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Non-Trivial Subring Perfect Codes in Unit Graph of Boolean Rings

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Abstract The aim of this paper is to investigate the non-trivial subring perfect codes in a unit graph associated with the Boolean rings. We prove a subring perfect code of size 2^{n-1}, where $n \geq 2$, in the unit graphs associated with the finite Boolean rings R. Moreover, we give a necessary and sufficient condition for a subring of an infinite Boolean ring R to be a perfect code of size infinity in the unit graph.

Keywords: subring, Boolean ring, unit graph, perfect code.

Introduction

A ring R with identity is called a Boolean ring if $r^2 = r$, for all $r \in R$ [1]. A non-empty subset H of R in which $x - y \in H$ and $xy \in H$ for all $x, y \in H$ is called a subring of R [2]. A graph Γ is simply represented as a pair of (V, E), where the non-empty set V indicates the vertex set and E indicates the edge set of Γ.

The connection between the two branches of mathematics namely, graph theory and commutative ring theory was first studied by Beck [3] through constructing a zero divisor graph; it is a graph which is generated by setting all elements of the ring as the set of vertices and the vertex x which is distinct from the vertex y are connected if they are zero divisors. He was mainly interested in colouring of commutative rings and characterized all commutative rings which are finitely colourable. Later on, the zero divisor graph familiarized in [3] was redefined in [4] to presents better the zero divisor structure of commutative rings. Then, the girth and diameter properties of zero divisor graph have been determined in [5].

The notion of a prime graph associated with a ring R was proposed in [6]. Moreover, it was shown that if R is a semiprime ring, then the ring R is prime if and only if the prime graph associated with R is a tree. In the same year, Patra and Kalita [7] investigated the chromatic number of the prime graph of ring $\mathbb{Z}_n$ for $n = \prod_{i=1}^{r} p_i^{\alpha_i}$, where $1 \leq i \leq m$, p_i is a prime and α_i is an odd positive integer, and found that $\chi(\Gamma(\mathbb{Z}_n)) = \prod_{i=1}^{r} p^{\left\lfloor \frac{\alpha_i}{2} \right\rfloor} + m$. Later, Pawar and Joshi [8] presented the definition of complement prime graph and they mostly focused on finding the degree of the vertices and number of triangles in both prime and complement prime graphs associated with ring $\mathbb{Z}_n$. Recently, Joshi and Pawar [9] redefined the prime graph of ring introduced in [6] with some changes in the vertex set of the graph, called prime

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graph $PG_2(R)$ of a ring. They concentrated on determining the girth and degree of vertices in $PG_2(R)$ as well as provided a sufficient condition for $PG_2(R)$ to be Eulerian and a necessary and sufficient condition for PG2 (R) to be planar..

A unit graph associated with ring R is denoted by $\Gamma(R)$ is a graph, where the vertices of $\Gamma(R)$ are elements of ring R and the edges are all pair of elements (x,y) of ring R such that $x+y \in U(R)$. This definition for $\Gamma(R)$ was introduced in [10]. Maimani *et al.* [11] provided some necessary and sufficient conditions for $\Gamma(R)$ to be Hamiltonian. Later, a research conducted in [12] presents the finiteness property of $\Gamma(R)$ by assuming that R is a non-commutative ring. The girth of the unit graph is the length of the shortest cycle in $\Gamma(R)$, this property of $\Gamma(R)$ was investigated in [13]. The diameter of a unit graph is the maximum eccentricity of all vertices in $\Gamma(R)$. Su and Wei [14] concentrated on the diameter of $\Gamma(R)$ and found that $diam(\Gamma(R)) \in \{1,2,3,\infty\}$. Su *et al.* [15] focused on the planarity of unit graph and characterized rings which their associated unit graphs are planar. However, Das [16] investigated the non-planarity of unit graph of rings and provided some necessary and sufficient conditions for the unit graph to be non-planar. Dejter and Giudici [17] introduced the concept of unitary Cayley graph associated with ring of integer modulo n, $\mathbb{Z}_n$. They provided some sufficient conditions for the unitary Cayley graph, $\Gamma(\mathbb{Z}_n)$ to be regular as well as bipartite. Then, Akhtar *et al.* [18] generalized the definition of unitary Cayley graph of ring of integer modulo n to an arbitrary finite commutative ring R with identity. In addition, some results were established which determine the diameter, girth, independence number, chromatic number, clique number, planarity and connectivity of this graph. Su and Zhou [19] also continued research on the unitary Cayley graph associated with R and provided some conditions for this graph to be planar.

A subset C of $V(\Gamma(R))$ is called a code. The code C is called perfect if the ball with centres at C and radius 1 form a partition of $V(\Gamma(R))$. In other words, if for any two distinct $c \in C$, $\bigcap S_1(c) = \phi$ for all c in C $\bigcup S_1(c) = V(\Gamma(R))$, then C is called the perfect code accepted by $\Gamma(R)$, where $S_1(c)$ is the closed neighbourhood of code words c with radius 1 [20]. The notion of investigating perfect codes in graph was first established by Biggs in [21]. Then Huang *et al.* [22] extended this research to Cayley graph of a group. They proved some necessary and sufficient conditions for a normal subgroup of a group, subgroup of an abelian and non-abelian groups to be perfect codes admitted by some Cayley graph associated with groups. Ma *et al.* [23] concentrated on perfect codes in power graphs of groups and established some results which show the power graphs accept or do not accept the perfect codes. Recently, Ma [24] investigated the perfect codes in proper reduced power graphs of finite groups, and proved the perfect codes and total perfect codes in these graphs. However, Raja [25] enlarged the research of perfect codes in graphs to graph associated with rings by mainly focusing on total perfect codes in zero divisor graph.

The problem whether a subgring is a perfect code in graphs associated with rings can be thought as an attracted object of study. In this paper, we determine the subring perfect codes of unit graphs of finite and infinite Boolean rings.

Materials and methods

This study concentrates on characterization of non-trivial subring perfect codes in unit graphs of Boolean rings. A necessary and sufficient condition is given for the non-trivial subrings of the finite Boolean rings to be perfect code accepted by the unit graph (Theorem 1). Furthermore, a necessary and sufficient condition is established for a non-trivial subring of an infinite Boolean ring to be perfect code accepted by the unit graph (Theorem 2).

Results and discussion

Subring Perfect Codes in Unit Graph of Finite Boolean Rings

This section aims to present the subring perfect codes in unit graph associated with Boolean rings. Proposition 1, shows a subring perfect codes of size 2 in the unit graph $\Gamma(R)$.

Proposition 1 Let R be a Boolean ring and $\Gamma(R)$ be the unit graph associated with R. Then, $\Gamma(R)$ accepts a perfect code C of size 2 if and only if $O(R)=4$.

Proof. Assume $O(R)=4$ that is $R=\mathbb{Z}_2\times\mathbb{Z}_2$. Thus, R consists of $2^2=4$ distinct elements as represented in the following:

$$R=\{r_i=(x_1,x_2): x_1,x_2\in\mathbb{Z}_2 \text{ and } 1\le i\le 4\}.$$

Further, assume that $U(R)$ represent the set of unit elements of R, thus $U(R)$ contains all $r_i,r_j\in R$ that satisfies the condition $r_i,r_j=(1,1)$. By the definition of a unit graph of R, $V(\Gamma(R))=R$ and $E(\Gamma(R))=\{\{r_i,r_j\}: r_i+r_j\in U(R)\}$. Since for every r_i there exists a unique r_j such that $r_i+r_j\in U(R)$, this implies that there are $\frac{2^2}{2}$ pairs of vertices r_i and r_j such that $r_i+r_j\in U(R)$. Hence, $\Gamma(R)$ contains $\frac{2^2}{2}=2$ components of complete graph K_2, i.e. $\Gamma(R)=\Gamma_1(R)\cup\Gamma_2(R)=2K_2$. Assume $V(\Gamma_1(R))=\{r_1,r_4\}$ and $V(\Gamma_2(R))=\{r_2,r_3\}$. Thus, $V(\Gamma(R))=V(\Gamma_1(R))\cup V(\Gamma_2(R))$. Now we show that C is a subring perfect code of size 2 in $\Gamma(R)$. Let $C=\{r_m=(x_1,x_2): x_1,x_2\in\mathbb{Z}_2 \text{ and } x_1\neq 1 \text{ for all } 1\le m\le 2\}$ be a subring of R. Then by applying the definition of a perfect code on C, the closed neighbourhoods of code words in C with radius 1 can be obtained as follows:

$$\begin{aligned}S_1(r_1)&=\{r_i\in V(\Gamma(R)): d(r_1,r_i)\le 1\}\\&=\{r_1,r_4\}\\&=V(\Gamma_1(R)),\\S_1(r_2)&=\{r_i\in V(\Gamma(R)): d(r_2,r_i)\le 1\}\\&=\{r_2,r_3\}\\&=V(\Gamma_2(R)).\end{aligned}$$

The closed neighbourhoods of code words r_1 and r_2 yields that $\bigcap_{m=1}^{2}S_1(r_m)=\phi$ and $\bigcup_{m=1}^{2}S_1(r_m)=V(\Gamma(R))$. Hence, C is a subring perfect code of size 2 accepted by $\Gamma(R)$.

Conversely, if C is a subring perfect code of size 2, then we prove that $O(R)=4$. Suppose that $\Gamma(R)$

accepts a subring perfect code C of size 2, this means that $S_1(r_m)$, where $m=1,2$ partition the $V(\Gamma(R))$ into two disjoint sets. If $|S_1(r_m)|>2$, then there exist more than one elements in $V(\Gamma(R))\setminus C$ which are of distance not more than 1 to exactly one code word $r_m \in C$. This is a contradiction since for each element $r_m \in C$ there exists a unique element $r_i \in V(\Gamma(R))\setminus C$ such that $r_m + r_i \in U(R)$. Thus $|S_1(r_m)|$ is not greater than 2. If $|S_1(r_m)|=1$, then there is no elements in $V(\Gamma(R))\setminus C$ which are of distance 1 to exactly one code word $r_m \in C$. This is also a contradiction since for each element $r_m \in C$ there exists a unique element $r_i \in V(\Gamma(R))\setminus C$ such that $r_m + r_i \in U(R)$. Hence, $|S_1(r_m)|=2$, where $S_1(r_1)=\{r_1,r_4\}$ and $S_1(r_2)=\{r_2,r_3\}$. This implies that $O(R)=4$.

In the next proposition, we prove a subring of R to be a perfect code of size 4 accepted by $\Gamma(R)$.

Proposition 2 Let R be a Boolean ring and $\Gamma(R)$ be the unit graph associated with R. Then, $\Gamma(R)$ accepts a perfect code C of size 4 if and only if $O(R)=8$.

Proof. Assume $O(R)=8$ that is $R=\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_2$. Thus, R contains $2^3=8$ distinct elements, as represented in the following:

$$R=\left\{r_i=(x_1,x_2,x_3):x_1,x_2,x_3\in\mathbb{Z}_2 \text{ for all } 1\le i\le 8\right\}.$$

Meanwhile, assume that $U(R)$ is the set of unit elements of R, therefore it can be represented in the following:

$$U(R)=\left\{r_i : r_i\cdot r_j=(1,1,1) \text{ for all } 1\le i\le j\le 8\right\}.$$

By the definition of a unit graph $\Gamma(R)$, $V(\Gamma(R))=R$ and $E(\Gamma(R))=\left\{\{r_i,r_j\}:r_i+r_j\in U(R)\right\}$. Since $|V(\Gamma(R))|=8$ and also for each $r_i\in V(\Gamma(R))$, there exists a unique $r_j\in V(\Gamma(R))$ such that $r_i+r_j\in U(R)$. This implies that there are $\frac{2^3}{2}=4$ pairs of vertices r_i and r_j such that $r_i+r_j\in U(R)$. Hence, $\Gamma(R)$ contains $\frac{2^3}{2}=4$ copies of complete graph K_2, i.e. $\Gamma(R)=\bigcup_{m=1}^{4}\Gamma_m(R)=4K_2$. Assume the vertex sets $\{r_1,r_8\}$, $\{r_2,r_7\}$, $\{r_3,r_6\}$ and $\{r_4,r_5\}$ form $\Gamma_m(R)$, respectively, then $V(\Gamma(R))=\bigcup_{m=1}^{4}V(\Gamma_m(R))=\{r_1,r_8\}\cup\{r_2,r_7\}\cup\{r_3,r_6\}\cup\{r_4,r_5\}$. Next we prove that $\Gamma(R)$ accepts a subring perfect code C of size 4. Suppose that $C=\left\{r_m=(x_1,x_2,x_3):x_1,x_2,x_3\in\mathbb{Z}_2 \text{ and } x_1\ne 1 \text{ for all } 1\le m\le 4\right\}$ be a subring of R. Then, by applying the definition of a perfect code on C, the closed neighbourhoods of code words in C with radius 1 can be obtained as follows:

$$\begin{aligned}
S_1(r_1)&=\left\{r_i\in V(\Gamma(R)):d(r_1,r_i)\le 1\right\}\\
&=\{r_1,r_8\}=V(\Gamma_1(R)),\\
S_1(r_2)&=\left\{r_i\in V(\Gamma(R)):d(r_2,r_i)\le 1\right\}\\
&=\{r_2,r_7\}=V(\Gamma_2(R)),
\end{aligned}$$

$$S_1(r_3) = \{r_i \in V(\Gamma(R)) : d(r_3, r_i) \le 1\}$$
$$= \{r_3, r_6\} = V(\Gamma_3(R)),$$
$$S_1(r_4) = \{r_i \in V(\Gamma(R)) : d(r_4, r_i) \le 1\}$$
$$= \{r_4, r_5\} = V(\Gamma_4(R)).$$

Therefore, the above results imply that $\bigcap_{m=1}^{4} S_1(r_m) = \phi$ for any two distinct rm in C and $\bigcup_{m=1}^{4} S_1(r_m) = V(\Gamma(R))$. Hence, C is a subring perfect code of size 4 accepted by $\Gamma(R)$.

Conversely, if $\Gamma(R)$ accepts a subring perfect code C of size 4, then we show that $O(R) = 8$. Suppose that C is a subring perfect code of size 4, this means that $S_1(r_m), m = 1,2,3,4$ partition the vertex set $V(\Gamma(R))$ into 4 distinct sets. If $|S_1(r_m)| > 2$, then there exist more than one elements in $V(\Gamma(R)) \setminus C$ which are of distance not more than 1 to exactly one code word $r_m \in C$. This is a contradiction since for each element $r_m \in C$ there exists a unique element $r_i \in V(\Gamma(R)) \setminus C$ such that $r_m + r_i \in U(R)$. Thus $|S_1(r_m)|$ is not greater than 2. If $|S_1(r_m)| = 1$, then there is no elements in $V(\Gamma(R)) \setminus C$ which are of distance 1 to exactly one code word $r_m \in C$. This is also a contradiction since for each element $r_m \in C$ there exists a unique element $r_i \in V(\Gamma(R)) \setminus C$ such that $r_m + r_i \in U(R)$. Hence, $|S_1(r_m)| = 2$, where $S_1(r_1) = \{r_1, r_8\}$, $S_1(r_2) = \{r_2, r_7\}$, $S_1(r_3) = \{r_3, r_6\}$ and $S_1(r_4) = \{r_4, r_5\}$. This gives that $O(R) = 8$.

The results obtained in Propositions 1 and 2 form the generalization of a subring perfect codes of size 2^{n-1} accepted by $\Gamma(R)$. This result is presented in the following theorem.

Theorem 1 Let R be a Boolean ring and $\Gamma(R)$ be the unit graph associated with R. Then, $\Gamma(R)$ accepts a subring perfect code of size 2^{n-1} if and only if $O(R) = 2^n, n \ge 2$.

Proof. Assume $O(R) = 2^n, n \ge 2$, i.e. $R = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \ldots \times \mathbb{Z}_2$. Thus, R contains 2^n distinct elements, as represented in the following:

$$R = \left\{ r_i = (x_1, x_2, x_3, \ldots x_i, \ldots, x_n) : x_i \in \mathbb{Z}_2 \ \text{ for all } 1 \le i \le 2^n \right\}.$$

Moreover, assume that $U(R)$ represent the set of unit elements of R, therefore $U(R) = \left\{ r_i : r_i \cdot r_j = (1,1,1,\ldots,1,\ldots,1) \text{ for all } 1 \le i \le j \le 2^n \right\}$. According to the definition of a unit graph, $V(\Gamma(R)) = R$ and $E(\Gamma(R)) = \left\{ \{r_i, r_j\} : r_i + r_j \in U(R) \right\}$. Since $|V(\Gamma(R))| = 2^n$ and also for every $r_i \in V(\Gamma(R))$ there exists a unique $r_j \in V(\Gamma(R))$ such that $r_i + r_j \in U(R)$. This implies that $\Gamma(R)$ is a graph consisting of $\frac{2^n}{2}$ copies of complete graph K_2. That is, $\Gamma(R) = \bigcup_{m=1}^{\frac{2^n}{2}} K_2$. Suppose that each of the set of vertices $\{r_1, r_{2^n}\}, \{r_2, r_{2^n-1}\}, \ldots, \{r_{2^{n-1}}, r_{2^{n-1}} + 1\}$, respectively form a K_2. Thus,

$$V(\Gamma(R))=\bigcup_{m=1}^{\frac{2^n}{2}}V(K_2)=\{r_1,r_{2^n}\}\cup\{r_2,r_{2^n-1}\}\cup\ldots\cup\{r_{2^{n-1}},r_{2^{n-1}}+1\}.$$

Now we prove that $\Gamma(R)$ accepts a subring perfect code C of size 2^{n-1}. Suppose that $C=\{r_m=(x_1,x_2,x_3,\ldots,x_n):x_1,x_2,x_3,\ldots,x_n\in\mathbb{Z}_2 \text{ and } x_1\neq 1 \text{ for all } 1\leq m\leq 2^{n-1}\}$ be a subring of R. Then the subring C is perfect if it satisfies the following two conditions:

1. $\bigcap_{m=1}^{2^{n-1}}S_1(r_m)=\phi,$
2. $\bigcup_{m=1}^{2^{n-1}}S_1(r_m)=V(\Gamma(R)).$

To prove the claim, we find the closed neighbourhoods of code words in C with radius 1.

$$\begin{aligned}
S_1(r_1)&=\{r_i\in V(\Gamma(R)):d(r_1,r_i)\leq 1\}\\
&=\{r_1,r_{2^n}\}=V(K_2),\\
S_1(r_2)&=\{r_i\in V(\Gamma(R)):d(r_2,r_i)\leq 1\}\\
&=\{r_2,r_{2^n-1}\}=V(K_2),\\
S_1(r_3)&=\{r_i\in V(\Gamma(R)):d(r_3,r_i)\leq 1\}\\
&=\{r_3,r_{2^n-2}\}=V(K_2),\\
&\vdots\\
S_1(r_{2^{n-1}})&=\{r_i\in V(\Gamma(R)):d(r_{2^{n-1}},r_i)\leq 1\}\\
&=\{r_{2^{n-1}},r_{2^{n-1}+1}\}=V(K_2).
\end{aligned}$$

The results yield that $\bigcap_{m=1}^{2^{n-1}}S_1(r_m)=\phi$ for any two distinct rm in C and $\bigcup_{m=1}^{2^{n-1}}S_1(r_m)=V(\Gamma(R))$ for all $r_m\in C$. Hence, it satisfies the conditions of perfect code, thus C is a subring perfect code of size 2^{n-1} accepted by $\Gamma(R)$. Conversely, if $\Gamma(R)$ accepts a subring perfect code C of size 2^{n-1}, then we show that $O(R)=2^n$. Suppose that C is a subring perfect code of size 2^{n-1}, this means that $S_1(r_m), m=1,2,3,\ldots,2^{n-1}$partition the vertex set $V(\Gamma(R))$ into 2^{n-1} distinct sets. If $|S_1(r_m)|>2$, then there exist more than one elements in $V(\Gamma(R))\setminus C$ which are of distance not more than 1 to exactly one code word $r_m\in C$. This is a contradiction since for each element $r_m\in C$ there exists a unique element $r_i\in V(\Gamma(R))\setminus C$ such that $r_m+r_i\in U(R)$. Thus $|S_1(r_m)|$ is not greater than 2. If $|S_1(r_m)|=1$, then there is no elements in $V(\Gamma(R))\setminus C$ which are of distance 1 to exactly one code word $r_m\in C$. This is also a contradiction since for each element $r_m\in C$ there exists a unique element $r_i\in V(\Gamma(R))\setminus C$ such that $r_m+r_i\in U(R)$. Hence, $|S_1(r_m)|=2$, where $S_1(r_1)=\{r_1,r_{2^n}\}$, $S_1(r_2)=\{r_2,r_{2^n-1}\}$, $S_1(r_3)=\{r_3,r_{2^n-2}\}$, …, and $S_1(r_{2^{n-1}})=\{r_{2^{n-1}},r_{2^{n-1}+1}\}$. This gives that $O(R)=2^n$.

Example 1 Let $R=\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_2$ be a Boolean ring. Then the elements of R can be listed as follows:

$$R=\{(0,0,0,0),(0,0,0,1),(0,0,1,0),(0,0,1,1),(0,1,0,0),(0,1,0,1),(0,1,1,0),(0,1,1,1),\\ (1,0,0,0),(1,0,0,1),(1,0,1,0),(1,0,1,1),(1,1,0,0),(1,1,0,1),(1,1,1,0),(1,1,1,1)\}.$$

Let we represent the elements of R by $r_1,r_2,r_3,\ldots,r_{16}$, respectively. Then $R=\{r_1,r_2,r_3,\ldots,r_{16}\}$. According to the definition of a unit graph, $V(\Gamma(R))=R$ and $E(\Gamma(R))=\{\{r_1,r_{16}\},\{r_2,r_{15}\},\ldots,\{r_8,r_9\}\}$. Hence, $\Gamma(R)$ is a simple graph consists of 16 vertices and 8 edges, given in the following figure.

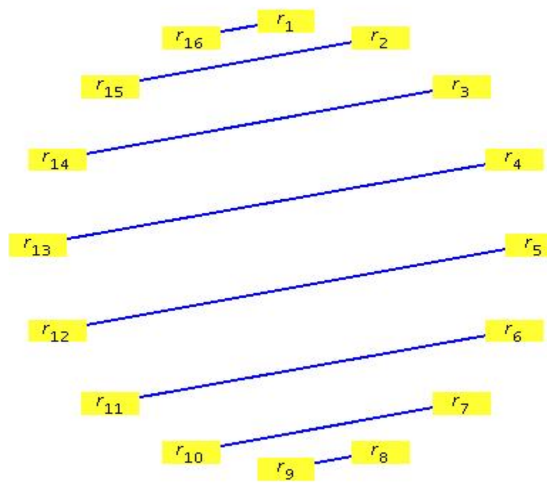

Figure 1. The unit graph of Boolean ring $R=\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_2$.

By Theorem 1, $C=\{r_1,r_2,r_3,r_4,r_5,r_6,r_7,r_8\}$ is a subring perfect code of size 8 in $\Gamma(R)$.

Subring Perfect Codes in Unit Graph of Infinite Boolean Ring

In this section, we give a necessary and sufficient condition for the subring of infinite Boolean ring to be a perfect code of size infinity in the unit graph.

Theorem 2 Let R be an infinite Boolean ring and $\Gamma(R)$ be the unit graph associated with R. Then, $\Gamma(R)$ accepts a subring perfect code of size infinity if and only if $O(R)=\infty$.

Proof. Suppose $O(R)=\infty$ that is $R=\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_2\times\ldots$, which is given as Cartesian product of infinite copies of $\mathbb{Z}_2$. Therefore, R has infinite distinct elements of length infinity as $R=\{r_i=(x_1,x_2,x_3,\ldots,x_i,\ldots):x_i\in\mathbb{Z}_2\}_{i=1}^{\infty}$. Assume that $U(R)$ denote the set of units of R, thus $U(R)=\{r_i:r_i\cdot r_j=(1,1,1,\ldots,1,\ldots)\text{ for all }1\le i\le j<\infty\}$. By the definition of a unit graph, $V(\Gamma(R))=R$ and $\{r_i,r_j\}$ forms an edge of $\Gamma(R)$ if and only if $r_i+r_j=(1,1,1,\ldots,1,\ldots)$. That is, for every r_i, there exists a unique r_j such that $r_i+r_j=(1,1,1,\ldots,1,\ldots)$. This implies that $\Gamma(R)$ contains infinite copies of complete graph K_2 with r_i and r_j, where $r_i \neq r_j$ as their endpoints. Hence, $\Gamma(R) = \bigcup_{m=1}^{\infty} K_2$. Assume that each of the vertex set $\{r_1, r_2\}, \{r_3, r_4\}, \ldots, \{r_{2m-1}, r_{2m}\}, \ldots$ form a K_2, thus $V(\Gamma(R)) = \bigcup_{m=1}^{\infty} \{r_{2m-1}, r_{2m}\}$. Now we prove that $\Gamma(R)$ accepts a subring perfect code C of size infinity. Let $C = \left\{ r_{2m-1} = (x_1, x_2, x_3, \ldots, x_n, \ldots) : x_1, x_2, x_3, \ldots, x_n, \ldots \in \mathbb{Z}_2 \text{ and } x_1 \neq 1 \text{ for all } 1 \le m \le \infty \right\}$ be a subring of R. Then by applying the definition of a perfect code on C, the closed neighbourhood of code words $r_{2m-1} \in C$ with radius 1 are obtained as follows:

$$\begin{aligned}
S_1(r_1) &= \left\{ r_i \in V(\Gamma(R)) : d(r_1, r_i) \le 1 \right\} \\
&= \{r_1, r_2\}, \\
S_1(r_3) &= \left\{ r_i \in V(\Gamma(R)) : d(r_3, r_i) \le 1 \right\} \\
&= \{r_3, r_4\}, \\
S_1(r_5) &= \left\{ r_i \in V(\Gamma(R)) : d(r_5, r_i) \le 1 \right\} \\
&= \{r_5, r_6\}, \\
&\vdots \\
S_1(r_{2m-1}) &= \left\{ r_i \in V(\Gamma(R)) : d(r_{2m-1}, r_i) \le 1 \right\} \\
&= \{r_{2m-1}, r_{2m}\} \\
&\vdots
\end{aligned}$$

The sets of closed neighbourhoods of any two distinct rm in C of $r_{2m-1} \in C$ show that $\bigcap_{m=1}^{\infty} S_1(r_{2m-1}) = \phi$ and $\bigcup_{m=1}^{\infty} S_1(r_{2m-1}) = V(\Gamma(R))$. Hence, C is a subring perfect code of size infinity accepted by $\Gamma(R)$.

Conversely, if $\Gamma(R)$ accepts a subring perfect code C of size ∞, then we show that $O(R) = \infty$. Suppose that C is a subring perfect code of size ∞, this means that $S_1(r_{2m-1}), m = 1, 2, 3, \ldots$ partition the vertex set $V(\Gamma(R))$ into infinite distinct sets namely $\{r_1, r_2\}, \{r_3, r_4\}, \ldots, \{r_{2m-1}, r_{2m}\}, \ldots$. This implies that $O(R) = \infty$.

Conclusions

This research concentrates on determining non-trivial subring perfect codes in unit graph associated with the finite and infinite Boolean rings. Some necessary and sufficient conditions are provided as propositions and theorems which demonstrate the non-trivial subring perfect codes in unit graph, $\Gamma(R)$. The findings of this research show that if $O(R) = 2^n, n \ge 2$ then $\Gamma(R)$ accepts a subring perfect code C of size 2^{n-1} and if $O(R) = \infty$, then $\Gamma(R)$ accepts subring perfect code C of size ∞, that is $|C| \in \left\{2, 4, 8, \ldots, 2^n, \infty\right\}$.

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