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PERFECT CODES IN GRAPH THEORY: A RING-THEORETIC PERSPECTIVE

(*Kod-kod Sempurna dalam Teori Graf: Perspektif Teori Gelanggang*)

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ABSTRACT

A graph is a mathematical structure that represents a network described between lines and points. Due to the graph's many fascinating features, its characteristics are widely studied by many researchers. One of the popular topics studied on graphs is perfect code. A set V of vertices in a graph G is called a perfect code if every vertex in G is either in V or is adjacent to exactly one vertex in V. Originally, perfect codes were used in coding theory. It is then extended to other fields including graph theory. In this paper, mathematical concepts of perfect codes of graphs are bridged using ring theory. The perfect codes are determined for the zero divisor graph of some finite rings of matrices with dimension two. First, the zero divisor graph of the finite rings of matrices is constructed where its vertices are all zero divisors of the ring and two distinct vertices are adjacent if their product is zero. Then, from the graph's vertex set, the vertices' neighborhood elements are determined to compute the graph's perfect codes.

Keywords: perfect codes; graph theory; ring theory

ABSTRAK

Graf merupakan sebuah struktur matematik yang mewakili rangkaian yang digambarkan menggunakan garisan dan titik. Oleh kerana graf mempunyai banyak sifat yang menarik, sifat-sifat tersebut dikaji secara meluas oleh ramai penyelidik. Salah satu topik popular yang dikaji ke atas graf ialah kod sempurna. Sebuah set bucu V dalam sebuah graf G digelar kod sempurna jika setiap bucu di dalam G adalah sama ada berada dalam set V atau bersebelahan secara tepatnya dengan satu bucu dalam V. Pada asalnya, kod sempurna digunakan dalam teori pengekodan. Kemudian, ianya telah diluaskan ke bidang-bidang yang lain, termasuk teori graf. Di dalam naskhah ini, konsep-konsep matematik bagi kod sempurna dalam graf dihubungkan menggunakan teori gelanggang. Kod sempurna tersebut ditentukan untuk graf pembahagi sifar bagi beberapa gelanggang matriks terbatas dengan dua dimensi. Pertama sekali, graf pembahagi sifar bagi gelanggang tersebut telah dibina di mana bucu-bucunya adalah semua pembahagi sifar bagi gelanggang tersebut dan dua bucu yang berbeza adalah bersebelahan jika hasil darab mereka adalah sifar. Kemudian, dari set bucu graf tersebut, unsur-unsur kejiranan bagi setiap bucu telah ditentukan bagi mengira kod-kod sempurna graf tersebut.

Kata kunci: kod sempurna; theory graf; teori gelanggang

1. Introduction

A graph is a mathematical object which consists of vertices and edges (Fraleigh 2003). Graphs are capable of modeling and addressing real-world problems by representing complex systems and relationships in a structured and analyzable manner. For example, Tola *et al.* (2024) uses line graphs of neutrosophic graph to represent and analyze uncertain or indeterminate information about edge relationships and complex networks in graphs.

The study of graphs has been widely applied in various branches of mathematics, including algebra, specifically in ring theory. In ring theory, graphs have been used mainly to visualize the properties of a finite ring. For example, the edges of a commuting graph of a ring depict that two adjacent vertices (which are ring elements) commute (Omidi & Vatandoost 2011). This interesting visualization of ring properties has increased the enthusiasm of many graph theorists to study graph properties and applications.

Graph properties and applications are also interesting subjects to investigate since they help in providing a better understanding of the algebraic properties of the ring and graph itself. One of the most commonly studied graph properties is the chromatic number of a graph. The chromatic number, which is the minimum coloring of graph vertices, shows the adjacency between two vertices of a graph (Erfanian & Tolue 2012). In addition, studies have also been done on the size and shape of graphs which provide even more insights into the algebraic structure of graphs.

On the other hand, several properties of rings, such as the annihilating property (Zaid *et al.* 2020) and the properties of ideals (Alsarahead & Ahmad 2017), are fundamental in understanding the structural behavior of rings. These properties have been instrumental in connecting algebraic concepts to graph theory, such as in constructing and analyzing zero divisor graphs, where the interactions of ring elements are visualized.

In the study of zero divisor graphs, previous research has primarily focused on properties such as connectivity, chromatic numbers, and domination numbers. For instance, Al-Janabi and Bacsó (2023) analyzed the chromatic properties of zero divisor graphs over commutative rings, while Akhtar and Lee (2016) investigated the structural aspects, specifically the vertex-connectivity and edge-connectivity of the zero divisor graph of some commutative rings. Besides that, Ali *et al.* (2024) studied the dominant metric dimension of the zero divisor graph of some finite rings, expressed in terms of the maximum degree, girth, clique number, and diameter of the graph.

Compared to these studies, our research extends the analysis by introducing perfect codes within zero divisor graphs. Unlike previous works that primarily explored graphical properties, our study applies coding theory principles to the algebraic structure of zero divisor graphs, providing a novel perspective. This approach is particularly useful in error correction and network design, where efficient data transmission relies on well-structured codes.

Furthermore, while past research has largely considered undirected zero divisor graphs, our study examines the directed case, adding a new dimension to the theoretical framework. The results obtained align with existing findings in terms of graph connectivity and structure but introduce new insights into how perfect codes can be derived within algebraic graphs.

This paper is divided into four sections, beginning with the introduction, followed by some preliminaries and literature reviews on the said topics. Then, the third section provides the results and discussions on the zero divisor graph and its perfect codes. Finally, the conclusion of this study is given in the last section.

2. Preliminaries

In this section, some definitions, propositions, and literature reviews on ring theory, graph theory as well as perfect codes are presented. First, the definition of a ring is provided below.

Definition 2.1. Ring (Fraleigh 2003). A ring R is defined as a nonempty set with two operations $+, \cdot \colon R \times R \to R$ with the properties, such that for all $a, b, c \in R$:

- $\langle R, +\rangle$ is an Abelian group;
- $\langle R, \cdot\rangle$ is associative: $a \cdot (b \cdot c) = (a \cdot b) \cdot c$;
- the distributive laws are valid: $(a + b) \cdot c = (a \cdot c) + (b \cdot c)$ and $a \cdot (b + c) = (a \cdot b) + (a \cdot c)$

Next, the definition of the zero divisors of a ring is given as follows.

Definition 2.2. Zero Divisors of a Finite Ring (Fraleigh 2003). Let a and b be the nonzero elements of a finite ring R. If $ab = 0$, then a and b are the zero divisors of R.

The annihilating property of the zero divisors of a ring has sparked the interest of many researchers to study this topic. One of the most significant studies on this subject is the zero divisor graph. Beck (1988) did research on a graph with the zero divisors of a ring as its vertices, and two vertices are adjacent when their product is zero. Although the main objective of the study by Beck (1988) was to explore the graph colorings, the study has successfully attracted the interest of many graph theorists to investigate this graph. Years later, Anderson and Livingston (1999) formally defined the graph as the following:

Definition 2.3. Zero Divisor Graph (Anderson & Livingston 1999). Let R be a commutative ring and $Z(R)$ be its set of zero divisors. Then, zero divisor graph is a simple graph where $Z(R)$ are the graph vertices, and two distinct vertices are adjacent if and only if their product is zero.

Subsequently, many studies have been done on the zero divisor graph. While all of the studies were done on commutative rings, Redmond (2002) was the first person who extended the graph to noncommutative rings, and concluded that the graph must be a directed graph. The definition of the graph is given as follows.

Definition 2.4. Zero Divisor Graph of Noncommutative Rings (Redmond 2002). Let R be a noncommutative ring. Then, its zero divisor graph, $\Gamma(R)$ is a directed graph where the zero divisors $Z(R)$ are its vertices, and an arc $a \to b$ is formed between two distinct vertices a and b if and only if $ab = 0$.

Afterward, studies were done on zero divisor graphs of noncommutative rings, including the study by Wu (2005), who focused on the Artinian rings, Han (2008), who studied the group actions of some ring of matrices, and Zaid *et al.* (2021) who investigated on the annihilating property of some ring of matrices.

In addition, various extensions of the zero divisor graph have been introduced by many researchers, on both commutative and noncommutative rings. Some examples of the extensions are the weakly zero divisor graph (Nikmehr *et al.* 2021), ideal-based zero divisor graph (Redmond 2003), extended zero divisor graph (Bennis *et al.* 2016), compressed zero divisor graph (Spiroff & Wickham 2011) and commuting zero divisor graph (Zaid *et al.* 2020).

Meanwhile, graph properties and applications have also been the topic that gained the attention of various graph theorists. Some commonly studied graph properties include the chromatic number, clique number, planarity, and bipartiteness of finite graphs. In this study, the graph application in focus is the perfect code of the zero divisor graph.

Perfect codes originated from the coding theory, where a study was initiated to correct errors in noisy communication channels by determining its error-correcting codes (Hamming 1950). The study of perfect codes eventually evolved in graph theory, starting from the research done by Biggs (1973). Some researches done on perfect codes of graph includes the study done by Ma (2020), where the author investigated the perfect codes in proper reduced power graphs of finite groups. Besides that, Banerjee (2023) studied the perfect codes on the commuting graphs of groups, Zai *et al.* (2022) studied the total perfect codes of the non-zero divisor graph of the ring of integers, and Mudaber *et al.* (2024) studied the perfect codes in unity product graphs of some commutative rings.

One of the ways of computing the perfect codes of a graph is by calculating the distance between two vertices of the graph and determining the vertices' neighborhood elements. The definitions of distance and neighborhood of a graph are given as follows.

Definition 2.5. Distance between Two Vertices of a Graph (Rosen 2011). The distance between two distinct vertices a and b of a connected simple graph is the number of edges of the shortest path between a and b.

Definition 2.6. Neighborhood of a Graph Vertex (Rosen 2011). The neighborhood of a vertex v in a graph G is the set of all vertices adjacent to v.

Then, by using the distance and neighborhood of a graph vertex, Van Lint (1975) proposed a notion called the e-error-correcting code. First, the definition of the code of a graph is given below.

Definition 2.7. Code (Van Lint 1975). A subset C of the vertex set of a graph, $V(\Gamma)$ is called a code.

Definition 2.8. e-Error-Correcting Code of a Graph (Van Lint 1975). If $S_e(a)$ is the set of neighborhood elements of a with distance less than or equal to e, then a code C is an e-error-correcting code if for all a and b in C, $S_e(a) \cap S_e(b) = \emptyset$ for distinct vertices a and b of a graph G.

Ultimately, the notion of the e-perfect codes of a graph is then introduced with the following definition.

Definition 2.9. e-Perfect Codes of a Graph (Van Lint 1975). If for any e-error-correcting code C, the union $\bigcup\limits_{a \in C} S_e(a) = V$, then the code is called an e-perfect code.

3. The e-Perfect Codes of the Zero Divisor Graph

In this section, our main results, which are the e-error-correcting codes and e-perfect codes for the zero divisor graph of a finite ring of matrices, namely the ring of 2×2 upper triangular matrices over integers modulo three, are provided. First of all, the zero divisor graph of the ring is provided in the following theorem.

Theorem 3.1. *Let* $R = \left\{ \begin{bmatrix} x & y \\ 0 & z \end{bmatrix} \middle| x, y, z \in \mathbb{Z}_3 \right\}$ *be a finite ring. Then, its zero divisor graph,* $\Gamma(R)$ *is a directed graph with 14 vertices and 74 arcs.*

Proof. Suppose $R = \left\{ \begin{bmatrix} x & y \\ 0 & z \end{bmatrix} \middle| x, y, z \in \mathbb{Z}_3 \right\}$. Based on Definition 2.2, the zero divisors of R are determined and written in the following set.

$$Z(R) = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 2 \end{pmatrix}, \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 2 \\ 0 & 1 \end{pmatrix}, \right.$$
$$\left. \begin{pmatrix} 0 & 2 \\ 0 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 2 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 2 & 2 \\ 0 & 0 \end{pmatrix} \right\}.$$

Therefore, the number of zero divisors of R, $|Z(R)| = 14$. Based on Definition 2.4, the zero divisor graph of R, $\Gamma(R)$ has 14 vertices. The zero divisors in the set $Z(R)$ are then labeled as $1, 2, 3, \ldots, 14$ according to their sequence. Then, also based on Definition 2.4, two vertices are adjacent if the product of the matrices is the zero matrix. Hence, by manual calculations, it is found that $\Gamma(R)$ has 74 arcs. Figure 1 illustrates the zero divisor graph of R.

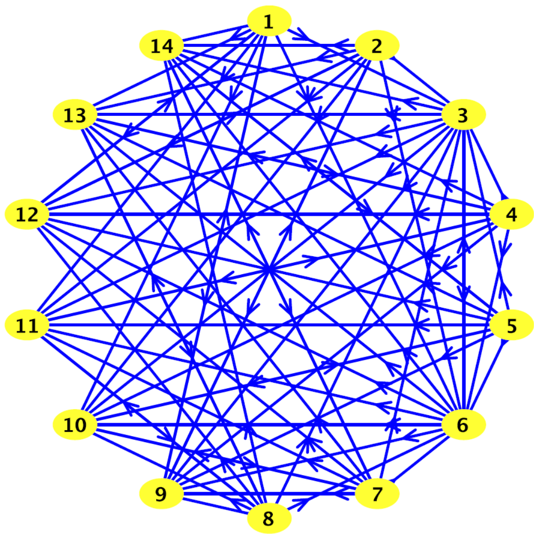

Figure 1: The zero divisor graph of R, $\Gamma(R)$

The zero divisor graph in Figure 1 can also be represented by the following adjacency matrix, where a value of 1 in position (i, j) indicates a directed arc from vertex i to vertex j, while 0 indicates no arc from vertex i to vertex j.

$$\Gamma(R) = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}. \quad \square$$

Based on the graph in Figure 1, the e-perfect codes, specifically for $e = 1$, $e = 2$ and $e = 3$ are determined for $\Gamma(R)$ in the following theorems.

Theorem 3.2. *Let $\Gamma(R)$ be the zero divisor graph of the ring of 2×2 upper triangular matrices over integers modulo three. Then, the zero divisor graph $\Gamma(R)$ has no 1-perfect code.*

Proof. Suppose $\Gamma(R)$ is as shown in Figure 1. To determine the 1-error-correcting codes, the neighborhood set of all vertices $a \in \Gamma(R)$ with distance less than and equal to one are found and listed as follows.

(1) $S_1(1) = \{1, 3, 6, 9, 10, 11, 12, 13, 14\}$
(2) $S_1(2) = \{2, 3, 6, 9, 10, 11, 12, 13, 14\}$
(3) $S_1(3) = \{3, 6, 9, 10, 11, 12, 13, 14\} = S_1(6)$
(4) $S_1(4) = \{3, 4, 6, 9, 10, 11, 12, 13, 14\}$
(5) $S_1(5) = \{3, 5, 6, 9, 10, 11, 12, 13, 14\}$

(6) $S_1(7) = \{3, 6, 7, 9, 10, 11, 12, 13, 14\}$
(7) $S_1(8) = \{3, 6, 8, 9, 10, 11, 12, 13, 14\}$
(8) $S_1(9) = \{1, 2, 9\}$
(9) $S_1(10) = \{5, 7, 10\}$
(10) $S_1(11) = \{4, 8, 11\}$
(11) $S_1(12) = \{1, 2, 12\}$
(12) $S_1(13) = \{4, 8, 13\}$
(13) $S_1(14) = \{5, 7, 14\}$

Based on the neighborhood sets with distance ≤ 1, it is found that there is no possible sets where $S_1(x) \cap S_1(y) = \emptyset$. Therefore, the zero divisor graph $\Gamma(R)$ has no 1-error-correcting code. Eventually, $\Gamma(R)$ has no 1-perfect code. □

Next, the 2-perfect codes are found for $\Gamma(R)$ in the following theorem.

Theorem 3.3. *Let $\Gamma(R)$ be the zero divisor graph of the ring of 2×2 upper triangular matrices over integers modulo three. Then, the number of 2-perfect codes of $\Gamma(R)$ is eight.*

Proof. Suppose $\Gamma(R)$ is the zero divisor graph illustrated in Figure 1. The neighborhood set of all vertices $a \in \Gamma(R)$ with distance less than and equal to two are found and listed as follows.

(1) $S_2(1) = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14\} = S_2(2) = S_2(3) = S_2(4) = S_2(5) = S_2(6) = S_2(7) = S_2(8)$
(2) $S_2(9) = \{1, 2, 3, 6, 9, 10, 11, 12, 13, 14\} = S_2(12)$
(3) $S_2(10) = \{3, 5, 6, 7, 9, 10, 11, 12, 13, 14\} = S_2(14)$
(4) $S_2(11) = \{3, 4, 6, 8, 9, 10, 11, 12, 13, 14\} = S_2(13)$

It is found that $S_2(a)$ for $a = \{1, 2, 3, 4, 5, 6, 7, 8\}$ is equal to $V[\Gamma(R)]$ itself. This shows that the maximum distance of the vertices $a = \{1, 2, 3, 4, 5, 6, 7, 8\} \in V[\Gamma(R)]$ is two. Meanwhile, there are no other possible combinations where $S_2(x) \cap S_2(y) = \emptyset$. Therefore, the 2-perfect codes of $\Gamma(R)$ are the singletons $\{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}, \{7\}$ and $\{8\}$. Hence, the number of 2-perfect codes of $\Gamma(R)$ is eight. □

Next, the following theorem presents the 3-perfect codes of $\Gamma(R)$.

Theorem 3.4. *Let $\Gamma(R)$ be the zero divisor graph of the ring of 2×2 upper triangular matrices over integers modulo three. Then, the number of 3-perfect codes of $\Gamma(R)$ is six.*

Proof. Suppose $\Gamma(R)$ is the zero divisor graph illustrated in Figure 1. It is found that $S_3(a)$ for $a = \{9, 10, 11, 12, 13, 14\}$ is the set of vertices of $\Gamma(R)$, $V[\Gamma(R)]$. This shows that the maximum distance of the vertices $a = \{9, 10, 11, 12, 13, 14\}$ is three. Therefore, the 3-perfect codes of $\Gamma(R)$ are the singletons $\{9\}, \{10\}, \{11\}, \{12\}, \{13\}$ and $\{14\}$. Hence, the number of 3-perfect codes of $\Gamma(R)$ is six. □

To validate the manual calculations, a script has been created in Groups, Algorithms, and Programming (GAP) software that defines the ring of 2×2 upper triangular matrices over $\mathbb{Z}_3$, computes the zero divisors of the ring, constructs the adjacency matrix for the zero divisor graph and verifies the e-perfect codes of the zero divisor graph of the ring. The GAP script is provided as follows.

```
# Define the Ring of Upper Triangular 2x2 Matrices over Z3
Z3 := Integers mod 3;
R := List([0..2], x -> List([0..2], y -> List([0..2], z -> [[x,y],[0,z]])));

# Convert nested lists into actual matrices
Matrices := List(R, x -> List(x, y -> ImmutableMatrix(Z3, y)));

# Flatten list to get all matrices
FlatMatrices := List(Concatenation(Matrices), x -> x);

# Define Zero Element
ZeroMat := ImmutableMatrix(Z3, [[0,0],[0,0]]);

# Compute the set of Zero Divisors
IsZeroDivisor := function(A)
    local B;
    for B in FlatMatrices do
        if A * B = ZeroMat then
            return true;
        fi;
    od;
    return false;
end;

ZeroDivisors := Filtered(FlatMatrices, IsZeroDivisor);

# Construct the adjacency matrix
AdjMatrix := List(ZeroDivisors, A -> List(ZeroDivisors, B ->
    IfThenElse(A * B = ZeroMat, 1, 0)));

# Display adjacency matrix
Print("Adjacency Matrix: ", AdjMatrix, "\n");

# Compute and verify e-perfect codes
ComputePerfectCodes := function(matrix, e)
    local i, j, N, perfectCodes;
    N := Length(matrix);
    perfectCodes := [];

    # Generate all possible code subsets
    for i in [1..N] do
        if Sum(List([1..N], j -> matrix[i][j])) = e then
            Add(perfectCodes, i);
        fi;
    od;

    Print("Perfect Codes for e=", e, ": ", perfectCodes, "\n");
end;

# Verify for e = 1, 2, 3
ComputePerfectCodes(AdjMatrix, 1);
ComputePerfectCodes(AdjMatrix, 2);
ComputePerfectCodes(AdjMatrix, 3);
```

Listing 1: GAP script for e-perfect code verification

4. Conclusion

In this paper, the zero divisor graph of the ring of 2×2 upper triangular matrices over integers modulo three, $\Gamma(R)$ are constructed. It is found that the graph is a directed graph with 14 vertices and 74 arcs. Besides that, the e-perfect codes where $e = 1, 2, 3$ are determined for $\Gamma(R)$. It is found that the graph has no 1-perfect code. Meanwhile, the 2-perfect codes of the graph are the singleton sets $\{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}, \{7\}$ and $\{8\}$, resulting that the number of 2-perfect codes of the graph is eight. In addition, the 3-perfect codes of the graph are the singletons sets $\{9\}, \{10\}, \{11\}, \{12\}, \{13\}$ and $\{14\}$, proving that the graph has six 3-perfect codes.

The findings of this study contribute to a broader understanding of the interplay between graph theory and ring theory. The identification of perfect codes in the zero divisor graph of a ring provides insight into error-correcting codes, a fundamental concept in information theory. Since perfect codes help minimize redundancy while maintaining error correction capabilities, the results can be extended to applications in network communication, data compression, and coding theory.

Additionally, the structural properties of the zero divisor graph may aid in cryptographic applications where algebraic structures are used for secure communication. Understanding the connection between perfect codes and algebraic structures allows the development of more efficient encoding schemes. Future research could explore similar constructions in different types of rings for broader applicability.

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