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ADVANCES OF SET THEORIES OF UNCERTAINTY APPROACHES ON DECISION-MAKING THEORY

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INTRODUCTION

Most of our real-life problems in engineering, science and technology and medical sciences involve imprecise data and their solutions involve the use of **mathematical principles**, based on uncertainty and imprecision to handle such uncertainties, some number of theories have been proposed.

Some of these are :

1) Fuzzy set theory

2) Rough set theory

3) Soft set theory

Decision Theories

In 1965, L. A. Zadeh introduced theory of **fuzzy sets** that deal with a kind of uncertainty known as **“fuzziness”.**

However, the fact that the membership degrees of the elements belonging to the universe set in the theory are expressed only with 0 or 1 makes it difficult to express the uncertainty in the most accurate way.

Although fuzzy set theory is very successful in handling uncertainties arising from vagueness, it cannot model all sorts of uncertainties prevailing in different real physical problems.

Thus, search for new theories has been continued.

In 1982, Z. Pawlak came up with **rough set theory.**

In 1990, Molodtsov initiated a novel concept called **soft set theory.**

FUZZY SET THEORY

01

The word ‘**fuzzy**’ means ‘vagueness (ambiguity)’

02 **Fuzziness** occurs when the boundary of a piece of information is not clear-cut.

03

Fuzzy set was introduced by Lotfi Zadeh in 1965 as an extension of classical notation set.

04

Classical set theory allows the membership of the elements in the set of binary terms.

FUZZY SET THEORY

- A membership function $\mu_x^{(x)}$ is associated with a fuzzy set *A* such that the function maps every element of universe of discourse *X* to the interval [0,1].
- The mapping is written as : $\mu_x(x): X \rightarrow [0,1]$.
- **A fuzzy set** is defined as follows:

  If *X* is a universe of discourse and *x* is a particular element of *X*, then a fuzzy set *A* defined on *X* can be written as a collection of ordered pairs

$$A = \{(x, \mu_x(x)), x \epsilon X\}$$

FUZZY SET THEORY

A fuzzy set operations are as follows:

i. Union: the union of two sets A and B is given as

$$A \cup B = \{x |\, x \in A \; or \; x \in B\}$$

ii. Intersection: the intersection of two sets A and B is given as

$$A \cap B = \{x |\, x \in A \; and \; x \in B\}$$

iii. Complement: it is denoted by Ă and is defined as

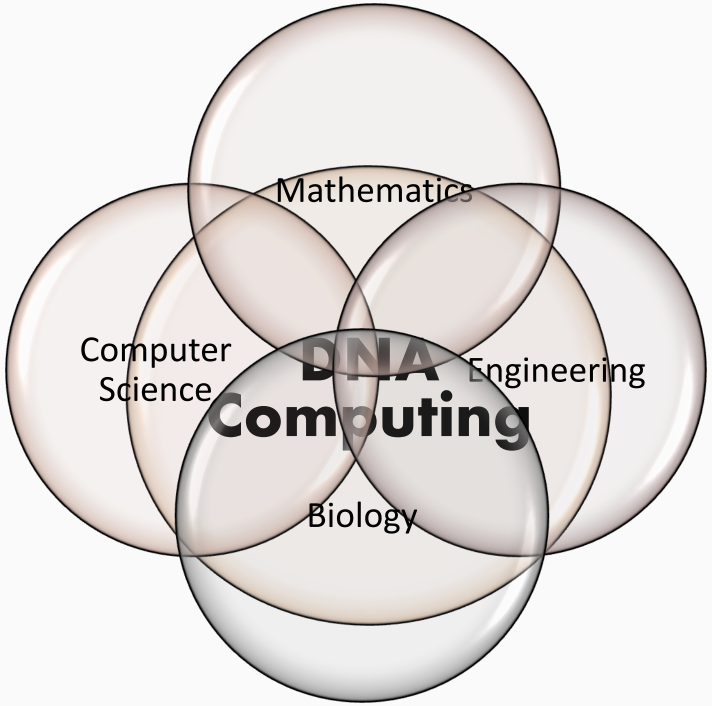
$$\breve{A} = \{x |\, x \; does \; not \; belong \; to \; A \; and \; x \in X\}$$

APPLICATION OF FUZZY SET THEORY

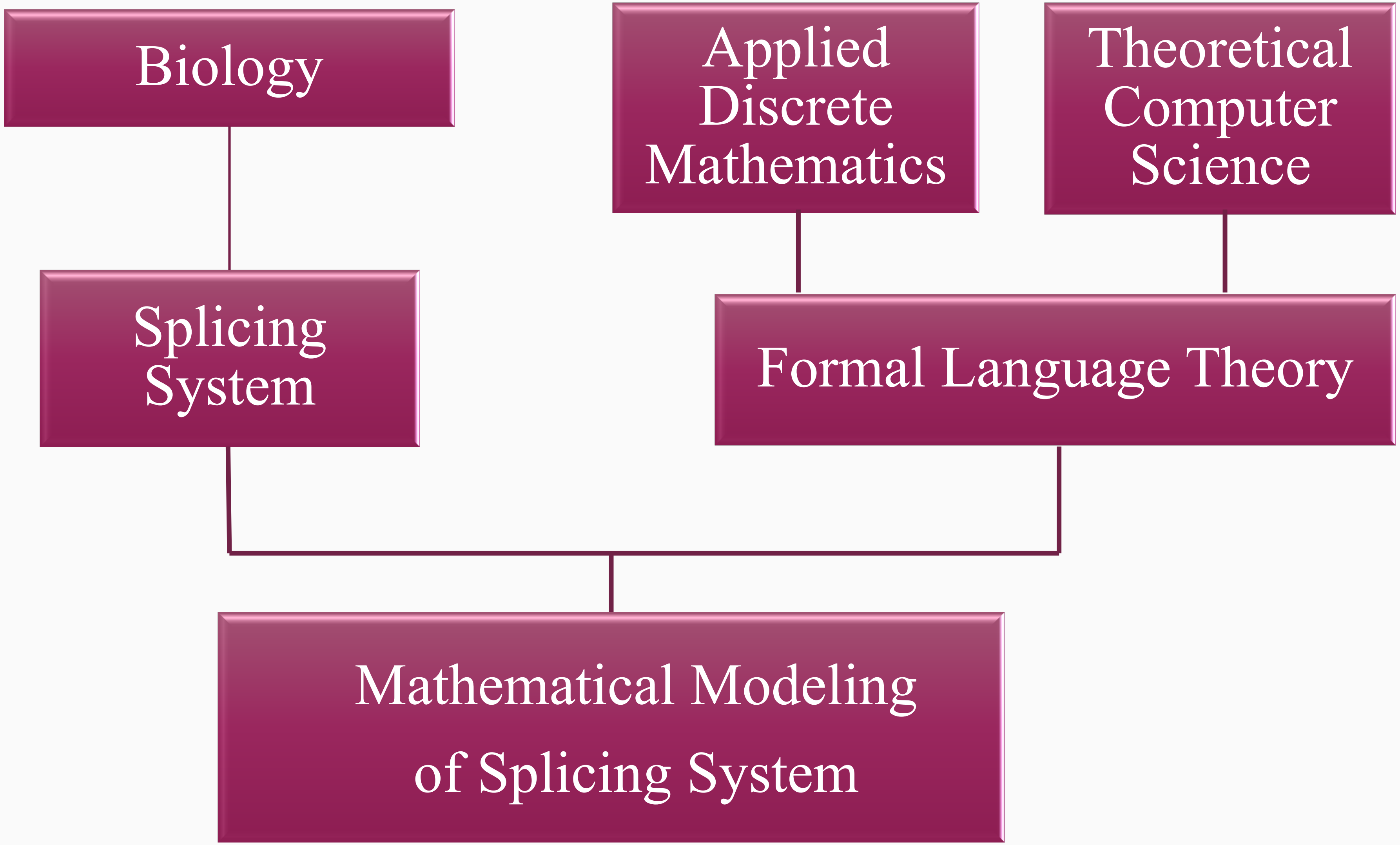
- One of the application of fuzzy set theory is fuzzy act as restriction in Splicing System.
- Splicing system is one of the DNA computing model which has grown significantly.

DNA COMPUTATION

- DNA computation has emerged in the last twenty years as an exciting new research field at the intersection of computer science, biology, engineering and mathematics.



SPLICING SYSTEM



APPLICATION OF FUZZY SET THEORY (Cont.)

Karimi *et. al.* in 2014 introduced fuzzy as restriction of splicing system is to increase the computational power of language generated by splicing systems.

Definition: A fuzzy extended splicing system is a 6-tuple $\gamma = (V, T, A, R, \mu, \odot)$ where V, T, R are defined as for usual extended splicing system, $\mu: V^* \rightarrow [0,1]$ is fuzzy membership function, A is a subset of $V^* \times [0,1]$ and $\odot$ is a fuzzy operation over [0,1].

Theorem: *Every fuzzy splicing system with the fuzzy operation: multiplication, max or min, and the cut-point: any number in* [0, 1] *or any subinterval of* [0, 1] *generates a regular language*.

Fariba Karimi, Sherzod Turaev, Nor Haniza Sarmin and Fong Wan Heng. “Fuzzy Splicing Systems”, Computational Collective Intelligience. Technologies and Applications (Lecture Notes in Computer Science), 2014, 8733: 20 – 29.

APPLICATION OF FUZZY SET THEORY (Cont.)

- As a result, fuzzy splicing systems can increase the computational power of generated languages up to context-sensitive languages.

- On the other hand, fuzzy splicing systems allow modeling molecular uncertainty processes appearing in molecular biology, systems biology and medicine.

- The study of fuzzy splicing systems in particular and the fuzzy variants of other theoretical models of DNA computing makes a significant contributions to formal language and automata theories.

ROUGH SET THEORY

- Rough set theory has been developed by Pawlak in 1982.
- It has emerged as major mathematical method to manage uncertainties from inexact, noisy and incomplete information.
- A rough set provides a representation of a given set using lower and upper approximations.
- The main objective of rough set analysis is to synthesis the approximation of concepts from the acquired data.
- It has been proven that rough set methods have enormous potential in dealing with uncertainties in decision making.

ROUGH SET THEORY (Cont.)

The definition of a rough set is given as follows:

Definition: Let U denote a finite and nonempty set called the universe. Suppose $R \rightarrow U \times U$ is an equivalence relation on U, i.e. R is reflexive, symmetrical and transitive. The equivalence relation R partitions the set U into disjoint subsets. It is a quotient set of the universe and is denoted by U/R. Elements in the same equivalence class are said to be indistinguishable. Equivalence classes of R are called elementary sets. Every union of elementary sets is called a definable set. The empty set is considered to be a definable set, thus all the definable sets form a Boolean algebra. (U, R) is called a Pawlak approximation space. Given an arbitrary set X of U, one can characterize X by a pair of lower and upper approximations. The lower approximation $\underline{\mathrm{R}}(X)$ is the greatest definable set contained in X, and the upper approximation $\hat{\mathrm{R}}(X)$ is the least definable set containing X. **The tuple composed of the lower and upper approximation** is called a rough set; thus, a rough set is composed of two crisp sets, one representing a lower boundary of the target set, and the other representing an upper boundary of the target set .

REDUCT AND CORE OF A KNOWLEDGE SYSTEM

- Reduct is the essential part; the set of attributes which supplies the same quality of classification as the original set of attributes.
- Core is the most important part of this knowledge; the collection of the most important attributes of a knowledge system.

$$CORE(P) = \cap\, RED(P),$$

where RED(P) is the family of all the “reducts” of P.

ROUGH SET THEORY

Definition: If R is an arbitrary crisp relation from U to V, then the triplet (U, V, R) is referred to as a generalized approximation space. For any set X $\subseteq$ V, a pair of lower and upper approximation, $\hat{R}(X)$ and $\underset{\cdot}{R}(X)$ are defined by

$$\underset{\cdot}{R}(X) = \{x \in U | R_s(x) \subseteq X\},$$
$$\hat{R}(X) = \{x \in U | R_s(x) \cap X \neq \emptyset\}.$$

The pair $\left(\underset{\cdot}{R}(X), \hat{R}(X)\right)$ is referred to as a generalized crisp rough set and $\underset{\cdot}{R}(X)$ and $\hat{R}(X) : P(V) \to P(U)$ are referred to as the lower and upper generalized crisp approximation operators, respectively.

SOME APPLICATIONS OF ROUGH SET THEORY

- Artificial intelligence (Pawlak, 1991; Pawlak et al., 1995)
- Knowledge discovery in clinical database (Tsumoto, 2000)
- Robotic systems (Bit & Beaubouef, 2008)
- The processing of large database (Lin, 2008)
- Treatment of imprecision in information systems (Gomes & Gomes, 2001)

Decision Rules for the Rough Set Theory

- S = (U, Q, V, f) where S is an information system; U is a finite set of objects; Q is a finite set of attributes; $V = \cup_{q \in Q} V_q$, where V_q is the domain of the attribute q and, $f: U \times Q \rightarrow V$ is a total function so that, $f(x, q) \in V_q$ for each $q \in Q,\ x \in U$.

- For every $P \subseteq Q$, we associate a formal language: a set of formulas *For* (P), built up from attribute-value pairs (a, v) where $a \in P$ & $v \in V$, and logical connectives $\wedge$ (and), $\vee$ (or) and $\sim$ (not).

- Decision rule in S is an expression $\phi \rightarrow \psi$, where $\phi \in For(C),\ \psi \in For(D)$; C and D are conditions and decisions attributes, respectively; ϕ and ψ are referred to as conditions and decisions of the rule, respectively.

Decision Rules for the Rough Set Theory (Cont.)

- With every decision rule $\phi \rightarrow \psi$, we can associate a conditionally probability:

$$\pi_S\,(\psi\,|\phi) = card\,(\,|\psi \wedge \phi\,|_S)\;/\;card\,(\,|\phi|_S), \text{ where } |\phi|_S \neq \emptyset,$$

  called uncertainty factor of the decision rule (the frequency of objects having the property ψ in the set of objects having the property ϕ).

- We also use the coverage factor of the decision rule (for estimation of the quality of the decision rule):

$$\pi_S\,(\phi|\psi) = card\,(\,|\phi \wedge \psi|_S)\;/\;card\,(\,|\psi|_S)$$

- The certainty and the coverage factors of decision rules express how exact is your knowledge (data) about the considered reality.
- Finally, the strength factor of the decision rule:

$$\sigma_S\,(\phi,\psi) = \pi_S(\psi\,|\phi)\,\pi_S\,(\phi), \text{ where } \pi_S\,(\phi) = card\,(|\phi|_S)\;/\;card\,(|\cup|)$$

APPLICATION OF ROUGH SET THEORY (Cont.)

- New rough set model based on the binary relation $\hat{\mathrm{R}}_{\alpha}^{\beta}$ was introduced by Z. Zhang in 2012.
- It was defined by an intuitionistic fuzzy relation $\hat{\mathrm{R}}$ between two different non-empty universes U and V and a threshold value pair (α, β).
- Some properties of the new rough sets model between two different universes was given.
- The definition and theorem that were introduced are given next.

Zhang, Z. (2012). A rough set approach to intuitionistic fuzzy soft set based decision making. *Applied Mathematical Modelling*, *36*(10), 4605-4633.

APPLICATION OF ROUGH SET THEORY (Cont.)

Definition: Denote $L = \{(\alpha, \beta)|\alpha\epsilon[0,1], \beta\epsilon[0,1], \alpha + \beta \leq 1\}$. Relation $\leq_L$ was defined as follows:

$$\forall(\alpha, \beta), (\xi, \eta) \in L,\ (\alpha, \beta) \leq_L (\xi, \eta) \leftrightarrow\ \alpha \leq \xi \text{ and } \beta \geq \eta.$$

Then the relation $\leq_L$ is a partial ordering on L and the pair $(L, \leq_L)$ Is a complete lattice with the smallest element 0_L = (0, 1) and the greatest element $1_L = (0,1)$. The meet operator $\wedge$ and the join operator $\vee$ on $(L, \leq_L)$ which are linked to the ordering $\leq_L$ are, respectively, defined as follows:

- $\forall(\alpha, \beta), (\xi, \eta)\ \in L$
- $(\alpha, \beta)\wedge(\xi, \eta) = \{\min(\alpha, \xi), \max(\beta, \eta)\},$
- $(\alpha, \beta)\vee(\xi, \eta) = \{max(\alpha, \xi), min(\beta, \eta)\}.$

Zhang, Z. (2012). A rough set approach to intuitionistic fuzzy soft set based decision making. *Applied Mathematical Modelling*, *36*(10), 4605-4633.

APPLICATION OF ROUGH SET THEORY (Cont.)

Theorem : Let U, V be two non-empty finite universes, $(\alpha_1, \beta_1), (\alpha_2, \beta_2) \in L$, and, $(\alpha_1, \beta_1) \leq_L (\alpha_2, \beta_2)$. Then for any X $\subseteq$ V, we have

$$\hat{R}_{\alpha 1}^{\beta 1}(X) \subseteq \hat{R}_{\alpha 2}^{\beta 2}(X)$$

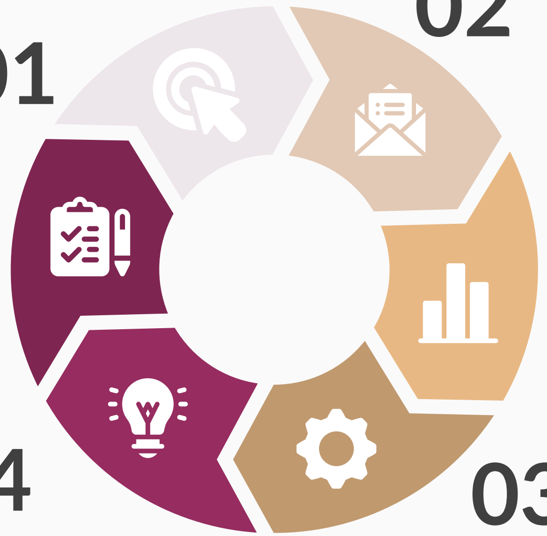
$$\hat{R}_{\alpha 2}^{\beta 2}(X) \subseteq \hat{R}_{\alpha 1}^{\beta 1}(X)$$

where $\hat{R}_{\alpha}^{\beta}$ is the intuitionistic fuzzy relation $\hat{R}$ between U and V and a threshold value pair (α, β).

The outcome of the study above is to show that approach proposed is more suitable for the decision making.

Zhang, Z. (2012). A rough set approach to intuitionistic fuzzy soft set based decision making. *Applied Mathematical Modelling*, *36*(10), 4605-4633.

SOFT SET THEORY



01

One of the deficiencies of fuzzy set theory is how to set the membership function.

02

Given the various observations on the efficiency of the existing tools for solving uncertainty problems, soft set theory emerged to soften these limitations.

03

Soft set is a parametrized general mathematical tool which deal with a collection of approximate descriptions of objects.

04

Approximate description has two parts:

1) Predicate
2) Approximate value

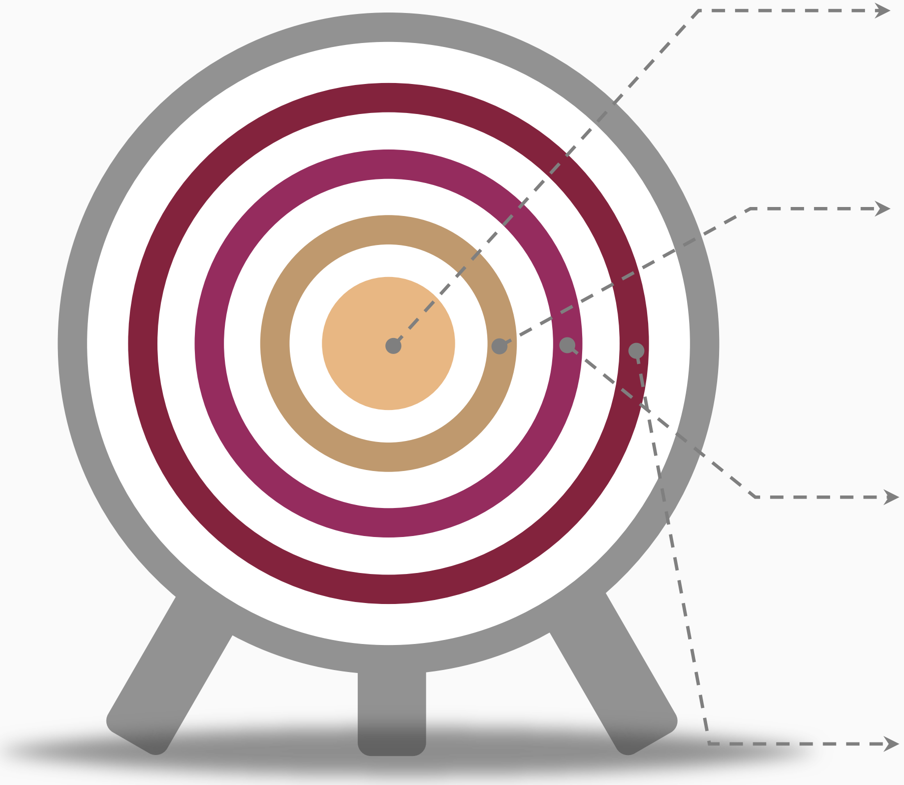
SOFT SET THEORY (Cont.)

Soft set is defined as follows:

Definition: Let U be the universal set and let E be a set of parameters. Let $P(U)$ denotes the power set of U.

A pair (F,E) is called a soft set over a given universal set U, if and only if F is a mapping of a set of parameters E, into the power set of U. That is, $F : E \to P(U)$. Clearly, a soft set over U is a parametrized family of subsets of a given universe U. Also, for any $e \in E$, $F(e)$ is considered as the set of e-approximate elements of the soft set (F,E).

Applications of Soft Set Theory



Game Theory

Operations Research

Riemann integration

Probability theory

FUZZY SOFT SETS

The definition of fuzzy soft set is as follows:

Definition: Let U be an initial universal set and let E be a set of parameters. Let I^U denote the power set of all fuzzy subsets of U. Let $A \subseteq E$.

A pair (F, E) is called a fuzzy soft set over U, where F is a mapping denoted by

$$F : A \to I^U$$

APPLICATION IN DECISION MAKING PROBLEM

An example

- Let $U=\{o_1,o_2,o_3,o_4,o_5,o_6\}$ he set of objects having different colors, sizes and surface texture features.
- The parameter set, E={blackish, dark brown, yellowish, reddish, large, small, very small, average, very large, course, moderately course, fine, extra fine}.
- Let A, B and C denote three subsets of the set of parameters E.

Also let A represent the color space and B represents the size of the object.

A = {blackish, dark brown, yellowish, reddish},

B = {large, very large, small, very small, average}.

The subset C represents the surface texture granularity .

C ={course, moderately course, fine, extra fine}.

Assuming that the fuzzy soft sets

(F, A) describe the objects having color space,

(G, B) describes the objects having size and

(H, C) describes the texture feature of the object surface.

The problem is to identify an unknown object from the multi observers' fuzzy data, specified by different observers, in terms of fuzzy soft sets (F, A), (G, B) and (H, C), as specified earlier. These fuzzy soft sets may be computed as follows.

The fuzzy-soft-set (F, A) is defined as follows :

$(F, A) = \{$objects having blackish colour$\}$

$= \{o_1/0.3, o_2/0.3, o_3/0.4, o_4/0.8, o_5/0.7, o_6/0.9\}$,

objects having dark brown colour$\}$

$= \{o_1/0.4, o_2/0.9, o_3/0.5, o_4/0.2, o_5/0.3, o_6/0.2\}$,

objects having yellowish colour$\}$

$= \{o_1/0.6, o_2/0.3, o_3/0.8, o_4/0.4, o_5/0.6, o_6/0.4\}$,

objects having reddish colour$\}$

$= \{o_1/0.9, o_2/0.5, o_3/0.7, o_4/0.8, o_5/0.5, o_6/0.3\}$.

U	blackish $= a_1$	*dark brown* $= a_2$	*yellowish* $= a_3$	*reddish* $= a_4$
o_1	0.3	0.4	0.6	0.9
o_2	0.3	0.9	0.3	0.5
o_3	0.4	0.5	0.8	0.7
o_4	0.8	0.2	0.4	0.8
o_5	0.7	0.3	0.6	0.5
o_6	0.9	0.2	0.4	0.3

The fuzzy-soft-set *(G,B) is*

U	'large = b_1 '	'very large = b_2 '	'small = b_3 '	'very small = b_4 '	'average = b_5 '
o_1	0.4	0.2	0.8	0.6	0.5
o_2	0.8	0.6	0.3	0.1	0.7
o_3	0.6	0.4	0.4	0.1	0.7
o_4	0.9	0.8	0.2	0.1	0.4
o_5	0.2	0.1	0.9	0.8	0.7
o_6	0.3	0.2	0.8	0.6	0.5

The fuzzy-soft-set *(H,C) is*

U	'course = c_1 '	'moderately course = c_2 '	'fine = c_3 '	'extra fine = c_4 '
o_1	0.3	0.4	0.1	0.9
o_2	0.6	0.5	0.4	0.5
o_3	0.5	0.6	0.3	0.6
o_4	0.7	0.6	0.6	0.3
o_5	0.6	0.6	0.5	0.4
o_6	0.8	0.7	0.7	0.9

Let (F, A) and (G, B) be any two fuzzy-soft-sets over the common universe U. After performing some operations (like AND, OR etc.) on the fuzzy-soft-sets for some particular parameters of A and B, we obtain another fuzzy-soft-set. The newly obtained fuzzy-soft-set is termed as resultant-fuzzy-soft-set of (F, A) and (G, B).

Considering the above two fuzzy-soft-sets (F, A) and (G, B) if we perform (F, A) AND (G, B) then we will have $4 \times 5 = 20$ parameters of the form e_{ij}, where $e_{ij} = a_i \wedge b_j$ all i = 1, 2, 3, 4 and j = 1, 2, 3, 4, 5. If we require the fuzzy soft set for the parameters $R = \{e_{11}, e_{15}, e_{21}, e_{24}, e_{33}, e_{44}, e_{45}\}$, then the resultant fuzzy soft set for the fuzzy soft sets (F, A) and (G, B) will be (K, R), say. So, after performing the (F, A) AND (G, B) for some parameters the tabular representation of the resultant fuzzy soft set will take the form as

U	'e11'	'e15'	'e21'	'e24'	'e33'	'e44'	'e45'
O1	0.3	0.3	0.4	0.4	0.6	0.6	0.5
o2	0.3	0.3	0.8	0.1	0.3	0.1	0.5
o3	0.4	0.4	0.5	0.1	0.4	0.1	0.7
o4	0.8	0.4	0.2	0.1	0.2	0.1	0.4
o5	0.2	0.7	0.2	0.3	0.6	0.5	0.5
o6	0.3	0.5	0.2	0.2	0.4	0.3	0.3

Let us now see how the algorithm may be used to solve our original problem.

Consider the fuzzy soft sets (F, A), (G,B) and (H,C) as defined above. Suppose that

$$P = \left\{ \begin{array}{l} e_{11} \wedge c_1, e_{15} \wedge c_3, e_{21} \wedge c_2, e_{24} \wedge c_4, \\ e_{33} \wedge c_3, e_{44} \wedge c_3, e_{45} \wedge c_4 \end{array} \right\}$$

be the set of choice parameters of an observer. On the basis of this parameter we have to take the decision from the availability set U. The tabular representation of resultant fuzzy soft set (S, P) will be as

U	$e_{11} \wedge c_1$	$e_{15} \wedge c_3$	$e_{21} \wedge c_2$	$e_{24} \wedge c_4$	$e_{33} \wedge c_3$	$e_{44} \wedge c_3$	$e_{45} \wedge c_4$
o_1	0.3	0.1	0.4	0.4	0.1	0.1	0.5
o_2	0.3	0.3	0.5	0.1	0.3	0.1	0.5
o_3	0.4	0.3	0.5	0.1	0.3	0.1	0.6
o_4	0.7	0.4	0.2	0.1	0.2	0.1	0.3
o_5	0.2	0.5	0.2	0.3	0.5	0.5	0.4
o_6	0.3	0.5	0.2	0.2	0.4	0.3	0.3

The Comparison table of the above resultant fuzzy soft set is as below

Table 1

	o_1	o_2	o_3	o_4	o_5	o_6
o_1	7	4	2	4	4	4
o_2	6	7	5	5	3	3
o_3	6	7	7	5	3	3
o_4	4	4	4	7	2	3
o_5	3	4	4	6	7	6
o_6	4	5	4	6	3	7

Next, we compute the row sum, column sum, and the score for each o_i as shown below:

Table 2

	Row sum (ri)	column sum (ti)	Score (Si)
o_1	25	30	−5
o_2	29	31	−2
o_3	31	26	5
o_4	24	33	−9
o_5	30	22	8
o_6	29	26	3

From the above score table, it is clear that the maximum score is 8, scored by o_5 and the decision is in favour of selecting o_5.

CONCLUSION

- It has been reported that a few application software tools based on the rough set theory have been developed and proven effective for specific applications. Hence, there are researches that can be best handled using rough set theories, especially in decision making.
- The development of integrated application software of rough sets should be a focus of future research by developers to offer standard application tools suitable for a broad range of fields.

CONCLUSION

- The fact that the membership degrees of the elements belonging to the universe set in the theory are expressed only with 0 or 1, hence makes it difficult to express the uncertainty in the most accurate way. In this talk, in order to solve this problem, the concepts of relational membership function and inverse relational membership function were considered and some decision theories and the type of problems they tackled were presented.
- In addition, using the concepts of soft set theory, some new technical formulations can be established and some algorithms for decision-making under uncertainty can be proposed.

CONCLUSION (Cont.)

- The proposed approach to other types of soft-set based decision making in the future need to be developed, such as interval-valued fuzzy soft set based decision making, ambiguous soft set based choice making, and interval-valued intuitionistic fuzzy soft set based decision making.
- Researchers in different fields can actively conduct researches that involves soft set application in various context of modern world issues.

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