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WIENER INDEX AND MEAN DISTANCE OF ZERO-DIVISOR TYPE GRAPH OF RING OF INTEGERS MODULO *n*

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Abstract

The zero-divisor type graph was first introduced as the compression of zero-divisor graph by partitioning the vertices. For ring of integers modulo n, Z_n, the zero-divisor type graph of is a graph with vertex set contains T_d, where d is nontrivial divisor of n. Two distinct vertices, T_i and T_j are adjacent if $i \cdot j \neq 0$. The study of Wiener index and mean distance of a graph serves as a tool to calculate the sum of the distances between vertices in graph. The objective of this research is to compute the Wiener index and mean distance of the zero-divisor type graph of Z_{p^3q}. The Wiener index and mean distance of zero-divisor type graph of Z_n have been found to be constant for each factorization of n.

1. Motivation and Main Results Introduction

The study of zero divisor has been introduced in [1]. In 1988, Beck defined a zero-divisor graph of a ring R from the point view of graph coloring. Many researches have been conducted on these graph modification and

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various graph properties. Based on Anderson and Livingston in [2], the zero-divisor graph $\Gamma[R]$ of a ring R is a graph where the vertices are nonzero zero-divisor of the ring denoted as $Z^*[R]$ and two distinct vertices x and y are adjacent if $xy = 0$. Mulay in [3] study the graph of equivalence classes of zero-divisors of a ring R denoted as $\Gamma_E(R)$ which the zero-divisors is determined by the annihilator ideals. Smith in [5] introduced the zero-divisor type graph of Z_n denoted as $\Gamma^T[Z_n]$ where the same concept of the compression zero-divisors graph is applied. Other study related to the graph of equivalence classes of zero divisors of ring integer modulo n is as in [6].

A graph G is connected if each pair of the vertices is adjacent to other vertices in G. The order of finite graph G is the number of the vertices. A complete graph, K_n is a complete n-vertex graph which every vertex is adjacent to any other vertex in this graph. The distance between x and y in G is the shortest length path from x to y denoted as $d(x, y)$. If there exist no path, then $d(x, y) = \infty$ and $d(x, x) = 0$. A subgraph H of G is an induced subgraph of G if two vertices are adjacent in H if and only if the vertices are adjacent in G. The Wiener index of a graph G denoted as the total distances between all unordered pairs of vertices denoted as $W(G) = \sum_{\{x, y\} \in V(G)} d_G(x, y)$. Mean distance of a graph is the average distance between all vertices denoted as $\sigma(G) = \frac{W(G)}{k(k-1)}$, where k is the order of the graph. Let Z_n be the ring of integer modulo n. The study of the structure of $\Gamma^T[Z_n]$ associated to primes n has become the study motivation. In particular, this study demonstrates the Wiener index and mean distance for $\Gamma^T[Z_{p^3q}]$. Some problems related with zero-divisor graph are complicated to be solved even for ring with small values of n. Therefore, the vertices of the graph is divided by partition into cosets.

Definition 1.1 [5]**.** Zero-Divisor Type Graph

The zero-divisor type graph denoted as $\Gamma^T[Z_n]$ has vertices of T_i where i is divisor of n that is neither 1 nor 0. The sets of T_i creates a partition of the zero divisor graph with $T_i = \{x \in Z^*(Z_n) \mid \gcd(x, n) = i\}$. Two vertices in $\Gamma^T[Z_n]$ are adjacent if and only if $i \cdot j = 0 \mod n$ where $i \neq j$.

Based on Definition 1.1, the following lemma is hold.

Lemma 1.1 [5]. *The nature of adjacency between vertices of* $\Gamma^T[Z_{p^a q^b}]$ *is defined as*:

Let $T_x, T_y \in V(\Gamma^T[Z_{p^a q^b}])$, *therefore* x, y *can be written as* $x = p^r q^s$ *and* $y = p^m q^n$. *Then,* T_x *and* T_y *are adjacent if and only if* $r + m \geq a$ *and* $s + n \geq b$.

In this section, some basic definition on graph theory and zero-divisor type graph of Z_n *is discussed. In Section* 2, *the Wiener index and mean distance of* $\Gamma^T[Z_{p^3 q}]$ *are calculated.*

2. Wiener index and Mean Distance of $\Gamma^T[Z_{p^3 q}]$

The first result in this section is on the Wiener index for $\Gamma^T[Z_{p^3 q}]$.

Lemma 2.1 *The distance between each pair of vertices in* $\Gamma^T[Z_{p^3 q}]$ *are as follows*:

$d(T_i, T_j) = 1$ *for* $(i, j) \in \{(p, p^2 q), (q, p^3), (p^2, p^2 q), (pq, p^2 q)\}$,

$d(T_i, T_j) = 2$ *for* $(i, j) \in \{(p, p^2), (p, pq), (p, p^3), (q, pq), (q, p^2 q),$
$(p^2, p^3)\}$,

$d(T_i, T_j) = 3$ *for* $(i, j) \in \{(p, p^2), (p, pq)$.

Proof. Let i and j be any nontrivial divisors of $p^3 q$ and $i = j$, then it is clear that $d(T_i, T_j) = 0$. Suppose $i \neq j$, then

Case 1. $d(T_i, T_j) = 1$.

There exists exactly 8 pairs of $V(\Gamma^T[Z_{p^3q}])$, T_i and T_j which are adjacent since Lemma 1.1 is hold.

Case 2. $d(T_i, T_j) = 2$.

Note that T_{p^2q} is adjacent to $T_i \ni i \in \{p, p^2, pq, p^2q\}$ and T_{p^3} is adjacent to $T_j \ni j \in \{q, pq, p^2q\}$. For vertex T_p and T_q, these vertices are only adjacent to one vertex each which are T_{p^2q} and T_{p^3} respectively.

Case 3. $d(T_i, T_j) = 3$.

To hold Lemma 1.1, vertex T_p only adjacent to T_{p^2q} and vertex T_q only adjacent to T_{p^3}. Therefore, for T_p to reach vertex T_q it must pass through both T_{p^2q} and T_{p^3}. As for T_{p^2} to reach T_q, it must pass through T_p (to hold Lemma 2.1) and T_{p^3}. ∎

Theorem 2.1. *Let* $\Gamma^T[Z_{p^3q}]$ *be the zero the zero-divisor type graph of* Z_{p^3q} *with* p *and* q *are distinct prime numbers. Then, the Wiener index of every* $\Gamma^T[Z_{p^3q}]$ *is* 25.

Proof. Let p and q be two prime numbers with $p < q$. From Lemma 2.1, the following results is obtained.

$$W(\Gamma^T[Z_{p^3q}]) = \sum_{i \neq j} (T_i, T_j) = \frac{1}{2}[50] = 25. \quad ∎$$

Next, the mean distance for $\Gamma^T[Z_{p^3q}]$ is given in the next theorem.

Theorem 2.2. *The mean distance of* $\Gamma^T[Z_{p^3q}]$ *is* $\frac{5}{6}$.

Proof. By Theorem 2.1, $W(\Gamma^T[Z_{p^3q}]) = 25$ and $|W(\Gamma^T[Z_{p^3q}])| = 6$. Hence,

$$\sigma(\Gamma^T[Z_{p^a q}]) = \frac{W(\Gamma^T[Z_{p^a q}])}{k(k-1)} = \frac{25}{6(6-1)} = \frac{5}{6}. \qquad \blacksquare$$

3. Conclusion

In this paper, the Wiener index and mean distance of $\Gamma^T[Z_{p^3 q}]$ has been described. The Wiener index and mean distance for $\Gamma^T[Z_{p^3 q}]$ has been found to be constant for any prime p and q.

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