Proceedings of Science and Mathematics
Faculty of Science, Universiti Teknologi Malaysia
Vol. 15, 2023, page 1-7

The Energy and The Maximum Degree Energy of The Cayley Graph Associated to The Alternating Group of Order 12 With Subsets of Order One

Ahmad Farhan Rooh Azzam, Nor Haniza Sarmin*
Department of Mathematical Sciences, Faculty of Science, Universiti Teknologi Malaysia
*Corresponding author: nhs@utm.my

Abstract
The energy of a simple graph is defined as the summation of the absolute value of the eigenvalues of the adjacency matrix of the graph. It was motivated by the Hückel Molecular Orbital theory. The theory was used by chemists to estimate the energy associated with π-electron orbitals of molecules which is called as conjugated hydrocarbons. Meanwhile, the maximum degree energy is defined as the summation of the absolute value of the eigenvalues of the maximum degree matrix of the graph. Besides, a Cayley graph associated to a finite group is defined as a graph where its vertices are the elements of a group, and two vertices v_1 and v_2 are joined with an edge if and only if v_2 is equal to the product of s and v_1 for some elements s in the subset S of the group. The computation of the energy and the maximum degree of the Cayley graph associated to the alternating group of order 12 with respect to the subset of order one has been carried out by using some concepts and properties in graph theory, group theory, linear algebra and results from previous studies. Furthermore, some propositions and figures are provided to illustrate the results for the energy and the maximum degree energy of the Cayley graph associated to the alternating group of order 12 with subset of order one. The Cayley graph related to the alternating group of order 12 with respect to the subset S of order one is found. The obtained graph is then mapped onto the adjacency matrix and the maximum degree matrix respectively to obtain the adjacency eigenvalues and the maximum degree eigenvalues of the graph. Finally, the energy and the maximum degree energy for the Cayley graph associated to the alternating group of order 12 with subset of order one is obtained by using the adjacency eigenvalues and maximum degree eigenvalues of the graph, respectively. The results show that the energy and the maximum degree energy of the Cayley graph associated to the alternating group of order 12 with subset of order one are the same.

Keywords: energy; maximum degree energy; Cayley group; alternating group

Introduction
The energy of a graph was first defined by Gutman [1] in 1978 inspired from the Hückel Molecular Orbital Theory proposed in 1930s by Hückel as the sum of all the absolute values of the eigenvalues of adjacency matrix of the graph. The approximation of the total π-electrons energy of molecules [2] in chemistry served as the driving force for the investigation in [1].

Meanwhile, the maximum degree energy of a graph was first described by Adiga and Smitha [3] as the total of the absolute values of the eigenvalues of the maximum degree matrix of the graph. Like the energy, the maximum degree energy is a measure of the complexity and structure of the graph and is related to its spectral properties. The maximum degree energy has been studied in a variety of contexts, including chemistry, physics, and mathematics. In chemistry, it is used to model the molecular structure of chemical compounds and has been used to predict the reactivity and properties of molecules. In physics, it is used to model the behaviour of complex systems such as quantum networks and electrical circuits. In mathematics, it is studied as a fundamental quantity in graph theory and related fields. One important property of the maximum degree energy is that it is always bounded below by a function of the maximum degree of the graph.

Furthermore, Arthur Cayley was the first person to initiate the study on Cayley graphs in 1878 [4]. Then, the research on Cayley graph has grown rapidly in the area of group theory and their application. Alspach & Mishna [5] are the researchers that study the enumeration of various family of Cayley graph and digraph. Since then, there are many researchers started to study on the terminology of Cayley graph and its application to the real-world problems [6, 7].

Based on the previous studies on the maximum degree energy of graphs, it is found that there is no research done yet on the maximum degree energy of Cayley graph associated to the alternating group of order 12. Therefore, this research aims to determine the energy of Cayley graphs associated to the alternating group of order 12 with subset of order one and further to compute the maximum degree energy of the Cayley graphs associated to the alternating group of order 12 with subset of order one.

The methodology consists of constructing the Cayley graph with respect to the subset of order one, finding their eigenvalues and finally computing the energy and the maximum degree energy of the Cayley graph related to the alternating group of order 12 with subset of order one. This paper is structured as follows: in Section1, some introductions and previous studies on the topics are explained. In Section 2, the results on the Cayley graphs of the alternating groups with subset of order one is presented in the form of propositions, theorem and illustrated by some figures. Meanwhile, the energy of the Cayley graphs related to the alternating group of order 12 with subset of order one is presented in several theorems in Section 3. In Section 4, the maximum degree energy of the Cayley graph associated to the alternating group, A_4 with subset of order one is presented in the form of propositions and theorem. The last section contains conclusion of the research.

2. The Cayley Graph Associated to the Alternating Group of Order 12 With Subsets of Order One

In this section, the Cayley graph associated to the alternating group with respect to subsets of order one is presented by some propositions and illustrated by some figures. By the definition of the Cayley graph, the subset S of G satisfying $1 \notin S$ and $S = S^{-1}$; $s \in S$ *if* and only *if* $s^{-1} \in S$. Thus, for the subset S of order one, there are three choices:

1. S_1 = {(12)(34)}.
2. S_2 = {(13)(24)}.
3. S_3 = {(14)(23)}.

In the following three propositions, the vertex set and the edge set of the Cayley graph with respect to the subsets S of order one are presented.

Proposition 2.1. Let A_4 be the alternating group of order 12 and $Cay(A_4$, {(12)(34)}) be the Cayley graph of A_4 with respect to the subset S_1 where S_1= {(12)(34)}. Then, the vertex set of the Cayley graph, $V(Cay(A_4$, {(12)(34)})) = {(1), (123), (132), (124), (142), (134), (143), (234), (243), (12)(34), (13)(24), (14)(23)} and the edge set of the Cayley graph, $E(Cay(A_4$,{(12)(34)})) = {{(1), (12)(34)}, {(123),(243)}, {(132),(143)}, {(124),(234)}, {(142),(134)}, {(13)(24),(14)(23)}}.

Proof By the definition of Cayley graph, the vertices of $Cay(A_n, S)$ are the elements of A_n. Thus, the vertex set of the Cayley graph, $V(Cay(A_4$, {(12)(34)})) = {(1), (123), (132), (124), (142), (134), (143), (234), (243), (12)(34), (13)(24), (14)(23)}. Next, by the definition of Cayley graph, there is an edge joining the vertices v_1 and v_2 in A_n, if and only if $v_2(v_1)^{-1} = s$ for $s \in S^{(1)}$ and v_1, v_2 in A_n. Thus, the edge set of the Cayley graph, $E(Cay(A_4$,{(12)(34)})) = {{(1),(12)(34)}, {(123),(243)}, {(132),(143)}, {(124),(234)}, {(142),(134)}, {(13)(24),(14)(23)}}. ■

Using similar method, the vertex set and the edge set for the other two subsets of order one are presented and stated in the following two propositions.

Proposition 2.2. Let A_4 be the alternating group of order 12 and $Cay(A_4$, {(13)(24)}) be the Cayley graph of A_4 with respect to the subset S_2 where S_2 = {(13)(24)}. Then, the vertex set of the Cayley graph, $V(Cay(A_4$, {(13)(24)})) = {(1), (123), (132), (124), (142), (134), (143), (234), (243), (12)(34), (13)(24), (14)(23)} and the edge set of the Cayley graph,

$E(Cay(A_4,\{(13)(24)\}))$ = {{(1), (13)(34)}, {(123),(142)}, {(132),(234)}, {(124),(143)}, {(134),(243)}, {(12)(34),(14)(23)}}.

Proposition 2.3 Let A_4 be the alternating group of order 12 and $Cay(A_4, \{(14)(23)\})$ be the Cayley graph of A_4 with respect to the subset S_3 where S_3 = {(14)(23)}. Then, the vertex set of the Cayley graph, $V(Cay(A_4, \{(14)(23)\}))$ = {(1), (123), (132), (124), (142), (134), (143), (234), (243), (12)(34), (13)(24), (14)(23)} and the edge set of the Cayley graph, $E(Cay(A_4,\{(14)(23)\}))$ = {{(1), (14)(23)}, {(123),(134)}, {(132),(124)}, {(143),(234)}, {(243),(142)}, {(12)(34),(13)(24)}}.

Next, the Cayley graph of the alternating group of order 12 with respect to the subset S_1 of order one is determined.

Proposition 2.4. Let A_4 be the alternating group of order 12. Then, the Cayley graph of A_4 with respect to the subset S_1 where S_1 = {(12)(34)}, $Cay(A_4, \{(12)(34)\}) = 6K_2$.

Proof From Proposition 2.1., the vertex set of the Cayley graph, $V(Cay(A_4, \{(12)(34)\}))$ = {(1), (123), (132), (124), (142), (134), (143), (234), (243), (12)(34), (13)(24), (14)(23)} and the edge set of the Cayley graph, $E(Cay(A_4,\{(12)(34)\}))$ = {{(1),(12)(34)}, {(123),(243)}, {(132),(143)}, {(124),(234)}, {(142),(134)}, {(13)(24),(14)(23)}}. Thus, the Cayley graph can be drawn as in Figure 2.1.

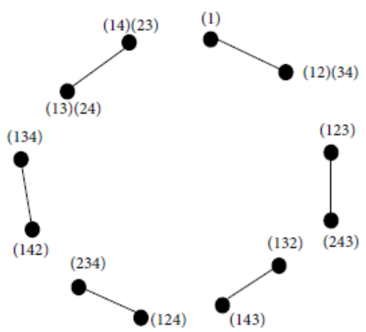

Figure 2.1 The Cayley graph of A_4 with respect to the subset S = {(12),(34)}.

This shows that $Cay(A_4, \{(12)(34)\}) = 6K_2$. ■

Using the similar method, the Cayley graph of A_4 with respect to the other two subsets of order one are found in the following two propositions.

Proposition 2.5 Let A_4 be the alternating group of order 12. Then, the Cayley graph of A_4 with respect to the subset S_2 where S_2 = {(13)(24)}, $Cay(A_4,\{(13)(24)\}) = 6K_2$.

Proof From Proposition 2.2, the vertex set of the Cayley graph, $V(Cay(A_4, \{(13)(24)\}))$ = {(1), (123), (132), (124), (142), (134), (143), (234), (243), (12)(34), (13)(24), (14)(23)} and the edge set of the Cayley graph, $E(Cay(A_4,\{(13)(24)\}))$ = {{(1), (13)(34)}, {(123),(142)}, {(132),(234)}, {(124),(143)}, {(134),(243)}, {(12)(34),(14)(23)}}.

Thus, the Cayley graph is presented in Figure 2.2.

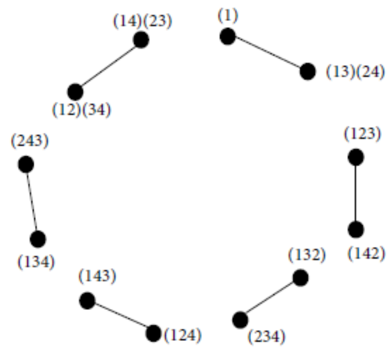

Figure 2.2 The Cayley graph of A_4 with respect to the subset S_2

This shows that $Cay(A_4, \{(13)(24)\}) = 6K_2$. ■

Proposition 2.6 Let A_4 be the alternating group of order 12. Then, the Cayley graph of A_4 with respect to the subset S_3 where $S_3 = \{(14)(23)\}$, $Cay(A_4,\{(14)(23)\}) = 6K_2$.

Proof From Proposition 2.3, the vertex set of the Cayley graph, $V(Cay(A_4, \{(14)(23)\})) = \{(1), (123), (132), (124), (142), (134), (143), (234), (243), (12)(34), (13)(24), (14)(23)\}$ and the edge set of the Cayley graph, $E(Cay(A_4,\{(14)(23)\})) = \{\{(1), (14)(23)\}, \{(123),(134)\}, \{(132),(124)\}, \{(143),(234)\}, \{(243),(142)\}, \{(13)(24),(12)(34)\}\}$.

Thus, the Cayley graph is presented in Figure 2.3.

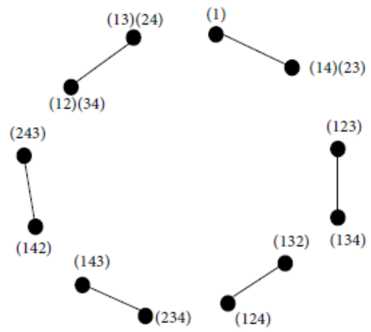

Figure 2.3 The Cayley graph of A_4 with respect to subset S_3

This shows that $Cay(A_4, \{(14)(23)\}) = 6K_2$. ■

From Proposition 2.4, Proposition 2.5, and Proposition 2.6, the Cayley graph of A_4 with respect to the subsets of order one can be summarized in the following theorem.

Theorem 2.1. Let A_4 be the alternating group of order 12. Then, the Cayley graph of A_4 with respect to the subsets S of order one, $Cay(A_4, S\} = 6K_2$.

3. The Energy of the Cayley Graph Associated to the Alternating Group of Order 12 with Subsets of Order One

In this section, the energy of the Cayley graph of the alternating group, A_4 with respect to the subset of order one is presented in the following propositions.

Proposition 3.1. Let A_4 be the alternating group of order 12 and R be the Cayley graph of A_4 with respect to the subset $S_1 = \{(12)(34)\}$. Then, the adjacency matrix of R,

$$A(R)=\begin{bmatrix} 0&1&0&0&0&0&0&0&0&0&0&0\\ 1&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&1&0&0&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&1&0&0&0&0&0&0\\ 0&0&0&0&1&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&1&0&0&0&0\\ 0&0&0&0&0&0&1&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&0&0&1&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&1\\ 0&0&0&0&0&0&0&0&0&0&1&0 \end{bmatrix}.$$

Proof Let A_4 be the alternating group of order 12 and R be the Cayley graph of A_4 with respect to the subset $S_1 = \{(12)(34)\}$ as given in Figure 2.1. By the definition of the adjacency matrix, the rows and columns of $A(R)$ are indexed by $V(R)$, namely v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_9, v_{10}, v_{11} and v_{12} where $v_1 = (1)$, $v_2 = (12)(34)$, $v_3 = (123)$, $v_4 = (243)$, $v_5 = (132)$, $v_6 = (143)$, $v_7 = (124)$, $v_8 = (234)$, $v_9 = (142)$, $v_{10} = (134)$, $v_{11} = (13)(24)$, $v_{12} = (14)(23)$. Since $R = 6K_2$, hence the corresponding adjacent vertices will have the entry 1, otherwise 0. ■

Next, the characteristic polynomial of the adjacency matrix is computed.

Proposition 3.2. Let A_4 be the alternating group of order 12 and R be the Cayley graph of A_4 with respect to the subset $S_1 = \{(12)(34)\}$. Then, the characteristic polynomial of $A(R)$ is $(\lambda+1)^6(\lambda-1)^6$.

Proof Let A_4 be the alternating group of order 12 and R be the Cayley graph of A_4 with respect to the subset $S_1 = \{(12)(34)\}$. By the definition of characteristic polynomial, the determinant, $det(A(R)-\lambda I)$ is the characteristic polynomial of $A(R)$. By solving $det(A(R)-\lambda I)$, the characteristic polynomial of $A(R)$ is $(\lambda+1)^6(\lambda-1)^6$. ■

From the characteristic polynomial of the adjacency matrix, the eigenvalues are then obtained.

Proposition 3.3. Let A_4 be the alternating group of order 12 and R be the Cayley graph of A_4 with respect to the subset $S = \{(12)(34)\}$. The adjacency eigenvalues of R are $\lambda_1=1$ with multiplicity 6 and $\lambda_2 = -1$ with multiplicity 6.

Proof By using the definition of the eigenvalues of a matrix, the roots of the characteristic equation is the eigenvalues of the matrix. From Proposition 3.2., since the characteristic equation of R is $(\lambda+1)^6(\lambda-1)^6 = 0$ then, the adjacency eigenvalues of R are $\lambda_1 = 1$ with multiplicity 6 and $\lambda_2 = -1$ with multiplicity 6. ■

Finally, the energy of the Cayley graph associated to the alternating group of order 12 with the first subset of order one is computed.

Proposition 3.4. Let A_4 be the alternating group of order 12 and R be the Cayley graph of A_4 with respect to the subset S = {(12)(34)}. Then, the energy of R, ε(R) = 12.

Proof By using the definition of the energy of a graph, the sum of absolute values of the eigenvalues of the adjacency matrix is the energy of a graph. From Proposition 3.3., the adjacency eigenvalues of R are $\lambda_1 = 1$ with multiplicity 6 and $\lambda_2 = -1$ with multiplicity 6. Then, the energy of R, $\epsilon(R) = 6|1| + 6|-1| = 12$. ■

Using the similar method, the energy of the Cayley graph of A_4 with respect to the other two subsets can be computed. Hence, the result can be summarized in the following theorem.

Theorem 3.1. Let A_4 be the alternating group of order 12. Then, the energy of the Cayley graph of A_4 with respect to the subsets S of order one, ε(Cay(A_4, S)) = 12.

4. The Maximum Degree Energy of the Cayley Graph Associated to the Alternating Group of Order 12 with Subsets of Order One

In this section, the maximum degree energy of the Cayley graph of the alternating group, A_4 with respect to the subsets of order one is presented in the following propositions.

Proposition 4.1. Let A_4 be the alternating group of order 12 and R be the Cayley graph of A_4 with respect to the subset S_1 = {(12)(34)}. Then, the maximum degree matrix of R is given in the following :

$$M(R)=\begin{bmatrix} 0&1&0&0&0&0&0&0&0&0&0&0\\ 1&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&1&0&0&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&1&0&0&0&0&0&0\\ 0&0&0&0&1&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&1&0&0&0&0\\ 0&0&0&0&0&0&1&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&0&0&1&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&1\\ 0&0&0&0&0&0&0&0&0&0&1&0 \end{bmatrix}.$$

Proof By the definition of the maximum degree matrix, consider the graph, R in Figure 2.1. The rows and columns of M(R) are indexed by V(R), namely v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_9, v_{10}, v_{11} and v_{12} where v_1 = (1), v_2 = (12)(34), v_3 = (123), v_4 = (243), v_5 = (132), v_6 = (143), v_7 = (124), v_8 = (234), v_9 = (142), v_{10} = (134), v_{11} = (13)(24), and v_{12} = (14)(23). The corresponding maximum degree of a vertices will have the entry 1, otherwise 0 because R is the union of six complete graphs of two vertices, $6K_2$ where the maximum degree of all vertices of R are 1. ■

Proposition 4.2. Let A_4 be the alternating group of order 12 and R be the Cayley graph of A_4 with respect to the subset S_1 = {(12)(34)}. The characteristic polynomial of M(R) is $(\lambda+1)^6(\lambda-1)^6$.

Proof Let A_4 be the alternating group of order 12 and R be the Cayley graph of A_4 with respect to the subset S_1 = {(12)(34)}. By the definition of characteristic polynomial, the determinant, $det(M(R)-\lambda I)$ is the characteristic polynomial of M(R). By solving $det(M(R)-\lambda I)$, the characteristic polynomial of M(R) is $(\lambda+1)^6(\lambda-1)^6$. ■

Proposition 4.3. Let A_4 be the alternating group of order 12 and R be the Cayley graph of A_4 with respect to the subset S_1 = {(12)(34)}. The maximum degree eigenvalues of R are $\lambda_1 = 1$ with multiplicity 6 and $\lambda_2 = -1$ with multiplicity 6.

Proof By definition, the eigenvalues of a matrix are the roots of the characteristic equation.

From Proposition 4.2., since the characteristic equation of R is $(\lambda+1)^6(\lambda-1)^6 = 0$ then, the eigenvalues of $Cay(A_4,\{(12)(34)\})$ are $\lambda_1 = 1$ with multiplicity 6 and $\lambda_2 = -1$ with multiplicity 6. ■

Proposition 4.4. Let A_4 be the alternating group of order 12 and R be the Cayley graph of A_4 with respect to the subset $S_1 = \{(12)(34)\}$. The maximum degree energy of R is 12.

Proof By the definition of maximum degree energy of a graph, the maximum degree energy of a graph is the sum of absolute values of the eigenvalues of the maximum degree matrix. From Proposition 4.3., the maximum degree eigenvalues of R are $\lambda_1 = 1$ with multiplicity 6 and $\lambda_2 = -1$ with multiplicity 6. Then, the maximum degree energy of $Cay(A_4,\{(12)(34)\})$, $\varepsilon_M(Cay(A_4, \{(12)(34)\})) = 6|1| + 6|-1| = 12$. ■

Using similar method, the maximum degree energy of the Cayley graph of A_4 with respect to the subsets S of order one can be summarized in the following theorem.

Theorem 4.1. Let A_4 be the alternating group of order 12. Then, the maximum degree energy of the Cayley graph of A_4 with respect to the subsets S of order one, $\varepsilon_M(Cay(A_4,S)) = 12$.

Conclusion

In this paper, since the Cayley graph of the alternating group of order 12 with subsets S of order one is the union of six complete graphs of two vertices, $6K_2$, then the adjacency matrix and the maximum degree matrix of the graph are identical. The findings obtained show that the energy and the maximum degree energy of the Cayley graph associated to the alternating group, A_4 with subsets of order one are also identical in which the values are 12.

Acknowledgement

The authors would like to express their sincere gratitude to all who have contributed throughout the course of this work.

References

[1] Gutman, I. The energy of graph. *Der Mathematisch-Statistischen Sektion im Forschungszentrum Graz*. 1978. 103: 1-22.

[2] Ramaswamy, H. and Veena, C. On the energy of unitary Cayley graphs. *The Electronic Journal of Combinatorics*, 2009. 16(1): N24

[3] Adiga, C. and Smitha, M. On maximum degree energy of a graph. *Int. J. Contemp. Math. Sciences*. 2009. 4(8): 385–396.

[4] Beineke, L. W. and Wilson, R. J. *Topics in algebraic graph theory*. 102. USA : Cambridge University Press. 2004

[5] Alspach, B., & Mishna, M. Enumeration of Cayley graphs and digraphs. *Discrete Mathematics*, 2002. 256(3), 527-539.

[6] Tripi, A. *Cayley Graphs of Groups and Their Applications*. Master Degree. Missouri State University; 2017.

[7] Kovács, I., & Servatius, M. On Cayley digraphs on nonisomorphic 2 groups. *Journal of Graph Theory*, 2011. 70(4), 435-448.