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General Zeroth-order Randić Index of Zero Divisor Graph for the Ring of Integers Modulo p^n

Ghazali Semil Ismail[1, 2, a)], Nor Haniza Sarmin[1, b)], Nur Idayu Alimon[2, c)] and Fariz Maulana[3, d)]

[1]Department of Mathematical Sciences, Faculty of Science, Universiti Teknologi Malaysia, 81310 Johor Bahru, Johor, Malaysia
[2]Mathematical Sciences Studies, College of Computing, Informatics and Media, Universiti Teknologi MARA Johor Branch, Pasir Gudang Campus, 81750, Masai, Johor, Malaysia
[3]Faculty of Mathematics and Natural Sciences, Institut Teknologi Bandung, 40132, Bandung, Indonesia

[a)] Corresponding author: ghazali85@graduate.utm.my
[b)]nhs@utm.my
[c)]idayualimon@uitm.edu.my
[d)]farizholmes@gmail.com

Abstract. A simple graph is a set of vertices, $V(\Gamma)$ and a set of edges, $E(\Gamma)$, where each edge $\langle u - v\rangle$ connects two different vertices u and v (there are no self-loops). In topological index, the general zeroth-order Randić index is defined as the sum of the degree of each vertex to the power of $\alpha \neq 0$. Given a ring R, let $\Gamma(R)$ denote the graph whose vertex set is R, such that the distinct vertices a and b are adjacent provided that $ab = 0$ for the zero-divisor graph of a ring. In this paper, we present the general formula of the general zeroth-order Randić index of the zero-divisor graph for some commutative rings. The commutative ring in the scope of this research is the ring of integers modulo p^n, where p is a prime number and n is a positive integer. The general zeroth-order Randić index is found for the cases $\alpha = 1, 2$ and 3.

INTRODUCTION

Topological indices are formulated in mathematical chemistry to describe chemical compounds based on their molecular graphs, where a molecular graph of chemical compounds can be represented as a graph in graph theory. Different molecular structures are typically modeled using molecular graphs to understand a chemical compound's properties theoretically. As algebraic structures such as a ring are introduced and explored, the concept of graphs derived from them is also becoming more prevalent. The nature and applications of topological indices have been studied by Basak et al. [1] as it has close relations to chemistry. According to Mondal et al. [2], the topological indices can be used to model properties and biological activities of chemical compounds. For the aforesaid antiviral drugs, this study investigates some degree-based and neighborhood degree sum-based topological indices, using polynomial approach.

Topological index has been introduced in 1947 by a chemist, Harold Wiener which the Wiener index is the first type of topological indices that has been developed and the boiling point of paraffin can be predicted with it. Since that, there are many types of topological indices have been introduced by many researchers such as the Zagreb index, the Szeged index, the Harary index, the Kirchoff index, the eccentric connectivity index, the Randić connectivity index, the atom-bond connectivity index, and the harmonic index. Several studies have been conducted on the topological indices of graphs including Zhong in [3] who described the minimum and maximum values of the harmonic index for simple connected graphs and trees, and the corresponding extremal graphs. Das et al. in [4] have obtained lower and upper bounds on the first Zagreb index $M_1(G)$ of G in terms of the number of edges (m), the number of vertices

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(n), the minimum vertex degree (δ) and the maximum vertex degree (Δ). The Wiener index of line graphs and some classes of graphs is obtained by Ramane et al. [5]. Meanwhile, Alimon et al. in [6] have determined the general formula of the first Zagreb index, the Edge-Wiener index and the second Zagreb index of the non-commuting graphs of dihedral groups. The general formula of the first Zagreb index, the second Zagreb index and the Wiener index for the non-commuting graph associated with generalised quaternion group in terms of n are determined by Sarmin et al. [7].

Randić index has been developed by Randić in 1975 [8] where many studies have been conducted into the characterization of molecular branding, including pharmacology and chemistry. Particularly for designing of quantitative structure-property (QSPR) and structure-activity relations (QSAR) [9]. It is a degree-based topological indices which usually be used by chemists. Then, the general zeroth-order Randić index has been developed by Li and Zheng [10]. The general zeroth-order Randić index is defined as the sum of the degree of each vertex to the power of $\alpha \neq 0$. In [11], the maximum value of zeroth-order general Randić index on the graphs of order n with a given clique number is presented for any $\alpha \neq 0,1$ and $\alpha \notin (2; 2n-1]$ where $n = |V(G)|$. Ahmed et al. in [12] have developed some extremal results for the zeroth-order general Randić index of cactus graphs and determined some sharp bounds on the index. The various characteristics of graph for non-commutative gamma near ring is investigated and corresponding results related with the Γ-near-ring theoretic concepts are obtained by Rajeswari and Meenakumari in [13]. Shuker and Rasheed have defined the Maxideal and Minideal zero divisor graphs of the ring $\mathbb{Z}_n$ for any n, to be the zero divisor graph of maximal and minimal ideals of the ring $\mathbb{Z}_n$ in [14]. Smith in [15] concentrated on the zero divisor graph of the set of integers modulo n, $\mathbb{Z}_n$. It is precisely determined all values of n for which the zero divisor graph is perfect.

This paper presents the general formula of the general zeroth-order Randić index of a zero-divisor graph using the set of integers of modulo p^n. The general zeroth-order Randić index is found for the cases $\alpha = 1, 2$ and 3.

PRELIMINARIES

In this section, some definitions and basic concepts in ring theory that are applied in this research are included.

Definition 1 [16] Zero Divisor Graph
Let R be a commutative ring. The zero-divisor graph of R, denoted by $\Gamma(R)$, is the simple graph with the vertex set R and two distinct vertices a and b are joined by an edge whenever $ab = 0$.

Definition 2 [4] General Zeroth-order Randić Index
Let Γ be a connected graph and $deg(u)$ be the degree of vertex u in the graph. Then,

$$R_\alpha^0 = \sum_{u \in V} \left(deg(u)\right)^\alpha.$$

In the following propositions, the degree of a vertex, the number of vertices and edges in the zero-divisor graph of integers of modulo p have been determined by Maulana et al. in [17].

Proposition 1 [17] Let p be a prime number, $n \in \mathbb{N}$ and $a \in \mathbb{Z}_{p^n}$ with $gcd(a, p^n) = p^i$ for $i = 0,1,2, \ldots, n$. Then, the degree of a in the zero divisor graph of $\mathbb{Z}_{p^n}$ is

$$deg(a) = \begin{cases} p^i, & i \leq \left\lfloor \frac{n-1}{2} \right\rfloor, \\ p^i - 1, & i > \left\lfloor \frac{n-1}{2} \right\rfloor. \end{cases}$$

Proposition 2 [17] Let $V'_i = \{a \in V(\Gamma\mathbb{Z}_{p^n}) : gcd(a, p^n) = p^i\}$, then $|V'_i| = p^{n-i} - p^{n-(i+1)}$ for $0 \leq i \leq n-1$ and $|V'_i| = 1$ for $i = n$.

Proposition 3 [17] The number of edges of $\Gamma(\mathbb{Z}_{p^n})$ is $\frac{1}{2}\left[(n+1)p^n - np^{n-1} - p^{\left\lceil \frac{n-1}{2} \right\rceil}\right]$.

By the definition, a perfect graph is a graph Γ for which every induced subgraph of Γ has chromatic number equal to its clique number. The following two terms define a perfect graph. The chromatic number of a graph Γ indicates the minimum number of colours that are required to colour the vertices of Γ such that no two adjacent vertices have the same colour. The clique number of graph Γ is the size of the largest complete subgraph of Γ [15].

Theorem 1 [15] Smith's Main Theorem
There are four forms in which the zero-divisor graph of $\mathbb{Z}_n$ is perfect:
(a) $n = p^a$ for positive integer a and a prime p,
(b) $n = p^a q^b$ for positive integers a, b and distinct primes p, q,
(c) $n = p^a q^b r^c$ for positive integers a, b, c and distinct primes p, q, r,
(d) $n = pqrs$ for distinct primes p, q, r, s.

The first form of the Smith's main theorem, namely when $n = p^a$ for positive integer a and a prime p is focused on this research.

RESULTS AND DISCUSSION

This section presents some results related to the general zeroth-order Randić index of zero divisor graph for the ring of integers modulo p^n for $\alpha = 1, 2$ and 3 given in the following three theorems.

Theorem 2 The general zeroth-order Randić index of zero divisor graph for $\mathbb{Z}_{p^n}$ when $\alpha = 1$ is $p^n - 1 + n(p^n - p^{n-1}) + \frac{p^n}{p-1}\left(\frac{p^{\left\lfloor \frac{n}{2} \right\rfloor - 1}}{p^{\left\lfloor \frac{2n+1}{2} \right\rfloor}} - \frac{p^{\left\lceil \frac{n-1}{2} \right\rceil - 1}}{p^{n-1}}\right)$ where $n \geq 1$.

Proof By Proposition 1 and Proposition 2, we have

$$
\begin{aligned}
R_1^0 &= \sum_{u \in V} \left(deg(u)\right)^1 \\
&= p^n - 1 + \sum_{i=0}^{\left\lfloor \frac{n-1}{2} \right\rfloor} (p^i)^1\left(p^{n-1} - p^{n-(i+1)}\right) + \sum_{i=1+\left\lfloor \frac{n-1}{2} \right\rfloor}^{n-1} (p^i - 1)^1\left(p^{n-i} - p^{n-(i+1)}\right) \\
&= p^n - 1 + \sum_{i=0}^{n-1} (p^n - p^{n-1}) - \sum_{i=1+\left\lfloor \frac{n-1}{2} \right\rfloor}^{n-1} (p^{n-i-1} - p^{n-i}) \\
&= p^n - 1 + n(p^n - p^{n-1}) + \frac{p^n}{p-1}\left(\frac{p^{\left\lfloor \frac{n}{2} \right\rfloor - 1}}{p^{\left\lfloor \frac{2n+1}{2} \right\rfloor}} - \frac{p^{\left\lceil \frac{n-1}{2} \right\rceil - 1}}{p^{n-1}}\right),
\end{aligned}
$$

where $n \geq 1$. □

Theorem 3 The general zeroth-order Randić index of zero divisor graph for $\mathbb{Z}_{p^n}$ when $\alpha = 2$ is $p^{2n} - 2p^n + p^{n-1}(p^n - 1) + p^{\left\lceil \frac{n-1}{2} \right\rceil} - 2(p^n - p^{n-1})\left\lceil \frac{n-1}{2} \right\rceil$ where $n \geq 1$.

Proof By Proposition 1 and Proposition 2, we have

$$
R_2^0 = \sum_{u \in V} \left(deg(u)\right)^2
$$

$$= (p^n - 1)^2 + \sum_{i=0}^{\left\lfloor\frac{n-1}{2}\right\rfloor} (p^i)^2\left(p^{n-1} - p^{n-(i+1)}\right) + \sum_{i=1+\left\lfloor\frac{n-1}{2}\right\rfloor}^{n-1} (p^i - 1)^2\left(p^{n-i} - p^{n-(i+1)}\right)$$

$$= p^{2n} - 2p^n + 1 + \sum_{i=0}^{n-1} (p^{2i})\left(p^{n-i} - p^{n-(i+1)}\right) + \sum_{i=1+\left\lfloor\frac{n-1}{2}\right\rfloor}^{n-1} (p^{2i} - 2p^i + 1)\left(p^{n-i} - p^{n-(i+1)}\right)$$

$$= p^{2n} - 2p^n + 1 + \sum_{i=0}^{n-1} (p^{n+i} - p^{n+i-1}) - 2\sum_{i=1+\left\lfloor\frac{n-1}{2}\right\rfloor}^{n-1} (p^n - p^{n-1}) + \sum_{i=1+\left\lfloor\frac{n-1}{2}\right\rfloor}^{n-1} (p^{n-i} - p^{n-i-1})$$

$$= p^{2n} - 2p^n + p^{n-1}(p^n - 1) + p^{\left\lceil\frac{n-1}{2}\right\rceil} - 2(p^n - p^{n-1})\left\lceil\frac{n-1}{2}\right\rceil,$$

where $n \geq 1$. □

Theorem 4 The general zeroth-order Randić index of zero divisor graph for $\mathbb{Z}_{p^n}$ when $\alpha = 3$ is $p^{3n} - 3p^{2n} + 3p^n\left(1 + \left\lceil\frac{n-1}{2}\right\rceil\right) - p^{\left\lceil\frac{n-1}{2}\right\rceil} + p^{n-1}\left[\frac{p^{2n}-1}{p+1} - 3\left(p^n - p^{1+\left\lceil\frac{n-1}{2}\right\rceil} + \left\lceil\frac{n-1}{2}\right\rceil\right)\right]$ where $n \geq 1$.

Proof By Proposition 1 and Proposition 2, we have

$$R_3^0 = \sum_{u \in V} \left(deg(u)\right)^3$$

$$= (p^n - 1)^3 + \sum_{i=0}^{\left\lfloor\frac{n-1}{2}\right\rfloor} (p^i)^3\left(p^{n-i} - p^{n-(i+1)}\right) + \sum_{i=1+\left\lfloor\frac{n-1}{2}\right\rfloor}^{n-1} (p^i - 1)^3\left(p^{n-i} - p^{n-(i+1)}\right)$$

$$= p^{3n} - 3p^{2n} + 3p^n - 1 + \sum_{i=0}^{\left\lfloor\frac{n-1}{2}\right\rfloor} (p^{3i})\left(p^{n-i} - p^{n-(i+1)}\right) + \sum_{i=1+\left\lfloor\frac{n-1}{2}\right\rfloor}^{n-1} (p^{3i} - 3p^{2i} + 3p^i - 1)\left(p^{n-i} - p^{n-(i+1)}\right)$$

$$= p^{3n} - 3p^{2n} + 3p^n - 1 + (p^n - p^{n-1})\left[\sum_{i=0}^{n-1} p^{2i} + \sum_{i=1+\left\lfloor\frac{n-1}{2}\right\rfloor}^{n-1} \left(3 - 3p^i - \frac{1}{p^i}\right)\right]$$

$$= p^{3n} - 3p^{2n} + 3p^n\left(1 + \left\lceil\frac{n-1}{2}\right\rceil\right) - p^{\left\lceil\frac{n-1}{2}\right\rceil} + p^{n-1}\left[\frac{p^{2n}-1}{p+1} - 3\left(p^n - p^{1+\left\lceil\frac{n-1}{2}\right\rceil} + \left\lceil\frac{n-1}{2}\right\rceil\right)\right],$$

where $n \geq 1$. □

An example of the general zeroth-order Randić index for a commutative ring $\mathbb{Z}_{p^n}$ is shown in the following.

Example 1 When $p = 2$ and $n = 3$, the zero-divisor graph of $\mathbb{Z}_8$, $\Gamma(\mathbb{Z}_8)$, is shown in the following figure.

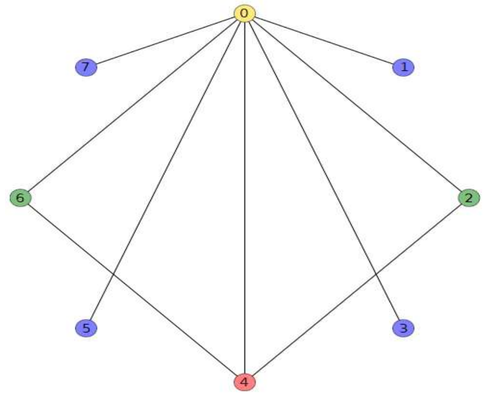

FIGURE 1. The zero divisor graph for $\mathbb{Z}_8$, $\Gamma(\mathbb{Z}_8)$

The vertices of $\Gamma(\mathbb{Z}_8)$ are the elements of $\mathbb{Z}_8$, and two vertices are adjacent if their product is zero.

By refering to Figure 1, the number of edges of $\Gamma(\mathbb{Z}_8)$ is nine.

By Definition 2, the general zeroth-order Randić index in the case $\alpha = 1$ can be calculated as follows:

$$\begin{aligned} R_1^0 &= \sum_{u \in V(\Gamma \mathbb{Z}_8)} deg(u) \\ &= deg\ (0) + deg\ (1) + deg\ (2) + deg\ (3) + deg\ (4) + deg\ (5) + deg\ (6) + deg\ (7) \\ &= 7 + 1 + 2 + 1 + 3 + 1 + 2 + 1 \\ &= 18. \end{aligned}$$

We can also calculate the general zeroth-order Randić index of $\Gamma(\mathbb{Z}_8)$ by using Theorem 2, as follows:

$$R_1^0 = 2^3 - 1 + 3(2^3 - 2^{3-1}) + \frac{2^3}{2-1}\left(\frac{2^{\left\lfloor \frac{3}{2} \right\rfloor} - 1}{2^{\left\lfloor \frac{2(3)+1}{2} \right\rfloor}} - \frac{2^{\left\lceil \frac{3-1}{2} \right\rceil} - 1}{2^{3-1}}\right) = 18.$$

CONCLUSION

In this paper, some results related to the general zeroth-order Randić index of zero divisor graph for the ring of integers modulo p^n are found. Then, the general formula of the general zeroth-order Randić index of the zero divisor graph for the ring of integers modulo p^n have been constructed for the cases $\alpha = 1, 2$ and 3. An example is given to illustrate the results.

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