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The Total Non-Zero Divisor Graph of Some Commutative Rings

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Abstract. In this study, we introduced a new concept of total non-zero divisor graph of a ring. The total non-zero divisor graph of a ring is defined as a simple undirected graph with its vertices are the non-zero elements of the ring and two distinct vertices are connected if and only if their product is not equal to zero, and their sum is in the its zero divisors sets. In this paper, the total non-zero divisor graph is constructed and the connectivity of the graph is explored. We prove that the total non-zero divisor graph is a null graph for the set of integers modulo p. The connectivity of the total non-zero divisor graph is also determined for the set of integers modulo n, where $n \neq p$.

INTRODUCTION

Over the past decades, the concept of graph theory associated with rings has received special attention especially in studying the graph and algebraic properties of the corresponding graph. This is mainly due to its applications in mathematics and other areas such as in the field of cryptography and computer science. Some of the types of graphs that have been explored extensively are Cayley graphs, bipartite graphs, conjugate and non-conjugate graphs associated with finite groups.

The concept of the zero-divisor graph of a ring R was first introduced by Beck [1] in 1988 where the author's main interest was on the coloring of the zero-divisor graph including clique, chromatic number and clique number. Suppose R is a commutative ring, the zero-divisor graph is a simple graph in which the vertices are the elements of R, such that two different elements x and y are adjacent if and only if $xy = 0$ [1]. A few decades later, the concepts of zero-divisor graph were further explored by using different types of groups, rings, ideals and fields. For example, see [2-7] for more related studies on the zero divisor graphs of finite commutative rings. In 2003, a simplified definition of the zero-divisor graph was introduced by Redmond [8]. Instead of having all elements of the ring as the vertices of the graph, Redmond only considered the non-trivial zero divisors as the vertices of the graph. The author believed that this definition illustrates better especially in studying the structure of zero-divisors of the ring. Based on the author's research, it is found that the diameter of the graph is less than or equal to three when the zero-divisor graph is a connected graph.

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Moreover, zero-divisor graph was also studied for non-commutative rings, modules and lattices. Akhbari and Mohammadian [9] explored the zero divisor graphs of finite non-commutative rings such as left Artinian rings and matrix rings. The zero-divisor graph under group actions in a non-commutative ring was investigated by Han [10]. Few years later, Ma *et al.* [11] investigated the automorphism group of the zero-divisor graph associated to matrix ring over a finite field.

In 2008, the total graph of a commutative ring is studied by Anderson and Badawi [12]. The vertices of this total graph are the elements of ring, R and for distinct $x, y \in R$, there will be an edge between vertices x and y if and only if $x + y \in Z(R)$ where $Z(R)$ is its set of zero divisors. The same authors then continued their research on the total graph by investigating the structure of generalized total graph of different types of rings such as quasilocal commutative ring with maximal ideal and integral domain [13]. Recently, Duric *et al.* [14] introduced the total zero divisor graph of a commutative ring with multiplicative identity element by combining the zero-divisor graph and total graph of a ring. In their research, the study reflects more structure of zero divisors of a ring since the graph engages both ring operations, addition and multiplication.

In this research, the main objective is to determine how the graph theoretical properties of $Z\Gamma(R)$ has an effect on the ring theoretical properties of R. All graphs that are being considered in this paper are simple, undirected graphs with no multiple edges or loops. Starting in Section 2, we presented all fundamental definitions that are related to the topics. In the next section, the total non-zero divisor graph is defined and constructed for the ring of integers modulo n. In general, for the ring $R = \mathbb{Z}_n$ where n is a prime number, we prove that the total non-zero divisor graph is a null graph.

PRELIMINARIES

In this section, we provide some fundamental definitions related to this research. First, the formal definition of the zero divisors of a finite ring is provided as follows.

Definition 1 [15] Zero Divisor
Zero divisors of a ring are the elements that have product zero when multiplied with each other. If p and q are two non-zero elements of a ring R, then p and q are known as the divisors of 0 when $pq = 0$,.

Graphs associated to groups and rings have been broadly studied by researchers and mathematicians since it is believed that studying the action of a ring on a graph is one of the comprehensive ways of analyzing the structure of the ring. An object with two sets, which are the edge set, $E(G)$ and the non-empty vertex set, $V(G)$ is called a graph. The $E(G)$ may be empty, but otherwise its elements are two-element subsets of the vertex set [16]. On the other hand, when the edge set of the graph is empty and each vertex of a graph is isolated, this graph is known as a null graph [17].

Motivated with the abundance research on the zero-divisor graph, the concepts of non-zero divisor graph were introduced in the early 2020. Kadem *et al.* [18] introduced the non-zero divisor graph and the authors proved that the graph is connected with diameter of almost two. In 2021, Zai *et al.* [19] introduced the non-zero divisor graph by considering all non-zero elements as the vertices of the graph and two distinct elements a and b are connected if and only if the product of the two elements is not equal to zero. This non-zero divisor graph is known as $\Gamma(R)$.

RESULTS AND DISCUSSIONS

In this section, the total non-zero divisor graph is defined and constructed for the ring of integers modulo n. We prove that the total non-zero divisor graph of ring of integers modulo p, where p is prime denoted by $Z\Gamma(\mathbb{Z}_p)$ is a null graph. It is also found that if $n \neq p$, then the total non-zero divisor graph is a connected graph.

The Constructions of The Total Non-Zero Divisor Graph of $\mathbb{Z}_n$

In this particular subsection, the total non-zero divisor graph is newly introduced and constructed. While there are quite considerable attentions given to the zero-divisor graph and its properties, the concepts on the non-zero divisor graphs are not widely explored yet. By combining the total graph and non-zero divisor graph, we established a new notion of graph, known as the total non-zero divisor graph. The formal definition of the graph is presented as follows.

Definition 2 Total Non-Zero Divisor Graph
A total non-zero divisor graph of R, $Z\Gamma(R)$, is a simple undirected graph whose vertex set is the set of all non-zero elements of R, and two distinct vertices x and y are adjacent if and only if $xy \neq 0$ and $x + y \in Z(R)$ where $Z(R)$ is the set of all zero divisors in R.

An example of a total non-zero divisor graph is given below. The type of ring that is used in this example is the commutative ring of integers modulo 10, $\mathbb{Z}_{10}$. The non-zero divisor graph of $\mathbb{Z}_{10}$ is constructed as well in Figure 1 to effectively compare these two graphs.

Example 1 Let $Z\Gamma(\mathbb{Z}_{10})$ be the total non-zero divisor graph of $\mathbb{Z}_{10}$, then $Z\Gamma(\mathbb{Z}_{10})$ is a simple undirected graph with nine vertices and 16 edges.

Solution The zero divisors needed to be found first so that we can find the elements where $x \cdot y \neq 0$. The zero divisors of $\mathbb{Z}_{10}$ are determined and the set of zero divisors of $\mathbb{Z}_{10}$ is {2,4,5,6,8} since $2 \cdot 5 = 5 \cdot 2, 4 \cdot 5 = 5 \cdot 4, 6 \cdot 5 = 5 \cdot 6$ and $8 \cdot 5 = 5 \cdot 8$. The total pairs of non-zero elements in $\mathbb{Z}_{10}$ are $9 \cdot 9 = 81$. Hence, the pairs of elements where the multiplication of these pairs of elements is not equal to zero is $81 - 8$ (pairs of elements from zero divisors) $= 73$. Now, we only need to check and determine the elements where $x + y \in Z(R)$. The edges $x \cdot x \neq 0$ are excluded from the graph to prevent the existence of loops. Figure 1 shows the non-zero divisor graph of $\mathbb{Z}_{10}$ and the total non-zero divisor graph is shown as in Figure 2.

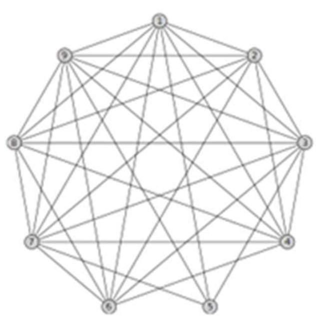

FIGURE 1. The non-zero divisor graph of $\mathbb{Z}_{10}$

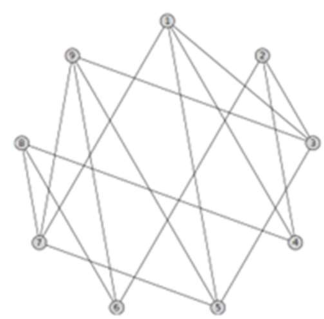

FIGURE 2. The total non-zero divisor graph of $\mathbb{Z}_{10}$

From Figure 2, it can be seen that this figure has lesser edges compared to the non-zero divisor graph in Figure 1. This is due to the additional condition from the total graph where the sum of two different elements in the ring must be in the set of zero divisors. The following theorem discusses on the connectivity of the total non-zero divisor graph of $\mathbb{Z}_p$.

Theorem 1 Let $Z\Gamma(\mathbb{Z}_p)$ be the total non-zero divisor graph of $\mathbb{Z}_p$ where p is a prime number, then $Z\Gamma(\mathbb{Z}_p)$ is a null graph.

Proof Since the ring $\mathbb{Z}_p$ has no zero-divisor, the set of zero divisor, denoted as $Z(\mathbb{Z}_p)$ is an empty set. Hence, the total non-zero divisor graph has no edge because the second condition of the graph, $x + y \in Z(R)$ is not satisfied. Therefore, the total non-zero divisor graph of $\mathbb{Z}_p$ is a null graph.

An example on the ring of integers modulo p is presented below to illustrate Theorem 1.

Example 2 The total non-zero divisor graphs of $\mathbb{Z}_7$ and $\mathbb{Z}_{23}$ are given as follows:

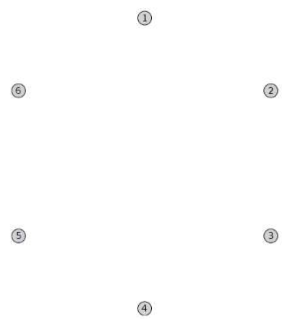

FIGURE 3. The total non-zero divisor graph of $\mathbb{Z}_7$

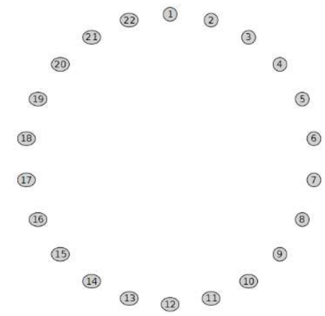

FIGURE 4. The total non-zero divisor graph of $\mathbb{Z}_{23}$

It is well known that 7 and 23 are prime numbers. Therefore, there is no zero divisors in the ring of integers modulo 7 and 23. Hence, there is no edge in the total non-zero divisor graph of both rings since the $Z(R) = \emptyset$. In the case where $n \neq p$, we found few examples when the total non-zero divisor graph is a connected graph. This is stated in the following conjecture.

Conjecture 1 The total non-zero divisor graph of $\mathbb{Z}_n$ where $n \neq p$ is a connected graph.

In the following, an example is presented to illustrate the conjecture.

Example 3 The total non-zero divisor graphs of $\mathbb{Z}_6$ and $\mathbb{Z}_{15}$ are constructed as follows:

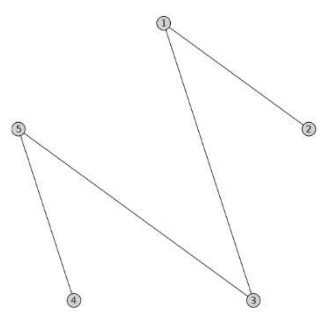

FIGURE 5. The total non-zero divisor graph of $\mathbb{Z}_6$

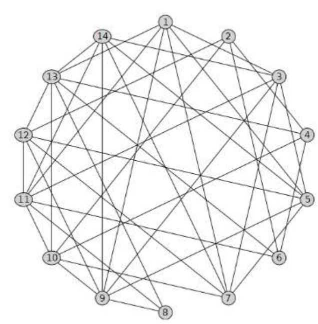

FIGURE 6. The total non-zero divisor graph of $\mathbb{Z}_{15}$

The above figures illustrate the total non-zero divisor graph of ring of integers modulo 6 and ring of integers modulo 15, respectively. We can see that the total non-zero divisor graph of $\mathbb{Z}_6$ and $\mathbb{Z}_{15}$ are connected graphs.

CONCLUSION

In conclusion, this paper studied different concepts of non-zero divisor graphs on different rings. By the idea of total graph and non-zero divisor graph, the total non-zero divisor graph is newly defined and constructed for the ring of integers modulo n. The result shows that if $n = p$ where p is a prime number, the total non-zero divisor graph of ring of integers modulo p is an empty graph. Therefore, this research opens up many ideas and research areas to determine the total non-zero divisor graphs associated to rings. The graph properties and other aspects of graphs can be determined in future research.

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