many studies have been done on the zero divisor graph, focusing on finite commutative rings. Then, Redmond [14] had extended the research on the zero divisor graph of finite noncommutative ring, where the author had found that the graph must be a directed graph. The definition is given in the following.

Definition 3. [14] *Let R be a ring. Then, $\Gamma(R)$ is defined as a graph where $Z(R)^* = Z(R) - \{0\}$ is its vertices, and $x \rightarrow y$ is an edge between distinct vertices x and y if and only if $xy = 0$.*

The study of zero divisor graph on noncommutative rings have been done by some researches, which include Wu [15], where the author focused on Artinian rings. Then, Zaid *et al.* [16] had also studied the graph on some finite rings of matrices. In addition, Han [17] had studied the zero divisor graph under group actions in some finite ring of matrices.

A number of studies had been done on the zero divisor graph which leads to its extensions. Some extensions of the graph includes the total zero divisor graph [18], compressed zero divisor graph [19] and commuting zero divisor graph [20]. In this article, the focus is given on the commuting zero divisor graph of some finite ring of matrices. The definition of the commuting zero divisor graph is given in the following.

Definition 4. [20] *The commuting zero divisor graph of a finite noncommutative ring R is a simple graph of R where its vertices is the set of zero divisors of R, $Z(R)$. A directed edge is constructed between two vertices, say x and y in $Z(R)$ if and only if $xy = yx = 0$.*

In this article, the commuting zero divisor graph of a finite ring will be denoted as $\Gamma_{comm}(R)$.

3. Perfect Codes of Finite Graphs

In this section, some definitions and concepts related to the perfect codes of finite graphs are presented.

Just like its name, perfect codes is originally associated with coding theory. In coding theory, the study started with error correcting codes which is used to correct errors on noisy communication channel [21]. This subject has then been extended to other fields, including graph theory. The study of perfect codes in graphs was started by Biggs [22] where the author investigated on the existence of nontrivial perfect codes of the distance-transitive graph, depending on the diameter of the graph.

Later, Kratichvil [23] extended the definition of perfect codes in graphs, where the author introduced the t-perfect codes, focusing on the complete bipartite graph and products of graphs. To determine the t-perfect codes, the independent sets of the graph are considered.

In this article, the perfect codes of the commuting zero divisor graph of some finite rings are determined using a method discussed by Van Lint [24], where the distance between vertices of a graph is considered. The related definitions are firstly given below.

Definition 5. [25] *The distance between two distinct vertices x and y of a connected simple graph is the length (number of edges) of the shortest path between x and y.*

Definition 6. [25] *The set of all neighbors (adjacent) vertex of a vertex v in a graph G is called the neighborhood of v.*

If any subset C of a vertex set V is called a code, then the definitions of the e-error-correcting code and e-perfect code as described by Van Lint [24] is as follows:

Definition 7. [24] *If $S_e(x)$ is the set of neighborhood elements of x with distance less than or equal to e, then a code C is an e-error-correcting code if for all x and y in C, $S_e(x) \cap S_e(y) = \emptyset$ when x and y are distinct.*

Definition 8. [24] *If for an e-error-correcting code C, the union, $\underset{x \in C}{\cup} S_e(x) = V$, then the code is called a perfect code.*

In this article, based on the e-error-correcting code, its perfect code is called the e-perfect code.

4. Results and Discussions

In this section, the main results of the perfect codes of the commuting zero divisor graphs are obtained for some finite rings. The finite rings considered are the ring of 2×2 matrices over $\mathbb{Z}_3$ and $\mathbb{Z}_4$.

First, the commuting zero divisor graphs are constructed based on Definition 4. Then, from the graphs constructed, the neighborhood elements of their vertices are analyzed and the e-perfect code of the graphs are determined using Definition 8. The results are presented in the following propositions.

Proposition 1. *Let $R_1 = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \middle| a, b, c, d \in Z_3 \right\}$. Then, its commuting zero divisor graph, $\Gamma_{comm}(R_1)$ is a directed graph of 32 vertices and 56 edges.*

Proof. Suppose $R_1 = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \middle| a, b, c, d \in Z_3 \right\}$. Referring to Definition 1 and the study done by Zaid *et al.* [16], it is found that the ring R_1 has 32 zero divisors, which are listed as follows.

1) $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$
9) $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$
17) $\begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix}$
25) $\begin{bmatrix} 2 & 0 \\ 1 & 0 \end{bmatrix}$

2) $\begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}$
10) $\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$
18) $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$
26) $\begin{bmatrix} 2 & 0 \\ 2 & 0 \end{bmatrix}$

3) $\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$
11) $\begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix}$
19) $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$
27) $\begin{bmatrix} 2 & 1 \\ 0 & 0 \end{bmatrix}$

4) $\begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}$
12) $\begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}$
20) $\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$
28) $\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$

5) $\begin{bmatrix} 0 & 0 \\ 1 & 2 \end{bmatrix}$
13) $\begin{bmatrix} 0 & 2 \\ 0 & 1 \end{bmatrix}$
21) $\begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$
29) $\begin{bmatrix} 2 & 1 \\ 2 & 1 \end{bmatrix}$

6) $\begin{bmatrix} 0 & 0 \\ 2 & 0 \end{bmatrix}$
14) $\begin{bmatrix} 0 & 2 \\ 0 & 2 \end{bmatrix}$
22) $\begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}$
30) $\begin{bmatrix} 2 & 2 \\ 0 & 0 \end{bmatrix}$

7) $\begin{bmatrix} 0 & 0 \\ 2 & 1 \end{bmatrix}$
15) $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$
23) $\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$
31) $\begin{bmatrix} 2 & 2 \\ 1 & 1 \end{bmatrix}$

8) $\begin{bmatrix} 0 & 0 \\ 2 & 2 \end{bmatrix}$
16) $\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$
24) $\begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$
32) $\begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$.

Hence, referring back to Definition 4, it is clear that the graph $\Gamma_{comm}(R_1)$ has 32 vertices. Two vertices are adjacent if and only if the elements commute and their product is the zero matrix. Therefore, the graph $\Gamma_{comm}(R_1)$ has 56 directed edges, as shown in figure 1. The vertices, numbered from 1 to 32, represent the zero divisors as listed previously.
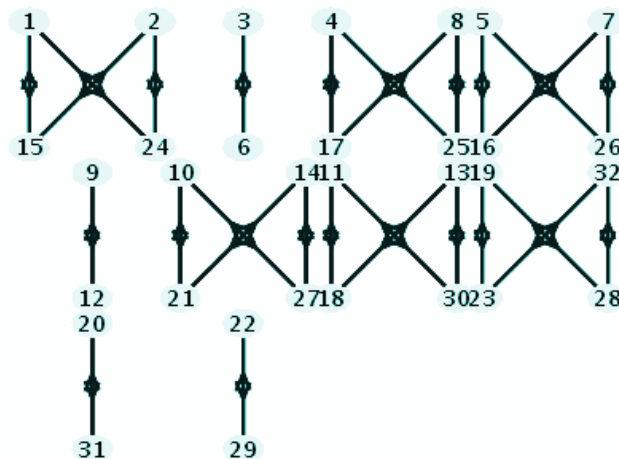

Figure 1. The commuting zero divisor graph of R_1

Based on $\Gamma_{comm}(R_1)$, the set of neighborhood elements of its vertices are determined and the e-perfect codes are then computed in the next two propositions.

Proposition 2. *Let* $\Gamma_{comm}(R_1)$ *be the commuting zero divisor graph of the ring* R_1. *Then,* $\Gamma_{comm}(R_1)$ *has no 1-perfect code.*

Proof. Suppose $\Gamma_{comm}(R_1)$ is a graph as shown in Figure 1. Based on Definition 7, the *e*-error-correcting codes of R_1 are determined. First, the set of all neighborhood elements in $V(\Gamma_{comm}(R_1))$ are determined.

When $e = 1$, the set $S_1(x)$, which is the set of all neighborhood elements with distance less than or equal to one (namely 0 and 1) are determined for all x in $V(\Gamma_{comm}(R_1))$. The sets are listed as follows.

- $S_1(1) = \{1, 15, 24\}$
- $S_1(2) = \{2, 15, 24\}$
- $S_1(3) = \{3, 6\}$
- $S_1(4) = \{4, 17, 25\}$
- $S_1(5) = \{5, 16, 26\}$
- $S_1(6) = \{3, 6\}$
- $S_1(7) = \{7, 16, 26\}$
- $S_1(8) = \{8, 17, 25\}$
- $S_1(9) = \{9, 12\}$
- $S_1(10) = \{10, 21, 27\}$
- $S_1(11) = \{11, 18, 30\}$

- $S_1(12) = \{9, 12\}$
- $S_1(13) = \{13, 18, 30\}$
- $S_1(14) = \{14, 21, 27\}$
- $S_1(15) = \{1, 2, 15\}$
- $S_1(16) = \{5, 7, 16\}$
- $S_1(17) = \{4, 8, 17\}$
- $S_1(18) = \{11, 13, 18\}$
- $S_1(19) = \{19, 23, 28\}$
- $S_1(20) = \{20, 31\}$
- $S_1(21) = \{10, 14, 21\}$
- $S_1(22) = \{22, 29\}$

- $S_1(23) = \{19, 23, 32\}$
- $S_1(24) = \{1, 2, 24\}$
- $S_1(25) = \{4, 8, 25\}$
- $S_1(26) = \{5, 7, 26\}$
- $S_1(27) = \{10, 14, 27\}$
- $S_1(28) = \{19, 28, 32\}$
- $S_1(29) = \{22, 29\}$
- $S_1(30) = \{11, 13, 30\}$
- $S_1(31) = \{20, 31\}$
- $S_1(32) = \{23, 28, 32\}$

Let $C = \{1, 3, 4\}$ be a code. Since $S_1(1) \cap S_1(3) = \emptyset$, $S_1(3) \cap S_1(4) = \emptyset$ and $S_1(1) \cap S_1(4) = \emptyset$, then the code C is an 1-error-correcting code of $\Gamma_{comm}(R_1)$. However, from Definition 8, since $S_1(1) \cup S_1(3) \cup S_1(4) = \{1, 3, 4, 6, 15, 17, 24, 25\} \neq V(\Gamma_{comm}(R_1))$, then the code $C = \{1, 3, 4\}$ is not a 1-perfect code of $\Gamma_{comm}(R_1)$.

Other possibilities of 1-error-correcting codes of $\Gamma_{comm}(R_1)$ includes $\{1, 3\}, \{7, 8, 9, 10\}$ and $\{1, 3, 4, 5, 9, 10, 11, 13, 19, 20, 22\}$. But since the union of the sets is not equal to the whole vertex set of $\Gamma_{comm}(R_1)$, then the codes are not 1-perfect codes of $\Gamma_{comm}(R_1)$.

Proposition 3. *Let* $\Gamma_{comm}(R_1)$ *be the commuting zero divisor graph of* R_1. *Then,* $C = \{1, 3, 4, 5, 9, 10, 11, 19, 20, 22\}$ *is a 2-perfect code of* $\Gamma_{comm}(R_1)$.

Proof. Suppose $\Gamma_{comm}(R_1)$ is as shown in Figure 1. To obtain the 2-error-correcting code, the set of neighborhood elements of the the vertices $V(\Gamma_{comm}(R_1))$ is firstly determined for distance less than or equal to two and are listed as follows.

- $S_2(1) = \{1, 2, 15, 24\} = S_2(2) = S_2(15) = S_2(24)$
- $S_2(3) = \{3, 6\} = S_2(6)$
- $S_2(4) = \{4, 8, 17, 25\} = S_2(8) = S_2(17) = S_2(25)$
- $S_2(5) = \{5, 7, 16, 26\} = S_2(7) = S_2(16) = S_2(26)$
- $S_2(9) = \{9, 12\} = S_2(12)$
- $S_2(10) = \{10, 14, 21, 27\} = S_2(14) = S_2(21) = S_2(27)$
- $S_2(11) = \{11, 13, 18, 30\} = S_2(13) = S_2(18) = S_2(30)$
- $S_2(19) = \{19, 23, 28, 32\} = S_2(23) = S_2(28) = S_2(32)$
- $S_2(20) = \{20, 31\} = S_2(31)$
- $S_2(22) = \{22, 29\} = S_2(29)$

Let $C = \{1, 3, 4, 5, 9, 10, 11, 19, 20, 22\}$ be a code. From the sets stated above, the intersections, $S_2(x_1) \cap S_2(x_2) = \emptyset$ for all x_1, x_2 in C. Therefore, C is a 2-error-correcting

code of $\Gamma_{comm}(R_1)$. In addition, the union $\underset{x \in C}{\cup} S_2(x) = V(\Gamma_{comm}(R_1))$. Hence, C is also a 2-perfect code of $\Gamma_{comm}(R_1)$.

Other possibilities of the 2-perfect codes of $\Gamma_{comm}(R_1)$ includes $\{1, 6, 8, 5, 9, 10, 11, 19, 20, 29\}$ and $\{2, 3, 12, 16, 17, 21, 29, 30, 31, 32\}$.

From Figure 1, the maximum distance of the graph is two. Therefore, when $e > 2$, the results of the e-perfect codes will be the same as $e = 2$.

Proposition 4. *Let* $R_2 = \left\{ \left[\begin{array}{cc} a & b \\ c & d \end{array} \right] \middle| a, b, c, d \in Z_4 \right\}$. *Then, its commuting zero divisor graph,* $\Gamma_{comm}(R_2)$ *is a directed graph of 159 vertices and 314 edges.*

Proof. Let $R_2 = \left\{ \left[\begin{array}{cc} a & b \\ c & d \end{array} \right] \middle| a, b, c, d \in Z_4 \right\}$. Based on Definition 1, it is found that the ring R_2 has 159 zero divisors. Hence, from Definition 4, its commuting zero divisor graph, $\Gamma_{comm}(R_2)$ has 159 vertices. Then, two vertices of the graph are adjacent if and only if the elements commute and the product is the zero matrix. Hence, it is found that $\Gamma_{comm}(R_2)$ has a total of 314 directed edges.

Proposition 5. *Let* $\Gamma_{comm}(R_2)$ *be the commuting zero divisor graph of* R_2. *Then,* $\Gamma_{comm}(R_2)$ *does not have any 1-perfect code and 2-perfect code.*

Proof. Suppose $\Gamma_{comm}(R_2)$ be the commuting zero divisor graph of R_2. The proof follows that of Proposition 2. By Definition 7, it is found that when $e = 1$, for all elements x in a code C, there is no 1-error-correcting code that satisfies the union condition as in Definition 8 where $\underset{x \in C}{\cup} S_1(x) = V(\Gamma_{comm}(R_2))$. Therefore, $\Gamma_{comm}(R_2)$ has no 1-perfect code. The same situation occurs when $e = 2$. Hence, $\Gamma_{comm}(R_2)$ does not have any 2-perfect code.

Proposition 6. *Let* $\Gamma_{comm}(R_2)$ *be the commuting zero divisor graph of* R_2. *Then,* $C = \{x\}$ *for all* $x \in V(\Gamma_{comm}(R_2))$ *is a 3-perfect code of* $\Gamma_{comm}(R_2)$.

Proof. Suppose $\Gamma_{comm}(R_2)$ be the commuting zero divisor graph of R_2. With similar method as in Proposition 2, the 3-error-correcting code of $\Gamma_{comm}(R_2)$ is determined. It is found that for all x in $V(\Gamma_{comm}(R_2))$, $S_3(x) = V(\Gamma_{comm}(R_2))$. Hence, the code $C = \{x\}$ is a perfect code since it satisfies the union operation as stated in Definition 8. This also shows that the maximum distance of the graph is three.

5. Conclusion

In this paper, the commuting zero divisor graph of the ring of 2×2 matrices over integer modulo three and four have been constructed. Based on the commuting zero divisor graphs, the e-perfect codes are determined. It has been found that the graph $\Gamma_{comm}(R_1)$ has no 1-perfect code. However, the graph has several 2-perfect codes. Meanwhile for the graph $\Gamma_{comm}(R_2)$, there is no 1-perfect code and 2-perfect code, but all singleton sets of $V(\Gamma_{comm}(R_2))$ is a 3-perfect code of the graph. From the results, it is concluded that if n is the maximum distance of a graph, then the e-perfect codes of the graph when $e > n$ will be the same as $e = n$.

Acknowledgments

References

[1] Bondy J and Murty U 1976 *Graph theory with application* (New York: Elsevier)
[2] Balaban A T 1985 *Journal of Chemical Information and Computer Sciences* **25** 334343
[3] Chalker J T, Pickles T S and Shukla P 2010 *Physical Review B* **82** 104209
[4] Zulkifli N and Ali N M M 2019 *Matematika* **35** 357369
[5] Iranmanesh A and Jafarzadeh A 2008 *Journal of Algebra and Its Applications* **7** 129146
[6] Abdollahi A, Akbari S and Maimani H R 2006 *Journal of Algebra* **298** 468492
[7] Alfandary G 1995 *Journal of Algebra* **176** 528 533
[8] El-Sanfaz M A, Sarmin N H and Omer S M S 2016 *International Journal of Pure and Applied Mathematics* **109** 235243
[9] Anderson D F and Livingston P S 1999 *Journal of Algebra* **217** 434447
[10] Akbari S, Habibi M, Majidinya A and Manaviyat R 2015 *Communications in Algebra* **43** 24572465
[11] Azimi A, Erfanian A and Farrokhi D G M 2013 *Journal of Algebra and Its Applications* **12** 1250179
[12] Fraleigh J and Katz V 2003 *A first course in abstract algebra* 7th ed (Addison-Wesley)
[13] Beck I 1988 *Journal of Algebra* **116** 208226
[14] Redmond S 2002 *International Journal of Commutative Rings* **1** 203211
[15] Wu T 2005 *Discrete Mathematics* **296** 7386
[16] Zaid N, Sarmin N H and Khasraw S M S 2021 *Jurnal Teknologi* **83** 127132
[17] Han J 2008 *Journal of the Korean Mathematical Society* **45** 16471659
[18] Duric A and Jevdenic S 2019 *Journal of Algebra and Its Applications* **18** 1950190
[19] Ma X, Wang D and Zhou J 2016 *Journal of the Korean Mathematical Society* **53** 519532
[20] Zaid N, Sarmin N H and Khasraw S M S 2020 *AIP Conference Proceedings* **2266** 060010
[21] Hamming R W 1950 *The Bell System Technical Journal* **29** 147160
[22] Biggs N 1973 *Journal of Combinatorial Theory (B)* **15** 289296
[23] Kratochvil J 1986 *Journal of Combinatorial Theory Series B* **40** 224228
[24] Lint J H V 1975 *Rocky Mountain Journal of Mathematics* **5** 199224
[25] Rosen K H 2007 *Discrete mathematics and its applications* 6th ed (New York: McGraw-Hill)