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# Laplacian Energy of Conjugacy Class Graph and Conjugate Graph Associated to Some Dihedral Groups

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## Abstract
The Laplacian energy of graphs, a fundamental concept in spectral graph theory, has recently gained significant attention among mathematicians. According to Huckel in the 1930s on the interplay between mathematics and chemistry, graph energy serves as an approximation for the total π-electron energy of molecules. Graphs associated to groups provide valuable insights into the structural characteristics and properties of the groups. The Laplacian energy of a graph is defined as the sum of the absolute deviations of the eigenvalues of its Laplacian matrix. Meanwhile, the graphs associated to groups in the scope of this research is conjugacy class graph and conjugate graph. The conjugacy class graph is defined as a graph with vertices of conjugacy class graph are non-central conjugacy classes and two vertices $x$ and $y$ of $\Gamma$ are adjacent if $xy$ is an edge of $\Gamma$. On the other hand, for the conjugate graph, the vertices are non-central elements and two vertices $x$ and $y$ are adjacent if they are conjugate. In this research, the Laplacian energy of graphs associated to groups is of interest, particularly for the dihedral groups. These characteristics and properties enable to derive the Laplacian energy by utilizing the eigenvalues of the Laplacian matrix of the graphs, providing a systematic framework for analysis. In this project report, the Laplacian energy of conjugacy class graph and conjugate graph associated to the dihedral groups from order six until order sixteen are presented. The main aim of this research is to find the Laplacian energy of these graphs by using the eigenvalues of the Laplacian matrix of the graphs. The computations of the Laplacian energy of conjugacy class graphs and conjugate graphs for selected dihedral groups are shown. Some examples are also provided to illustrate the results.

**Keywords:** Laplacian energy; conjugacy class graph, conjugate graph; finite group; dihedral group

## Introduction
The definition of the energy of a graph was introduced by Gutman in 1978 [1]. The energy of a graph is defined as the sum of the absolute values of its eigenvalues. In 2006, Gutman and Zhou [2] defined the Laplacian energy of a graph as the sum of the absolute deviations i.e., distance from the mean of the eigenvalues of its Laplacian matrix.

In other fields, graphs representing conjugated molecules, the Laplacian energy of the graph $LE(\Gamma)$ is closely related to their total π-electron energy, as calculated within Huckel molecular orbital approximation. In most cases the Laplacian energy can be used to calculate the energy of molecular structures in a much simpler way.

A graph $\Gamma$ is a finite nonempty set of objects called vertices together with a set of unordered pairs of distinct vertices of $\Gamma$ called the edges [3]. The vertex-set of $\Gamma$ is denoted by $V(\Gamma)$, while the edge-set is denoted by $E(\Gamma)$.

The conjugacy class graph is defined as a graph with vertices of conjugacy class graph are non-central conjugacy classes and two vertices $x$ and $y$ of $\Gamma$ are adjacent if $xy$ is an edge of $\Gamma$. On the other hand, for the conjugate graph, the vertices are non-central elements and two vertices $x$ and $y$ are adjacent if they are conjugate. In this research, the graph associated to groups are focused on the conjugacy class graph and the conjugate graph. Thus, the Laplacian energy of both graphs is of interest.

## Graph Theory
Graph theory is the study of points (vertices) and lines (edges), where the edges can be directed or undirected. However, all graphs used in this research are undirected. In this section, some definitions and notations in graph theory related to this research are presented. These definitions are used to find some results on conjugacy class graph and conjugate graph associated to all nonabelian dihedral groups up to order 16.

**Definition 1 [4] Graph**
A pair consisting of a set $V$ of vertices and a set $E$ of edges labelled as $\Gamma = (V, E)$ is called a graph. The elements of $E$ are the lines that combine two elements of $V$.

**Definition 2 [5] Connected and disconnected graph**
A graph is connected if for every pair of vertices $u$ and $v$, there is a path from $u$ to $v$, while a disconnected graph is made up of connected pieces called components.

## Group Theory

In this section, some basic definitions and fundamental concepts of group theory that are used in this research are presented, starting with the definition of the centre of the group $G$.

**Definition 3 [6] Conjugates and conjugacy class**
Let $a$ and $b$ be elements of group $G$. Then, $a$ and $b$ are conjugates in $G$ if $xax^{-1} = b$ for some $x$ in $G$. The conjugacy class of $a$ is the set $cl(a) = \{x^{-1}ax | x \in G\}$ , where the non-central conjugacy class is the conjugacy class which does not include the centre elements of the group.

**Definition 4 [7] Dihedral Group**
The dihedral group, denoted by $D_n$, is a group of symmetries of a regular polygon, which include rotations and reflections, and its order is $2n$ where $n$ is an integer $n \geq 3$. The dihedral groups can be presented in a form of generators and relations given as follows:

$$D_n = \langle a, b: a^n = b^2 = 1, bab = a^{-1} \rangle$$

## Graph Associated to Group

Recently, many researchers are interested in doing research on graphs associated to groups. For example, earliest research on the non-commuting graph has been done by Goodman in 2003 [12].

**Definition 5 [8] Conjugacy Class Graph**
Let $G$ be a finite group and let $Z(G)$ be the centre of $G$. The vertices of conjugacy class graph of $G$ are non-central conjugacy classes of $G$ i.e., $|V(G)| = K(G) - |Z(G)|$, where $K(G)$ is the class number of $G$. Two vertices are adjacent if their cardinalities are not coprime (i.e., have common factor).

**Definition 6 [9] Conjugate Graph**
Let $G$ be a finite non-abelian group with centre $Z(G)$. The conjugate graph is a graph whose vertices are noncentral elements of $G$ in which two vertices are adjacent if they are conjugate.

## The Energy and Laplacian Energy of a Graph

In this section, some basic definitions and concepts of energy and Laplacian energy of a graph are included. Some researchers on Laplacian energy are also included. To approximate the π-electron energy of molecules, Gutman established the energy of graphs. In [20], Koolen and Moulton established a new upper bound on a graph's energy in terms of the quantity *n* of vertices and *m* of edges.

**Definition 7 [10] Adjacency Matrix**
Let *R* be a graph with $V(R) = \{1, 2, \ldots, n\}$ and $E(R) = \{e_1, e_2, \cdots, e_m\}$. The adjacency matrix of $R$, denoted by $A(R)$, is the $n \times n$ matrix defined as follows:
The rows and the columns of $A(R)$ are indexed by $V(R)$. If $i \neq j$ then the (*i*, *j*) −entry of $A(R)$ is 0 for vertices $i$ and $j$ non-adjacent, and the (*i*, *j*) −entry is 1 for $i$ and $j$ adjacent. The (*i*, *i*) −entry of $A(R)$ is 0 for $i = 1, 2, \cdots, n$. *A*(*R*) is often simply denoted by $A$.

**Definition 8 [11] Characteristic Polynomial of a Matrix**
Let $L$ be an $n \times n$ matrix. The determinant, $det(L - \lambda I)$, is a polynomial in the (complex) variable $\lambda$ of degree $n$ and is called the characteristic polynomial of $L$. The equation $det(L - \lambda I) = 0$ is called the characteristic equation of $L$.

**Definition 9 [11] Eigenvalues of Matrix**
The roots of the characteristic equation $\det(L - \lambda I) = 0$ of $L$ are called the eigenvalues of $L$.

**Definition 10 [12] Energy of a Graph**
Consider $R$ to be a graph and $\lambda_1, \lambda_2, \cdots, \lambda_n$ be the eigenvalues of the adjacency matrix of $R$. The energy of $R$, $\varepsilon(R)$, is defined as follows,

$$\varepsilon(R) = \sum_{i=1}^{n} |\lambda_i|$$

**Definition 11 [13] Laplacian Matrix**
Given a simple graph $G$ with $n$ vertices $v_1, \cdots, v_n$, its Laplacian matrix $L_{n\times n}$ is defined elementwise as

$$L_{i,j} = \begin{cases} deg(v_i), & \text{if } i = j, \\ -1, & \text{if } i \neq j \text{ and } v_i \text{ is adjacent to } v_j, \\ 0, & \text{otherwise.} \end{cases}$$

**Definition 12 [14] Laplacian Energy**
Let $G$ be a graph with $n$ vertices and $m$ edges. Let $\mu_1, \mu_2, \cdots, \mu_n$ be the eigenvalues of the Laplacian matrix of $G$. The Laplacian energy of G, $LE(G) = \Sigma_{i=1}^{n} \left|\mu_i - \frac{2m}{n}\right|$.

**The Conjugacy Class Graph and Conjugate Graph Associated to Dihedral Group Up to Order 16**

**Proposition 1** Let $G$ be the dihedral group of order 8, $D_4$. Then, $G$ has two conjugacy classes of order one, and three conjugacy classes of order three.
**Proof** Let $G$ be the dihedral group of order 8, $D_4$. The elements of $G$ are $\{R_0, R_1, R_2, R_3, S_0, S_1, S_2, S_3\}$. The Cayley table for the dihedral group $D_4$ is given in the following for each rotation and reflection.

Table 3.2 The Cayley table of $D_4$

| $\cdot$ | $R_0$ | $R_1$ | $R_2$ | $R_3$ | $S_0$ | $S_1$ | $S_2$ | $S_3$ |
|---|---|---|---|---|---|---|---|---|
| $R_0$ | $R_0$ | $R_1$ | $R_2$ | $R_3$ | $S_0$ | $S_1$ | $S_2$ | $S_3$ |
| $R_1$ | $R_1$ | $R_2$ | $R_3$ | $R_0$ | $S_3$ | $S_2$ | $S_0$ | $S_1$ |
| $R_2$ | $R_2$ | $R_3$ | $R_0$ | $R_1$ | $S_1$ | $S_0$ | $S_3$ | $S_2$ |
| $R_3$ | $R_3$ | $R_0$ | $R_1$ | $R_2$ | $S_2$ | $S_3$ | $S_1$ | $S_0$ |
| $S_0$ | $S_0$ | $S_2$ | $S_1$ | $S_3$ | $R_0$ | $R_2$ | $R_1$ | $R_3$ |
| $S_1$ | $S_1$ | $S_3$ | $S_0$ | $S_2$ | $R_2$ | $R_0$ | $R_3$ | $R_1$ |
| $S_2$ | $S_2$ | $S_1$ | $S_3$ | $S_0$ | $R_3$ | $R_1$ | $R_0$ | $R_2$ |
| $S_3$ | $S_3$ | $S_0$ | $S_2$ | $S_1$ | $R_1$ | $R_3$ | $R_2$ | $R_0$ |

The conjugacy classes of dihedral group $D_4$ are found using the Cayley table and the definition of conjugacy class (Definition 3).

The conjugacy classes of each element in dihedral group $D_4$ are shown below:

$$cl(R_0) = \{x^{-1}R_0x | x \in G\} = \{R_0\},$$
$$cl(R_1) = \{x^{-1}R_1x | x \in G\} = \{R_1, R_3\},$$
$$cl(R_2) = \{x^{-1}R_2x | x \in G\} = \{R_2\},$$
$$cl(S_0) = \{x^{-1}S_0x | x \in G\} = \{S_0, S_1\},$$
$$cl(S_2) = \{x^{-1}S_2x | x \in G\} = \{S_2, S_3\}.$$

The conjugacy class of elements $R_3, S_1$ and $S_3$ are the same with the conjugacy class of elements $R_1, S_0$ and $S_2$ respectively. Thus, $G$ has two conjugacy classes of order one, and three conjugacy classes of order two.

**Proposition 2** Let $G$ be the dihedral group of order 8, $D_4$. Then the conjugacy class graph of $G$ is a complete graph with three vertices and three edges.
**Proof** Let $G$ be the dihedral group of order 8, $D_4$. From Proposition 1, the conjugacy class shows that $cl(R_1) = cl(R_3)$, $cl(S_0) = cl(S_1)$ and $cl(S_2) = cl(S_3)$. So, the order of each conjugacy class becomes $|cl(R_1)| = 2$, $|cl(S_0)| = 2$, and $|cl(S_2)| = 2$. Then, $gcd(|cl(R_1)|, |cl(S_0)|) = 2 \neq 1$ and $gcd(|cl(R_1)|, |cl(S_2)|) = 2 \neq 1$. So, the vertices of $cl(R_1)$, $cl(S_0)$ and $cl(S_2)$ are adjacent and connected to

each other. Figure 1 shows the graph of conjugacy class graph of dihedral group $D_4$.

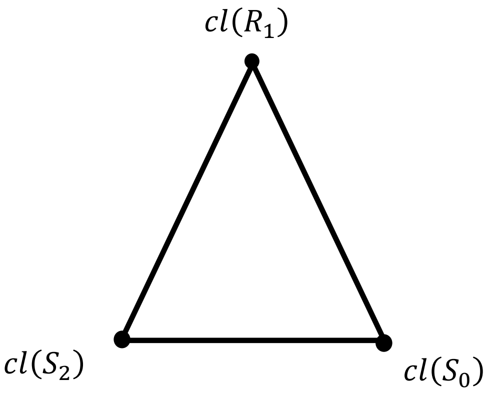



**Figure 1** The Conjugacy Class Graph of Dihedral Group $D_4$

Thus, the conjugacy class graph of $G$ is a complete graph with three vertices and three edges. □

**Proposition 3** Let $G$ be the dihedral group of order 8, $D_4$. Then the conjugate graph of G is a union of three complete graphs of order two.
**Proof** Let $G$ be the dihedral group of order 8, $D_4$. From Proposition 1, the conjugacy class shows that $cl\,(R_1) = cl(R_3)$, $cl(S_0) = cl(S_1)$ and $cl(S_2) = cl(S_3)$. So, the vertex $R_1$ is adjacent to $R_3$, the vertex $S_0$ is adjacent to $S_1$, and the vertex $S_2$ is adjacent to $S_3$. Figure 2 shows the conjugate graph of dihedral group $D_4$.

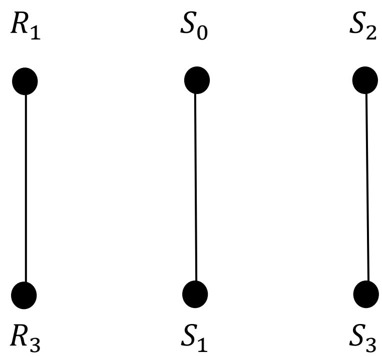
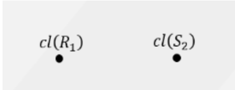
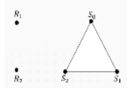
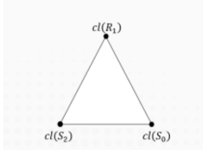
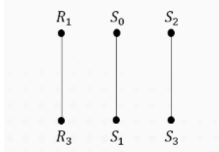
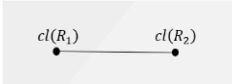
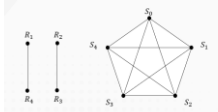
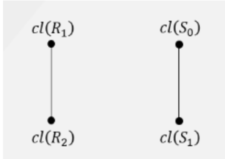
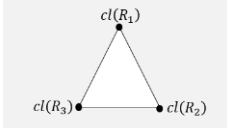
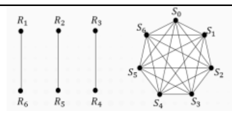
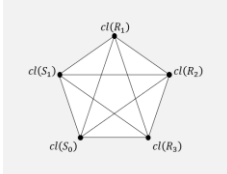
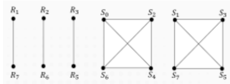



**Figure 2** The Conjugate Graph of Dihedral Group $D_4$

Thus, the conjugate graph of G is a union of three complete graph of order two.

Using similar methods, the conjugacy class graph and conjugate graph of each dihedral group up to order 16 can be found as given in Table 1.

**Table 1:** The Conjugacy Class Graph and Conjugate Graph of Each Dihedral Group

| Dihedral Group | Conjugacy Class | Conjugacy Class Graph | Conjugate Graph |
|---|---|---|---|
| $D_3$ | $cl(R_0) = \{x^{-1}R_0x \mid x \in G\} = \{R_0\}$,<br>$cl(R_1) = \{x^{-1}R_1x \mid x \in G\} = \{R_1, R_2\}$,<br>$cl(S_0) = \{x^{-1}S_0x \mid x \in G\} = \{S_0, S_1, S_2\}$. |  |  |
| $D_4$ | $cl(R_0) = \{x^{-1}R_0x \mid x \in G\} = \{R_0\}$,<br>$cl(R_1) = \{x^{-1}R_1x \mid x \in G\} = \{R_1, R_3\}$,<br>$cl(R_2) = \{x^{-1}R_2x \mid x \in G\} = \{R_2\}$,<br>$cl(S_0) = \{x^{-1}S_0x \mid x \in G\} = \{S_0, S_1\}$,<br>$cl(S_2) = \{x^{-1}S_2x \mid x \in G\} = \{S_2, S_3\}$. |  |  |
| $D_5$ | $cl(R_0) = \{x^{-1}R_0x \mid x \in G\} = \{R_0\}$,<br>$cl(R_1) = \{x^{-1}R_1x \mid x \in G\} = \{R_1, R_4\}$,<br>$cl(R_2) = \{x^{-1}R_2x \mid x \in G\} = \{R_2, R_3\}$,<br>$cl(S_0) = \{x^{-1}S_0x \mid x \in G\} = \{S_0, S_1, S_2, S_3, S_4\}$. |  |  |
| $D_6$ | $cl(R_0) = \{x^{-1}R_0x \mid x \in G\} = \{R_0\}$,<br>$cl(R_1) = \{x^{-1}R_1x \mid x \in G\} = \{R_1, R_5\}$,<br>$cl(R_2) = \{x^{-1}R_2x \mid x \in G\} = \{R_2, R_4\}$,<br>$cl(R_3) = \{x^{-1}R_3x \mid x \in G\} = \{R_3\}$,<br>$cl(S_0) = \{x^{-1}S_0x \mid x \in G\} = \{S_0, S_2, S_4\}$,<br>$cl(S_1) = \{x^{-1}S_1x \mid x \in G\} = \{S_1, S_3, S_5\}$. |  |  |
| $D_7$ | $cl(R_0) = \{x^{-1}R_0x \mid x \in G\} = \{R_0\}$,<br>$cl(R_1) = \{x^{-1}R_1x \mid x \in G\} = \{R_1, R_6\}$,<br>$cl(R_2) = \{x^{-1}R_2x \mid x \in G\} = \{R_2, R_5\}$,<br>$cl(R_3) = \{x^{-1}R_3x \mid x \in G\} = \{R_3, R_4\}$,<br>$cl(S_0) = \{x^{-1}S_0x \mid x \in G\} =$<br>$\{S_0, S_1, S_2, S_3, S_4, S_5, S_6\}$. |  |  |
| $D_8$ | $cl(R_0) = \{x^{-1}R_0x \mid x \in G\} = \{R_0\}$,<br>$cl(R_1) = \{x^{-1}R_1x \mid x \in G\} = \{R_1, R_7\}$,<br>$cl(R_2) = \{x^{-1}R_2x \mid x \in G\} = \{R_2, R_6\}$,<br>$cl(R_3) = \{x^{-1}R_3x \mid x \in G\} = \{R_3, R_5\}$,<br>$cl(R_4) = \{x^{-1}R_4x \mid x \in G\} = \{R_4\}$,<br>$cl(S_0) = \{x^{-1}S_0x \mid x \in G\} = \{S_0, S_2, S_4, S_6\}$,<br>$cl(S_1) = \{x^{-1}S_1x \mid x \in G\} = \{S_1, S_3, S_5, S_7\}$. |  |  |

**The Laplacian Energy of The Conjugacy Class Graph and Conjugate Graph Associated to Dihedral Group Up to Order 16**

**Proposition 4** Let $G$ be the dihedral group of order 8, $D_4$. Then the Laplacian matrix of $G$,

$$L_{D_4} = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}.$$

**Proof** Let $G$ be the dihedral group of order 8, $D_4$. The Laplacian matrix of the conjugacy class graph can be obtained by using the definition of Laplacian matrix (Definition 11), namely

$$L_{i,j} = \begin{cases} deg(v_i), & \text{if } i = j, \\ -1, & \text{if } i \neq j \text{ and } v_i \text{ is adjacent to } v_j, \\ 0, & \text{otherwise.} \end{cases}$$

For dihedral group $D_4$, the graph is from Figure 1. The diagonal of the matrix must be 2 by the definition. Since the graph is a complete graph, so other elements in the matrix are -1. The Laplacian matrix for conjugacy class graph associated to dihedral group $D_4$ is:

$$L_{D_4} = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}.$$

**Proposition 5** Let $G$ be the dihedral group of order 8, $D_4$. Then the eigenvalues of the Laplacian matrix for the conjugacy class graph of $G$ are 0 of multiplicity one and 3 of multiplicity two.
**Proof** The eigenvalues of the Laplacian matrix were obtained using the definition of eigenvalues of a matrix (Definition 9).

$$|L-\lambda I| = \left|\begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}\right| = \begin{vmatrix} 2-\lambda & -1 & -1 \\ -1 & 2-\lambda & -1 \\ -1 & -1 & 2-\lambda \end{vmatrix} = 0,$$

which gives

$$(2-\lambda)\begin{vmatrix} 2-\lambda & -1 \\ -1 & 2-\lambda \end{vmatrix} - (-1)\begin{vmatrix} -1 & -1 \\ -1 & 2-\lambda \end{vmatrix} + (-1)\begin{vmatrix} -1 & 2-\lambda \\ -1 & -1 \end{vmatrix} = 0$$

$$-\lambda^3 + 6\lambda^2 - 9\lambda = 0$$

$-\lambda(\lambda-3)^2 = 0.$

Thus, $\lambda_1 = 0, \lambda_2 = 3, \lambda_3 = 3$ are the eigenvalues.

**Proposition 6** Let $G$ be the dihedral group of order 8, $D_4$. Then the Laplacian energy of the conjugacy class graph of $G$ is 4.
**Proof** Let $G$ be the dihedral group of order 8, $D_4$. The Laplacian energy of the conjugacy class graph of dihedral group $D_4$ is calculated by using the definition of Laplacian energy (Definition 12).

$$LE(G) = \sum_{i=1}^{n} \left|\mu_i - \frac{2m}{n}\right|.$$

$$\begin{aligned} LE(D_4) &= \sum_{i=1}^{3} \left|\mu_i - \frac{2(3)}{3}\right| \\ &= \left|0 - \frac{2(3)}{3}\right| + \left|3 - \frac{2(3)}{3}\right| + \left|3 - \frac{2(3)}{3}\right| \\ &= 4\ . \end{aligned}$$

Thus, the Laplacian energy of the conjugacy class graph of the dihedral group of order 8, $D_4$ is 4. □

**Proposition 4** Let $G$ be the dihedral group of order 8, $D_4$. Then the Laplacian matrix of *G*,

$$L_{D_4} = \begin{bmatrix} 2 & -1 & 0 & 0 & 0 & 0 \\ -1 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & -1 & 0 & 0 \\ 0 & 0 & -1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & -1 \\ 0 & 0 & 0 & 0 & -1 & 2 \end{bmatrix}.$$

**Proof** Let $G$ be the dihedral group of order 8, $D_4$. The Laplacian matrix of the conjugacy class graph can be obtained by using the definition of Laplacian matrix (Definition 11), namely

$$L_{i,j} = \begin{cases} deg(v_i), & \text{if } i = j, \\ -1\ , & \text{if } i \neq j \text{ and } v_i \text{ is adjacent to } v_j, \\ 0\ , & \text{otherwise.} \end{cases}$$

For dihedral group $D_4$, the graph is from Figure 2. The diagonal of the matrix must be 2 by the definition. When two vertices were connected with an edge, then, that two vertices are adjacent. So, the elements in the matrix are -1. And if they are not connected with an edge, then the element in the matrix is 0. The Laplacian matrix for conjugate graph associated to dihedral group $D_4$ is:

$$L_{D_4} = \begin{bmatrix} 2 & -1 & 0 & 0 & 0 & 0 \\ -1 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & -1 & 0 & 0 \\ 0 & 0 & -1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & -1 \\ 0 & 0 & 0 & 0 & -1 & 2 \end{bmatrix}$$

.

**Proposition 5** Let $G$ be the dihedral group of order 8, $D_4$. Then the eigenvalues of the Laplacian matrix for the conjugacy class graph of $G$ are 1 with multiplicity three and 3 with multiplicity three.
**Proof** The eigenvalues of the Laplacian matrix were obtained using the definition of eigenvalues of a matrix (Definition 9).

$$|L-\lambda I| = \left\| \begin{bmatrix} 2 & -1 & 0 & 0 & 0 & 0 \\ -1 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & -1 & 0 & 0 \\ 0 & 0 & -1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & -1 \\ 0 & 0 & 0 & 0 & -1 & 2 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 & 0 & 0 \\ 0 & 0 & \lambda & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda & 0 & 0 \\ 0 & 0 & 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & 0 & 0 & \lambda \end{bmatrix} \right\|$$

$$= \begin{vmatrix} 2-\lambda & -1 & 0 & 0 & 0 & 0 \\ -1 & 2-\lambda & 0 & 0 & 0 & 0 \\ 0 & 0 & 2-\lambda & -1 & 0 & 0 \\ 0 & 0 & -1 & 2-\lambda & 0 & 0 \\ 0 & 0 & 0 & 0 & 2-\lambda & -1 \\ 0 & 0 & 0 & 0 & -1 & 2-\lambda \end{vmatrix} = 0.$$

Thus, $\lambda_1 = 1, \lambda_2 = 3, \lambda_3 = 1, \lambda_4 = 3, \lambda_5 = 1, \lambda_6 = 3$ are the eigenvalues.

**Proposition 6** Let $G$ be the dihedral group of order 8, $D_4$. Then the Laplacian energy of the conjugacy class graph of $G$ is 12.

**Proof** Let $G$ be the dihedral group of order 8, $D_4$. The Laplacian energy of the conjugacy class graph of dihedral group $D_4$ were obtained by using the definition of Laplacian energy (Definition 12).

$$LE(G) = \sum_{i=1}^{n} \left|\mu_i - \frac{2m}{n}\right|,$$

$$LE(D_4) = \sum_{i=1}^{n} \left|\mu_i - \frac{2(6)}{3}\right|$$

$$= \left|1-\frac{2(6)}{3}\right| + \left|3-\frac{2(6)}{3}\right| + \left|1-\frac{2(6)}{3}\right| + \left|3-\frac{2(6)}{3}\right| + \left|1-\frac{2(6)}{3}\right| + \left|3-\frac{2(6)}{3}\right|$$

$$= 12.$$

Thus, the Laplacian energy of the conjugate graph of the dihedral group of order 8, $D_4$ is 12. □

**Table 2:** The Laplacian Energy of The Conjugacy Class Graph and Conjugate Graph of Each Dihedral Group

| **Dihedral Group** | **Laplacian Energy of Conjugacy Class Graph** | **Laplacian Energy of Conjugate Graph** |
|---|---|---|
| $D_3$ | Cannot be solved | $LE(D_3) = \frac{11}{2}$ |
| $D_4$ | $LE(D_4) = 4$ | $LE(D_4) = 12$ |
| $D_5$ | $LE(D_5) = 4$ | $LE(D_5) = 8$ |
| $D_6$ | $LE(D_6) = 8$ | $LE(D_6) = 11$ |
| $D_7$ | $LE(D_7) = 4$ | $LE(D_7) = 12$ |
| $D_8$ | $LE(D_8) = 6$ | $LE(D_8) = \frac{122}{7}$ |

## Conclusion

To find the Laplacian energy of conjugacy class graph and conjugate graph, the first thing to do is find the conjugacy class of each dihedral group. Then, using the definition of Laplacian matrix, the Laplacian matrix of each dihedral group was obtained. After the Laplacian matrix was obtained, the eigenvalues of each graph were computed. Lastly, using the definition of Laplacian energy, the Laplacian energy of each conjugacy class graph and conjugate graph associated to dihedral group has been calculated.

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