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Research article

On properties and topological indices of the zero divisor graph for direct product of some commutative rings

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Abstract: Graph-theoretic representations of algebraic structures provide useful tools for studying algebraic properties and degree-distance-based invariants that arise, for example, in mathematical chemistry. Motivated by these applications and by recent work on zero divisor graphs over modular rings, this paper investigated the zero divisor graph $\Gamma(R)$ of a commutative ring R in the sense of Anderson and Livingston, where vertices are the zero divisors and two distinct vertices are adjacent when their product is zero. We focused on rings of the form $\mathbb{Z}_{p^m} \times \mathbb{Z}_{q^n}$, with p, q primes and $m, n \in \mathbb{N}$. This research employed a literature review combined with a deductive method, drawing specific conclusions from well-established algebraic and graph-theoretic principles. By explicitly describing the zero divisors of R and partitioning the vertex set according to greatest common divisor conditions, we determined several structural properties of $\Gamma(\mathbb{Z}_{p^m} \times \mathbb{Z}_{q^n})$. Furthermore, we obtained closed formulas for the general zeroth-order Randić index, the eccentric connectivity index, and the Schultz index of this graph. These results extended previous studies on zero divisor graphs over integers modulo rings and yield explicit families of graphs with exactly computable topological indices.

Keywords: zero divisor graph; integers modulo ring; graph properties; topological indices
Mathematics Subject Classification: 05C07, 05C09, 05C25, 13A70

1. Introduction

Graph theory has broad applications across various disciplines, including electrical engineering, chemistry, and computer science. In chemistry, graph theory is utilized to address molecular problems, where molecular structures are represented as graphs, with atoms as vertices and bonds between atoms as edges. Numerous applications of graph theory and group theory in chemistry have been identified, one of which is the use of topological indices that assign numerical values to represent chemical structures. These topological indices are instrumental in predicting the chemical and physical properties of molecular structures.

One of the earliest indices introduced is the Wiener index [1], which is defined as the sum of distances between all pairs of carbon atoms in a molecular graph. This index was subsequently used to estimate the boiling points of alkane compounds. Later, Hosoya [2] introduced a new index defined as the number of matchings in a graph, and found that the boiling points of several saturated hydrocarbon compounds show a strong correlation with this index. In 1975, Randić [3] introduced the Randić connectivity index, which is based on vertex degrees. Subsequently, Schultz et al. [4] introduced another index based on the adjacency matrix, valence matrix, and distance matrix of alkane molecular graphs. They also computed the values of Schultz index for several alkanes and compared them with their boiling points as well as with the Randić connectivity index. The correlation between the Schultz index and the boiling points of these compounds yielded a coefficient of determination of 0.887. Meanwhile, the Randić connectivity index produced a higher coefficient of determination, namely 0.976. Later, Li and Zheng [5] generalized this index into the general zeroth-order Randić index. Another index is the eccentric connectivity index, which is defined based on the eccentricity and degree of vertices in a graph [6]. They investigated the relationship between the eccentric connectivity index, the Wiener index, and several physical properties of various compounds. The results showed correlation coefficients ranging from 95% to 99% for different datasets related to diverse physical properties. These correlation values are higher than those obtained using the Wiener index, which only range around 92% to 97% [6].

In addition to the aforementioned applications of graphs, graphs are also employed to represent various mathematical systems such as groups, rings, and modules. Examples of graph that can represent a group include coprime graphs, non-coprime graphs, commuting graphs, non-commuting graphs, intersection graphs, and power graphs. For representing rings, relevant graphs include prime ideal graphs, prime graphs, and Jacobson graphs. In each of these graph types, the set of vertices and edges is determined by the specific definitions associated with the graph.

Motivated by this, the authors seek to investigate the properties of graphs that represent commutative rings, with particular emphasis on zero divisors. The graph being developed is known as the zero divisor graph, where the vertices correspond to all elements of a commutative ring, and two distinct vertices are adjacent if their product is the zero element of the ring.

The zero divisor graph was initially introduced by Beck in 1988 [7] in the context of ring theory. Beck defined this graph for a commutative ring with identity, where the vertices represent the non-zero elements of the ring, and an edge connects two distinct vertices if their product is zero element of the ring. This approach was initially designed to investigate the algebraic properties of rings through graph concepts. Over the subsequent decades, the notion of a zero-divisor graph has been refined and extended in many directions. In 1999, Anderson and Livingston modified Beck's construction by restricting the vertex set to the nonzero zero divisors and introduced the now standard zero-divisor graph $\Gamma(R)$, proving that it is always connected with a diameter at most three and small girth [8]. In the early 2000s, DeMeyer and Schneider and Mulay studied automorphisms, cycles, and girth of zero-divisor graphs, while Redmond extended the concept both to non-commutative rings and to ideal-based zero-divisor graphs $\Gamma_I(R)$, relating their structure to that of $\Gamma(R/I)$ [9–12]. Axtell and co-authors then considered polynomial and power-series rings and direct products of commutative rings, determining the diameter and girth of the associated zero-divisor graphs [13, 14]. For the specific case $R = \mathbb{Z}_n$ and related rings, Phillips, Cordova et al. and Aponte et al. obtained explicit formulas for the number of vertices and edges, vertex degrees, connectivity, planarity, and Eulerian and Hamiltonian properties of $\Gamma(\mathbb{Z}n)$ and its line graph [15–17]. More recently, Birch et al., Koam et al. and Akg"uneş and Nacaroğlu studied zero-divisor graphs of finite direct product rings such as $\Gamma(\mathbb{Z}_n)$, $\Gamma(\mathbb{Z}_{p_1 p_2}) \times \Gamma(\mathbb{Z}_{q^2})$, and $\Gamma(\mathbb{Z}_p \times \mathbb{Z}_q \times \mathbb{Z}_r)$, focusing on edge counts, eccentricity-based indices, and other distance parameters [18–20]. A comprehensive overview of these developments and further results on spectral and topological indices, including the Wiener index and energy, is provided in the survey of Singh and Bhat [21].

In 2024, Maulana et al. [22] explored the zero divisor graph based on Beck's definition and obtained six topological indices over integers modulo ring prime powers and their direct product. A year earlier, Ismail et al. also studied the zero divisor graph from integers modulo ring p^n, but using the definition from Anderson and Livingston and two different studies were produced, the first one focused on the first Zagreb index [23] and another focused on the general zeroth-order Randić index [24]. In the following year, the research focused on the same topic but involved some commutative rings [25, 26].

This research investigates several properties of zero divisor graphs (Anderson and Livingston's definition) of commutative rings, such as vertex degree, diameter (which represents distances within the graph), and topological indices. The topological indices to be examined are closely related to vertex degree and distance, including the general zeroth-order Randić index, the eccentric connectivity index, and the Schultz index.

2. Preliminaries and known results

Before going to the main results of this study, several necessary definitions are provided.

Definition 2.1. [7] The zero divisor graph of a commutative ring R, denoted by $\Gamma_0(R)$, is a simple graph with vertex set is R with two distinct vertices x and y are joined by an edge if

$$xy = 0_R.$$

Definition 2.2. [27] Consider R as a ring, 0_R is the zero element of R, then $x \in R \setminus \{0_R\}$ is a zero divisor if there exists $y \in R \setminus \{0_R\}$ such that

$$xy = 0_R.$$

Anderson and Livingston provided a new definition for the zero divisor graph by limiting its vertex set only to the set of zero divisors of the ring R. The formal definition is given as follows.

Definition 2.3. [8] The zero divisor graph of a commutative ring R, denoted as $\Gamma(R)$, is a simple graph where the vertex set is the collection the zero divisors of R and two distinct vertices are adjacent if their product is 0_R.

The zero divisor graph discussed in this study is the zero divisor graph according to Definition 2.3.

Definition 2.4. [28] In a graph G, the number of its vertices is called the order of G, denoted by $|G|$, while the number of edges is denoted by $||G||$.

Definition 2.5. [28] Let

$$G = (V, E)$$

be a graph. The number of $u \in V$ that is adjacent to $v \in V$ is the degree of v, denoted by $\deg(v)$. Furthermore, $N(v)$ is the collection of neighbors of v.

If we sum up all the vertex degrees in G, we count every edge exactly twice: once from each of its ends. Thus,

$$|E| = \frac{1}{2} \sum \deg(v).$$

This property is commonly known as the handshaking lemma.

Definition 2.6. [28] Let

$$G = (V, E)$$

be a graph. The distance $d(u, v)$ in G between two vertices u and v is the length of the shortest path in G; if no such path exists, set

$$d(u, v) := \infty.$$

The greatest distance between any two vertices in G is the diameter of G, denoted by $\mathrm{diam}(G)$.

Definition 2.7. [8] Consider

$$G = (V, E)$$

as a connected graph. The eccentricity of $v \in V$, denoted as $e(v)$, is the distance between v and a vertex farthest from v in G.

Definition 2.8. [28] Let

$$G = (V, E)$$

be a graph, $X \subseteq V$ is a separator for G if $G - X$ is a disconnected graph.

Definition 2.9. [28] Graph

$$G = (V, E)$$

is called k-connected (for some $k \in \mathbb{N}$) if $|G| > k$ and $G - X$ is a connected graph for every $X \subseteq V$ with $|X| \leq k$.The greatest integer k such that G is a k-connected graph is the connectivity $\kappa(G)$ of G.

Definition 2.10. [5] Consider

$$G = (V, E)$$

be a graph. For any $\alpha \in \mathbb{R}$, the general zeroth-order Randić index of G, denoted by $R^0_\alpha(G)$, is defined as

$$R^0_\alpha(G) = \sum_{v \in V} (\deg(v))^\alpha.$$

Definition 2.11. [4] The Schultz index of graph

$$G = (V, E)$$

is defined as

$$Sc(G) = \frac{1}{2} \sum_{u,v \in V, u \neq v} (\deg(u) + \deg(v)) d(u, v).$$

Definition 2.12. [6] The eccentric connectivity index of graph

$$G = (V, E),$$

denoted by $\xi^C(G)$, can be defined as the sum of the product of eccentricity and degree of each vertex in G.

Theorem 2.13. *[29] If*

$$n = \prod p^k$$

for some prime number p and $k \in \mathbb{N}$, then

$$\varphi(n) = n \prod_{p|n} \left(1 - \frac{1}{p}\right),$$

where $\varphi(n)$ is the number of positive integers up to n that are relatively prime to n.

The following are a few more theorems regarding to zero divisor graphs of some commutative rings that are essential for this research.

Theorem 2.14. *[30] Let R be a commutative ring and then $diam(\Gamma(R)) \leq 3$.*

Theorem 2.15. *[22] If p, q primes, m, n natural numbers,*

$$(a, b) \in \mathbb{Z}_{p^m} \times \mathbb{Z}_{q^n}, \quad \gcd(a, p^m) = p^i, \quad \textit{and} \quad \gcd(b, q^n) = q^j,$$

then $\deg((a, b))$ *in* $\Gamma_0(\mathbb{Z}_{p^m} \times \mathbb{Z}_{q^n})$ *is*

$$\deg(a, b) = \begin{cases} p^i q^j - 1, & i > \lfloor \frac{m-1}{2} \rfloor,\ j > \lfloor \frac{n-1}{2} \rfloor, \\ p^i q^j, & \textit{otherwise}. \end{cases}$$

Theorem 2.16. *[22] If p, q primes, m, n natural numbers then*

$$\left\|\Gamma_0(\mathbb{Z}_{p^m} \times \mathbb{Z}_{q^n})\right\| = \frac{1}{2}\Big(mn(p^m - p^{m-1})(q^n - q^{n-1}) + m(p^m - p^{m-1})q^n + n(q^n - q^{n-1})p^m + p^m q^n - p^{\lceil \frac{m-1}{2} \rceil} q^{\lceil \frac{n-1}{2} \rceil}\Big).$$

3. Results and discussion

This research specifically discusses the zero-divisor graph of the commutative ring $\mathbb{Z}_{p^m} \times \mathbb{Z}_{q^n}$ with p, q primes (either $p = q$ or $p \neq q$) and m, n natural numbers. The first part of this section examines the properties of this graph, such as its order, size, degree, and other characteristics. Furthermore, the properties obtained in the first part are used to determine its topological indices in the second part.

3.1. Properties of $\Gamma(\mathbb{Z}_{p^m} \times \mathbb{Z}_{q^n})$

Let $\Gamma(\mathbb{Z}_{2^3} \times \mathbb{Z}_{3^2})$ be the zero divisor graph of $\mathbb{Z}_{2^3} \times \mathbb{Z}_{3^2}$. By Definition 2.1,

$$V(\Gamma(\mathbb{Z}_{2^3} \times \mathbb{Z}_{3^2})) = Z^*(\mathbb{Z}_{2^3} \times \mathbb{Z}_{3^2})) = \mathbb{Z}_{2^3} \times \mathbb{Z}_{3^2} \setminus A$$

with

$$A = \{(0,0),(1,1),(1,2),(1,4),(1,5),(1,7),(1,8),(3,1),(3,2),(3,4),(3,5),(3,7),(3,8),(5,1),$$
$$(5,2),(5,4),(5,5),(5,7),(5,8),(7,1),(7,2),(7,4),(7,5),(7,7),(7,8)\}.$$

Hence,

$$|\Gamma(\mathbb{Z}_{2^3} \times \mathbb{Z}_{3^2})| = |\mathbb{Z}_{2^3} \times \mathbb{Z}_{3^2}| - |A| = 72 - 25 = 47.$$

Generalizing this result to any primes p and q and natural numbers m and n yields the following theorem.

Theorem 3.1. *If p, q are primes and $m, n \in \mathbb{N}$, then,*

$$|\Gamma(\mathbb{Z}_{p^m} \times \mathbb{Z}_{q^n})| = p^{m-1}q^{n-1}(p + q - 1) - 1.$$

Proof. The vertex set of the graph $\Gamma(\mathbb{Z}_{p^m} \times \mathbb{Z}_{q^n})$ is given by

$$V\Big(\Gamma(\mathbb{Z}_{p^m} \times \mathbb{Z}_{q^n})\Big) = \Big\{(a,b) \in \mathbb{Z}_{p^m} \times \mathbb{Z}_{q^n} \;\Big|\; \gcd(a, p^m) \neq 1 \text{ or } \gcd(b, q^n) \neq 1\Big\}.$$

Define the sets

$$A = \{\, a \in \mathbb{Z}_{p^m} \mid \gcd(a, p^m) = 1 \,\} \quad \text{and} \quad B = \{\, b \in \mathbb{Z}_{q^n} \mid \gcd(b, q^n) = 1 \,\}.$$

By Theorem 2.13,

$$|A| = \varphi(p^m) = p^m\left(1 - \frac{1}{p}\right) = p^{m-1}(p - 1)$$

and

$$|B| = \varphi(q^n) = q^n\left(1 - \frac{1}{q}\right) = q^{n-1}(q - 1).$$

We now compute the order of the graph. Since

$$|\mathbb{Z}_{p^m} \times \mathbb{Z}_{q^n}| = p^m q^n,$$

and the vertices excluded from $\Gamma(\mathbb{Z}_{p^m} \times \mathbb{Z}_{q^n})$ are precisely the identity element together with all pairs (a, b) such that $a \in A$ and $b \in B$, we obtain

$$\begin{aligned}\Big|\Gamma(\mathbb{Z}_{p^m} \times \mathbb{Z}_{q^n})\Big| &= p^m q^n - 1 - |A||B| \\ &= p^m q^n - 1 - p^{m-1}(p-1)q^{n-1}(q-1) \\ &= p^{m-1}q^{n-1}(p + q - 1) - 1.\end{aligned}$$

This completes the proof. □

The subsequent analysis concerns the degrees of vertices and the size of $\Gamma(\mathbb{Z}_{p^m} \times \mathbb{Z}_{q^n})$.Because $\Gamma(\mathbb{Z}_{p^m} \times \mathbb{Z}_{q^n})$ is an induced subgraph of $\Gamma_0(\mathbb{Z}_{p^m} \times \mathbb{Z}_{q^n})$, both the degrees of its vertices and its size may be determined by applying Theorems 2.15 and 2.16.

For instance, we present the graphs $\Gamma_0(\mathbb{Z}_2 \times \mathbb{Z}_3)$ and $\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_3)$ (see Figures 1 and 2).

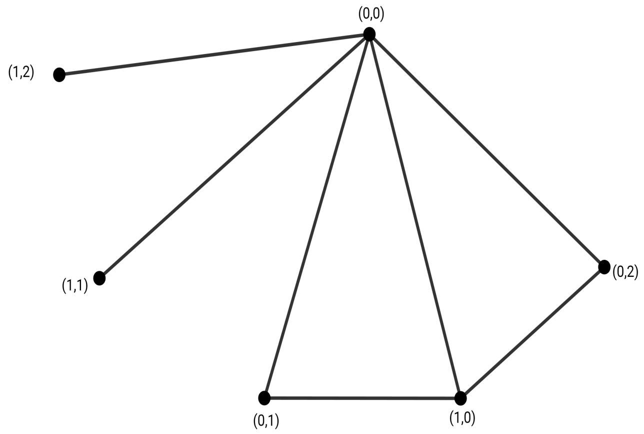

Figure 1. $\Gamma_0(\mathbb{Z}_2 \times \mathbb{Z}_3)$.

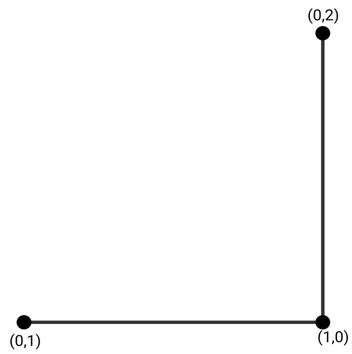

Figure 2. $\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_3)$.

It can be observed that each vertex in $\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_3)$ has a degree that is one less than its corresponding degree in $\Gamma_0(\mathbb{Z}_2 \times \mathbb{Z}_3)$.

The corresponding results are stated below.

Theorem 3.2. *Let p, q be primes, m, n be natural numbers, and $(a, b) \in V(\Gamma(\mathbb{Z}_{p^m} \times \mathbb{Z}_{q^n}))$. If*

$$\gcd(a, p^m) = p^i$$

and

$$\gcd(b, q^n) = q^j,$$

then

$$\deg(a, b) = \begin{cases} p^i q^j - 2, & i > \lfloor \frac{m-1}{2} \rfloor, j > \lfloor \frac{n-1}{2} \rfloor, \\ p^i q^j - 1, & \textit{otherwise.} \end{cases}$$

Proof. Partition the vertex set $V(\Gamma_0(\mathbb{Z}_{p^m} \times \mathbb{Z}_{q^n}))$ into the subsets

$$A = \{(0,0)\}, \quad B = Z^*(\mathbb{Z}_{p^m} \times \mathbb{Z}_{q^n}) = V(\Gamma(\mathbb{Z}p^m \times \mathbb{Z}q^n)) \quad \text{and} \quad C.$$

For any $(a_1, a_2) \in A$, $(b_1, b_2) \in B$, and $(c_1, c_2) \in C$, we have

$$(a_1, a_2).(b_1, b_2) = (0,0) \quad \text{and} \quad (b_1, b_2).(c_1, c_2) \neq (0,0).$$

Consequently, the degree of each vertex in $\Gamma(\mathbb{Z}_{p^m} \times \mathbb{Z}_{q^n})$ is one less than its degree in $\Gamma_0(\mathbb{Z}_{p^m} \times \mathbb{Z}_{q^n})$. Applying Theorem 2.15 completes the proof. □

Theorem 3.3. *If p, q are primes, $m, n \in \mathbb{N}$ then*

$$\left\|\Gamma(\mathbb{Z}_{p^m} \times \mathbb{Z}_{q^n})\right\| = \frac{1}{2}\Big(mn(p^m - p^{m-1})(q^n - q^{n-1}) + m(p^m - p^{m-1})q^n + n(q^n - q^{n-1})p^m + p^m q^n - p^{\lceil \frac{m-1}{2} \rceil} q^{\lceil \frac{n-1}{2} \rceil}\Big) - p^m q^n + 1.$$

Proof. Let p, q be primes and $m, n \in \mathbb{N}$. Since $\Gamma(\mathbb{Z}_{p^m} \times \mathbb{Z}_{q^n})$ is an induced subgraph of $\Gamma_0(\mathbb{Z}_{p^m} \times \mathbb{Z}_{q^n})$, $\Gamma(\mathbb{Z}_{p^m} \times \mathbb{Z}_{q^n})$ can be obtained by removing all vertices of $\Gamma_0(\mathbb{Z}_{p^m} \times \mathbb{Z}_{q^n})$ that are not a zero divisor. Let (a, b) be any non-zero divisor of $\mathbb{Z}_{p^m} \times \mathbb{Z}_{q^n}$. In $\Gamma_0(\mathbb{Z}_{p^m} \times \mathbb{Z}_{q^n})$, (a, b) is only adjacent to $(0, 0)$. This means that to obtain $\Gamma(\mathbb{Z}_{p^m} \times \mathbb{Z}_{q^n})$ we remove as many as degree of $(0, 0)$ (in $\Gamma_0(\mathbb{Z}_{p^m} \times \mathbb{Z}_{q^n})$) edges of $\Gamma_0(\mathbb{Z}_{p^m} \times \mathbb{Z}_{q^n})$. Therefore, $\left\|\Gamma(\mathbb{Z}_{p^m} \times \mathbb{Z}_{q^n})\right\|$ can be formulated as follows:

$$\begin{aligned}\left\|\Gamma(\mathbb{Z}_{p^m} \times \mathbb{Z}_{q^n})\right\| &= \left\|\Gamma_0(\mathbb{Z}_{p^m} \times \mathbb{Z}_{q^n})\right\| - \deg(0,0) \\ &= \frac{1}{2}\Big(mn(p^m - p^{m-1})(q^n - q^{n-1}) + m(p^m - p^{m-1})q^n \\ &\quad + n(q^n - q^{n-1})p^m + p^m q^n - p^{\lceil \frac{m-1}{2} \rceil} q^{\lceil \frac{n-1}{2} \rceil}\Big) - p^m q^n + 1.\end{aligned}$$

This completes the proof. □

An example that adheres to Theorem 3.3 is given below for illustration.

Example 3.1. Based on Theorem 3.3:

$$\begin{aligned}\|\Gamma(\mathbb{Z}_3 \times \mathbb{Z}_{2^3})\| &= \frac{1}{2}\Big(3(3 - 3^0)(2^3 - 2^2) + (3 - 3^0)2^3 \\ &\quad + 3(2^3 - 2^2)3 + 3.2^3 - 3^{\lceil \frac{1-1}{2} \rceil} 2^{\lceil \frac{3-1}{2} \rceil}\Big) - 3.2^3 + 1 \\ &= \frac{1}{2}\Big(3.2.4 + 2.8 + 3.4.3 + 3.8 - 2\Big) - 24 + 1 \\ &= \frac{1}{2}\Big(24 + 16 + 36 + 24 - 2\Big) - 24 + 1 \\ &= 26.\end{aligned}$$

By Definition 2.1,

$$\|\Gamma(\mathbb{Z}_3 \times \mathbb{Z}_{2^3})\| = 26$$

as shown in Figure 3.

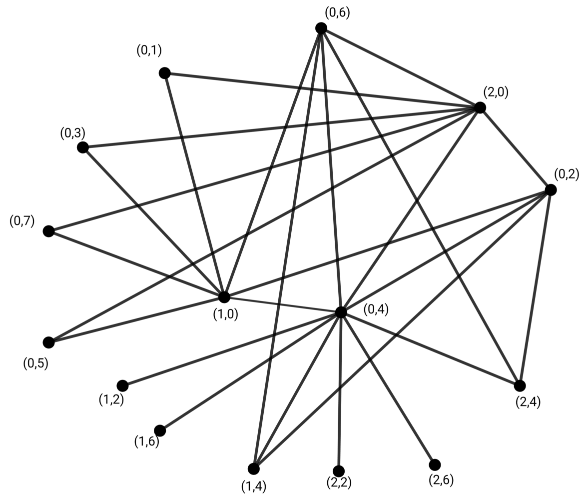

Figure 3. $\Gamma(\mathbb{Z}_3 \times \mathbb{Z}_{2^3})$.

The same result can be obtained whether applying the theorem or definition.

Furthermore, to facilitate the research process, define partitions of $V(\Gamma(\mathbb{Z}_{p^m} \times \mathbb{Z}_{q^n}))$ so that two distinct vertices in the same partition have similar properties including degree and eccentricity. Define

$$V_{i,j} = \{(a, b) \in V : \gcd(a, p^m) = p^i, \gcd(b, q^n) = q^j\},$$

where $i = 0, 1, 2, \cdots, m-1$ and $j = 0, 1, 2, \cdots, n-1$. Note that

$$V_{0,0} = \{(c, d) : \gcd(c, p^m) = p^0 = 1, \quad \gcd(d, q^n) = q^0 = 1\}$$

and

$$V_{m,n} = \{(0, 0)\}$$

are not the subset of the vertex set because its elements are not the zero divisor of $\mathbb{Z}_{p^m} \times \mathbb{Z}_{q^n}$. As a result, $V(\Gamma(\mathbb{Z}_{p^m} \times \mathbb{Z}_{q^n}))$ can be partitioned into $mn + m + n - 1$ sets.

Consider the case $p = 3$, $q = 2$, $m = 1$, and $n = 3$. The vertex set of $\Gamma(\mathbb{Z}_3 \times \mathbb{Z}_{2^3})$ can be expressed as

$$V(\Gamma(\mathbb{Z}_3 \times \mathbb{Z}_{2^3})) = V_{0,1} \cup V_{0,2} \cup V_{0,3} \cup V_{1,0} \cup V_{1,1} \cup V_{1,2},$$

with the partition defined as follows:

$$V_{0,1} = \{(1, 2), (1, 6), (2, 2), (2, 6)\}, \quad V_{0,2} = \{(1, 4), (2, 4)\}, \quad V_{0,3} = \{(1, 0), (2, 0)\},$$
$$V_{1,0} = \{(0, 1), (0, 3), (0, 5), (0, 7)\}, \quad V_{1,1} = \{(0, 2), (0, 6)\}, \quad V_{1,2} = \{(0, 4)\}.$$

Observe that every vertex in $V_{0,2}$ is adjacent to all vertices in $V_{1,1}$.

Below are the properties related to the adjacency of two distinct vertices based on the defined partitions.

Theorem 3.4. *Let*

$$V_{i,j}, V_{k,l} \subset V(\Gamma(\mathbb{Z}_{p^m} \times \mathbb{Z}_{q^n})).$$

Vertices $(a,b) \in V_{i,j}$ and $(c,d) \in V_{k,l}$ are adjacent if only if $i + k \geq m$ and $j + l \geq n$.

Proof. Consider any elements $(a,b) \in V_{i,j}$ and $(c,d) \in V_{k,l}$. By the definition of the partition, we have

$$\gcd(a, p^m) = p^i, \qquad \gcd(c, p^m) = p^k,$$

which implies

$$a = p^i x, \qquad c = p^k y,$$

for some integers x and y with

$$\gcd(x, p) = \gcd(y, p) = 1.$$

Similarly, we write

$$b = q^j s, \qquad d = q^l t,$$

for integers s and t with

$$\gcd(s, q) = \gcd(t, q) = 1.$$

If (a,b) and (c,d) are adjacent, then

$$ac \equiv 0 \pmod{p^m}, \qquad bd \equiv 0 \pmod{q^n},$$

which is equivalent to $p^m \mid ac, q^n \mid bd$. Since

$$ac = p^i x \cdot p^k y = p^{i+k} xy,$$

we have

$$p^m \mid p^{i+k} xy.$$

Similarly, $q^n \mid q^{j+l} st$. Because

$$\gcd(x, p) = \gcd(y, p) = 1,$$

it follows that

$$p^m \mid p^{i+k} \implies i + k \geq m,$$

and likewise,

$$j + l \geq n.$$

Conversely, if $(a,b) \in V_{i,j}$ and $(c,d) \in V_{k,l}$ satisfy

$$i + k \geq m \quad \text{and} \quad j + l \geq n,$$

then

$$ac = p^{i+k} xy \equiv 0 \pmod{p^m}, \qquad bd = q^{j+l} st \equiv 0 \pmod{q^n},$$

and therefore (a,b) and (c,d) are adjacent. This completes the proof. □

If the primes p and q in Theorem 3.4 are equal, we obtain the following corollary.

Corollary 3.5. *Let*

$$V_{i,j}, V_{k,l} \subset V(\Gamma(\mathbb{Z}_{p^m} \times \mathbb{Z}_{q^n})).$$

Two distinct vertices $(a,b),(c,d) \in V_{i,j}$ are adjacent if only if $2i \geq m$ and $2j \geq n$.

The following theorem presents the cardinality of the defined partition $V_{i,j}$.

Theorem 3.6. *[22] Let*

$$V_{i,j} = \{(a,b) \in V : \gcd(a,p^m) = p^i, \gcd(b,q^n) = q^j\},$$

then

$$|V_{i,j}| = \begin{cases} (p^{m-i} - p^{m-(i+1)})(q^{n-j} - q^{n-(j+1)}), & i = 0,1,2,\cdots,m-1, j = 0,1,2,\cdots,n-1, \\ p^{m-i} - p^{m-(i+1)}, & i = 0,1,2,\cdots,m-1, j = n, \\ q^{n-j} - q^{n-(j+1)}, & i = m, j = 0,1,2,\cdots,n-1. \end{cases}$$

By utilizing the partition properties of the vertex set defined above, we can analyze the connectivity properties of the graph. Let

$$G = (V,E)$$

be a simple graph and let $X \subseteq V$. The graph $G - X$ is the subgraph of G obtained by deleting the vertices in X together with all edges incident to those vertices. Furthermore, if G is a connected graph and $G - X$ is disconnected, then X is called a separator of G.

Theorem 3.7. *Both $V_{m,n-1}$ and $V_{m-1,n}$ are a minimal separator of $\Gamma(\mathbb{Z}_{p^m} \times \mathbb{Z}_{q^n})$.*

Proof. By Theorem 3.4, for any

$$(a,b) \in V_{0,1} \subset V(\Gamma(\mathbb{Z}_{p^m} \times \mathbb{Z}_{q^n})),$$

we have

$$N((a,b)) = V_{m,n-1}.$$

Consequently, the graph

$$\Gamma(\mathbb{Z}_{p^m} \times \mathbb{Z}_{q^n}) - V_{m,n-1}$$

is a disconnected graph. This implies that $V_{m,n-1}$ is a separator of $\Gamma(\mathbb{Z}_{p^m} \times \mathbb{Z}_{q^n})$.

Now let $X \subsetneq V_{m,n-1}$. Then there exists a vertex $(c,d) \in V_{m,n-1}$ such that $(c,d) \notin X$. Hence,

$$\Gamma(\mathbb{Z}_{p^m} \times \mathbb{Z}_{q^n}) \ - \ X$$

remains connected. In other words, X is not a separator of $\Gamma(\mathbb{Z}_{p^m} \times \mathbb{Z}_{q^n})$. Therefore, $V_{m,n-1}$ is a minimal separator of $\Gamma(\mathbb{Z}_{p^m} \times \mathbb{Z}_{q^n})$.

Similarly, by examining the neighbors of any vertex in $V_{1,0}$, $V_{m-1,n}$ is also a minimal separator of $\Gamma(\mathbb{Z}_{p^m} \times \mathbb{Z}_{q^n})$. This completes the proof. □

Here is an example of a minimal separator for a graph.

Example 3.2. Observe $\Gamma(\mathbb{Z}_3 \times \mathbb{Z}_{2^3})$ in Figure 3. It is clear that

$$V_{1,2} = \{(0,4)\} \quad \text{and} \quad V_{0,3} = \{(1,0),(2,0)\}$$

are a minimal separator for this graph.

From Theorem 3.7, the following connectivity result follows.

Theorem 3.8. *Let p and q be primes, and let m, n be natural numbers. Then $\Gamma(\mathbb{Z}_{p^m} \times \mathbb{Z}_{q^n})$ is a k-connected graph with*

$$k \leq \min\{p-1, q-1\}.$$

Moreover,

$$\kappa\Big(\Gamma(\mathbb{Z}_{p^m} \times \mathbb{Z}_{q^n})\Big) = \min\{p-1, q-1\}.$$

Proof. Based on Theorem 3.7, the sets $V_{m,n-1}$ and $V_{m-1,n}$ are minimal separators of $\Gamma(\mathbb{Z}_{p^m} \times \mathbb{Z}_{q^n})$. From Theorem 3.6, it follows that

$$|V_{m,n-1}| = q-1 \quad \text{and} \quad |V_{m-1,n}| = p-1,$$

and that the size of each partition is given by one of the following forms:

$$p^{m-i-1}(p-1)q^{n-j-1}(q-1), \quad p^{m-i-1}(p-1), \quad \text{or} \quad q^{n-j-1}(q-1),$$

for some $0 \leq i \leq m$ and $0 \leq j \leq n$. Consequently, the smallest possible size among all such partitions is $\min\{p-1, q-1\}$.

Hence, for any subset

$$X \subseteq V\Big(\Gamma(\mathbb{Z}_{p^m} \times \mathbb{Z}_{q^n})\Big)$$

satisfying

$$|X| \leq \min\{p-1, q-1\},$$

the graph $\Gamma(\mathbb{Z}_{p^m} \times \mathbb{Z}_{q^n}) - X$ remains connected. By Definition 2.9, this implies that $\Gamma(\mathbb{Z}_{p^m} \times \mathbb{Z}_{q^n})$ is k-connected for all

$$k \leq \min\{p-1, q-1\}.$$

Furthermore, the vertex-connectivity of the graph satisfies

$$\kappa\Big(\Gamma(\mathbb{Z}_{p^m} \times \mathbb{Z}_{q^n})\Big) = \min\{p-1, q-1\}.$$

This completes the proof. □

Example 3.3. $\Gamma(\mathbb{Z}_3 \times \mathbb{Z}_{2^3})$ in Figure 3 is a 1-connected graph and

$$\kappa(\Gamma(\mathbb{Z}_3 \times \mathbb{Z}_{2^3})) = \min\{3-1, 2-1\}.$$

3.2. *Topological Indices of* $\Gamma(\mathbb{Z}_{p^m} \times \mathbb{Z}_{q^n})$

This section focuses on the topological indices of $\Gamma(\mathbb{Z}_{p^m} \times \mathbb{Z}_{q^n})$ related to the degree and distance between two distinct vertices. The first one is the general zeroth-order Randić index for $\alpha = 1$, as detailed in the corollary below.

Corollary 3.9. *If p, q are primes, m, n are natural numbers, then*

$$R_1^0(\Gamma(\mathbb{Z}_{p^m} \times \mathbb{Z}_{q^n})) = \Big(mn(p^m - p^{m-1})(q^n - q^{n-1}) + m(p^m - p^{m-1})q^n + n(q^n - q^{n-1})p^m + p^m q^n - p^{\lceil \frac{m-1}{2} \rceil} q^{\lceil \frac{n-1}{2} \rceil}\Big) - 2p^m q^n + 2.$$

Proof. Apply handshaking lemma and Theorem 3.3. □

The following index is the eccentric connectivity index which is the sum of the product of degree and eccentricity for each vertex. The discussion is divided into two cases which are $\xi^C(\Gamma(\mathbb{Z}_{p^m} \times \mathbb{Z}_{q^n}))$ for $m = 1$ and $n = 3$ and $m, n \geq 2$. For the first case is provided by the theorem below.

Theorem 3.10. *If p, q are primes, then*

$$\xi^C(\Gamma(\mathbb{Z}_p \times \mathbb{Z}_{q^3})) = 15pq^3 - 16pq^2 - 11q^3 + 10q^2 - 2q + p + 3.$$

Proof. Let $\Gamma(\mathbb{Z}_p \times \mathbb{Z}_{q^3})$ denote the zero-divisor graph of the ring $\mathbb{Z}_p \times \mathbb{Z}_{q^3}$, where p and q are primes. The vertex set can be partitioned as

$$V(\Gamma(\mathbb{Z}_p \times \mathbb{Z}_{q^3})) = V_{0,1} \cup V_{1,0} \cup V_{0,2} \cup V_{0,3} \cup V_{1,2}.$$

By Theorem 3.6, the cardinalities of these sets are

$$|V_{0,1}| = (p-1)(q^2 - q), \quad |V_{1,0}| = q^3 - q^2, \quad |V_{0,2}| = (p-1)(q-1),$$

$$|V_{0,3}| = p - 1, \quad |V_{1,2}| = q - 1, \quad |V_{1,1}| = q^2 - q.$$

Moreover, by Theorem 3.2, the degrees of the vertices are

$$\deg(v_1) = q - 1, \quad \deg(v_2) = p - 1, \quad \deg(v_3) = q^2 - 1, \quad \deg(v_4) = q^3 - 1,$$

$$\deg(v_5) = pq^2 - 2, \quad \deg(v_6) = pq - 1,$$

where $v_1 \in V_{0,1},\ v_2 \in V_{1,0},\ v_3 \in V_{0,2},\ v_4 \in V_{0,3},\ v_5 \in V_{1,2},$ and $v_6 \in V_{1,1}$.

Since all vertices within the same partition have the same degree, the graph can be represented using partitions as vertices, with loops indicating adjacency between distinct vertices in the same partition (see Figure 4).

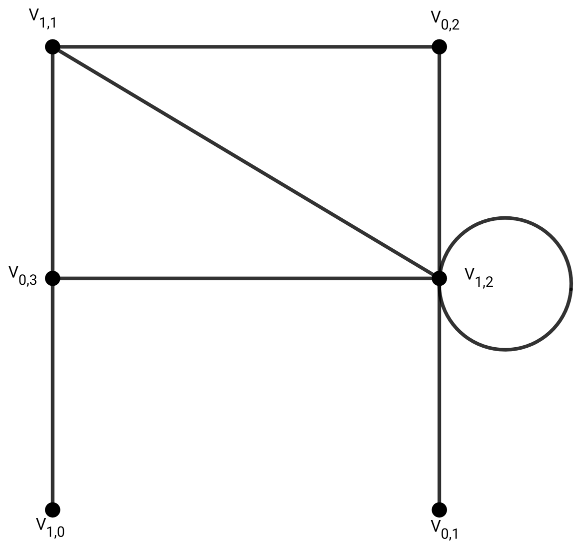

Figure 4. An illustration of $\Gamma(\mathbb{Z}_p \times \mathbb{Z}_{q^3})$.

From Figure 4, it follows that the eccentricities of the vertices are

$$e(v_1) = e(v_2) = e(v_3) = 3, \quad e(v_4) = e(v_5) = e(v_6) = 2.$$

By Definition 2.12, the eccentric connectivity index of $\Gamma(\mathbb{Z}_p \times \mathbb{Z}_{q^3})$ is

$$\begin{aligned}
\xi^C(\Gamma(\mathbb{Z}_p \times \mathbb{Z}_{q^3})) &= \sum_{v\in V} e(v)\deg(v) \\
&= \sum_{v_1\in V_{0,1}} e(v_1)\deg(v_1) + \sum_{v_2\in V_{1,0}} e(v_2)\deg(v_2) + \sum_{v_3\in V_{0,2}} e(v_3)\deg(v_3) \\
&\quad + \sum_{v_4\in V_{0,3}} e(v_4)\deg(v_4) + \sum_{v_5\in V_{1,2}} e(v_5)\deg(v_5) + \sum_{v_6\in V_{1,1}} e(v_6)\deg(v_6) \\
&= 3[\deg(v_1)|V_{0,1}| + \deg(v_2)|V_{1,0}| + \deg(v_3)|V_{0,2}|] \\
&\quad + 2[\deg(v_4)|V_{0,3}| + \deg(v_5)|V_{1,2}| + \deg(v_6)|V_{1,1}|] \\
&= 3[(q-1)(p-1)(q^2-q) + (p-1)(q^3-q^2) + (q^2-1)(p-1)(q-1)] \\
&\quad + 2[(q^3-1)(p-1) + (pq^2)(q-1) + (pq-1)(q^2-q)] \\
&= 15pq^3 - 16pq^2 - 11q^3 + 10q^2 - 2q + p + 3.
\end{aligned}$$

This completes the proof. □

Note that the values of p and q in the above theorem can be either equal or different. Consequently, if $p = q$, the following immediate result is obtained.

Corollary 3.11. *If p is prime then*

$$\xi^C(\Gamma(\mathbb{Z}_p \times \mathbb{Z}_{p^3})) = 15p^4 - 27p^3 - +10p^2 - p + 3.$$

Example 3.4 is presented to illustrate Theorem 3.10.

Example 3.4. Consider the zero-divisor graph $\Gamma(\mathbb{Z}_3 \times \mathbb{Z}_{2^3})$ as illustrated in Figure 3. The eccentricities of the vertices are

$$\begin{aligned} e((0,1)) &= e((0,3)) = e((0,5)) = e((0,7)) = e((1,2)) = e((1,6)) \\ &= e((2,2)) = e((2,6)) = e((1,4)) = e((2,4)) = 3 \end{aligned}$$

and

$$e((1,0)) = e((2,0)) = e((0,2)) = e((0,6)) = e((0,4)) = 2.$$

By Definition 2.12,

$$\begin{aligned} \xi^C(\Gamma(\mathbb{Z}_3 \times \mathbb{Z}_{2^3})) &= \sum_{v \in \Gamma(\mathbb{Z}_3 \times \mathbb{Z}_{2^3})} e(v)\deg(v) \\ &= 3(2+2+2+2) + 3(1+1+1+1) + 3(3+3) + 2(7+7) \\ &\quad + 2(5+5) + 2(10) \\ &= 122. \end{aligned}$$

By Theorem 3.10,

$$\begin{aligned} \xi^C(\Gamma(\mathbb{Z}_3 \times \mathbb{Z}_{2^3})) &= 15pq^3 - 16pq^2 - 11q^3 + 10q^2 - 2q + p + 3 \\ &= 15(24) - 16(12) - 11(8) + 10(4) - 4 + 3 + 3 \\ &= 360 - 192 - 88 + 40 - 4 + 6 \\ &= 122. \end{aligned}$$

Prior to discussing the eccentric connectivity index for the case $m, n \geq 2$, the following theorem is provided, which examines the eccentricity of each vertex in $V(\Gamma(\mathbb{Z}_{p^m} \times \mathbb{Z}_{q^n}))$.

Theorem 3.12. *If $m, n \geq 2$ and $v \in V(\Gamma(\mathbb{Z}_{p^m} \times \mathbb{Z}_{q^n}))$, then:*

$$e(v) = \begin{cases} 3, & v \in V_{i,0} \cup V_{0,j}; \\ 2, & \textit{otherwise.} \end{cases}$$

Proof. Consider any element $v \in V(\Gamma(\mathbb{Z}_{p^m} \times \mathbb{Z}_{q^n}))$. By Theorem 2.14, we have

$$diam(\Gamma(\mathbb{Z}_{p^m} \times \mathbb{Z}_{q^n})) \leq 3,$$

which means that for every pair of vertices $u, v \in V(\Gamma(\mathbb{Z}_{p^m} \times \mathbb{Z}_{q^n}))$,

$$d(u, v) \leq 3.$$

Since $\Gamma(\mathbb{Z}_{p^m} \times \mathbb{Z}_{q^n})$ is not a complete graph, for each $u \in V(\Gamma(\mathbb{Z}_{p^m} \times \mathbb{Z}_{q^n}))$, there exists $u \in V(\Gamma(\mathbb{Z}_{p^m} \times \mathbb{Z}_{q^n}))$ with $u \neq v$ and $d(u, v) \neq 1$.

(1) If $v \in V_{i,0}$ for some $1 \leq i \leq m$, then

$$N(v) = V_{m-i,n} \cup V_{m-i+1,n} \cup \cdots \cup V_{m-1,n}.$$

For any $u \in V_{0,1}$, we have

$$N(u) = V_{m,n-1}.$$

Hence, $d(u, v) \neq 1$, and since

$$N(u) \cap N(v) = \emptyset,$$

we also have $d(u, v) \neq 2$. Therefore,

$$d(u, v) = 3 = e(v).$$

(2) If $v \in V_{0,j}$ for some $1 \leq j \leq n$, then

$$N(v) = V_{m,n-j} \cup V_{m,n-j+1} \cup \cdots \cup V_{m,n-1}.$$

For any $u \in V_{1,0}$, we have

$$N(u) = V_{m-1,n},$$

so $d(u, v) \neq 1$. Since

$$N(u) \cap N(v) = \emptyset,$$

it follows that $d(u, v) \neq 2$. Hence,

$$d(u, v) = 3 = e(v).$$

(3) If $v \in V_{i,n}$ for some $1 \leq i \leq m - 1$, consider any $u \in V(\Gamma(\mathbb{Z}_{p^m} \times \mathbb{Z}_{q^n}))$ with $d(u, v) \neq 1$. Examine the following cases:

- If $u \in V_{k,0}$ for some $1 \leq k \leq m - i$, then

$$V_{m-1,n} \subseteq N(u) \cap N(v),$$

so $d(u, v) = 2$.

- If $u \in V_{0,l}$ with $l \neq 0$, then

$$V_{m,n-1} \subseteq N(u) \cap N(v),$$

so $d(u, v) = 2$.

- If $u \in V_{k,l}$ with $V_{0,l} \neq V_{k,l} \neq V_{k,0}$ and $1 \leq k + i \leq m$, then

$$V_{m-1,n-1} \subseteq N(u) \cap N(v),$$

so $d(u, v) = 2$.

Thus, $e(v) = 2$.

(4) If $v \in V_{m,j}$ for some $1 \leq j \leq n - 1$, consider any $u \in V(\Gamma(\mathbb{Z}_{p^m} \times \mathbb{Z}_{q^n}))$ with $d(u, v) \neq 1$. Examine the following cases:

- If $u \in V_{m,j}$, then

$$N(u) = N(v),$$

but $d(u, v) \neq 1$, hence $d(u, v) = 2$.

- If $u \in V_{k,0}$ with $k \neq 0$, then

$$V_{m-1,n} \subseteq N(u) \cap N(v),$$

so $d(u,v) = 2$.

- If $u \in V_{0,l}$ for some $1 \leq l \leq m - j$, then

$$V_{m,n-1} \subseteq N(u) \cap N(v),$$

so $d(u,v) = 2$.

- If $u \in V_{k,l}$ with

$$V_{0,l} \neq V_{k,l} \neq V_{k,0}$$

and $1 \leq l + j \leq n$, then

$$V_{m-1,n-1} \subseteq N(u) \cap N(v),$$

so $d(u,v) = 2$.

Therefore, $e(v) = 2$.

(5) If $v \in V_{i,j}$ for some $1 \leq i \leq m-1$ and $1 \leq j \leq n-1$, consider any $u \in V(\Gamma(\mathbb{Z}_{p^m} \times \mathbb{Z}_{q^n}))$ with $d(u,v) \neq 1$. Examine the following possibilities:

- If $u \in V_{i,j}$, then

$$N(u) = N(v),$$

but $d(u,v) \neq 1$, so $d(u,v) = 2$.

- If $u \in V_{k,l}$ with $l \neq 0$, then

$$V_{m,n-1} \subseteq N(u) \cap N(v),$$

so $d(u,v) = 2$.

- If $u \in V_{k,0}$, then

$$V_{m-1,n} \subseteq N(u) \cap N(v),$$

so $d(u,v) = 2$.

Hence, $e(v) = 2$.

This completes the proof. □

The following theorem gives the formula for the eccentric connectivity index of the graph.

Theorem 3.13. *If $m, n \in \mathbb{N}$ and $m, n \geq 2$ then*

$$\xi^C(\Gamma(\mathbb{Z}_{p^m} \times \mathbb{Z}_{q^n})) = (2mn + m + n)(p^m - p^{m-1})(q^n - q^{n-1}) + 2m(p^m - p^{m-1})q^n + 2n(q^n - q^{n-1})p^m - 2p^m q^n - 2p^{\lceil \frac{m-1}{2} \rceil} q^{\lceil \frac{n-1}{2} \rceil} + 4.$$

Proof. Let

$$V_1 = V_{1,0} \cup V_{2,0} \cup \cdots \cup V_{m,0}, \quad V_2 = V_{0,1} \cup V_{0,2} \cup \cdots \cup V_{0,n}$$

and

$$V' = V_1 \cup V_2.$$

Applying Theorems 3.6 and 3.12,

$$\begin{aligned}
\sum_{v\in V'}\deg(v) &= \sum_{v\in V_1}\deg(v) + \sum_{v\in V_2}\deg(v)\\
&= \sum_{i=1}^{m}|V_{i,0}|\deg(v) + \sum_{j=1}^{n}|V_{0,j}|\deg(v)\\
&= (q^n - q^{n-1})(q^n - 1) + \sum_{i=1}^{m-1}(q^n - q^{n-1})(p^{m-i} - p^{m-i-1})(p^i - 1)\\
&\quad + (p^m - p^{m-1})(p^m - 1) + \sum_{j=1}^{n-1}(p^m - p^{m-1})(q^{n-j} - q^{n-j-1})(q^j - 1)\\
&= m(p^m - p^{m-1})(q^n - q^{n-1}) + n(p^m - p^{m-1})(q^n - q^{n-1})\\
&= (m+n)(p^m - p^{m-1})(q^n - q^{n-1}).
\end{aligned}$$

Next, it remains to formulate the eccentricity connectivity index.

$$\begin{aligned}
\xi^C(\Gamma(\mathbb{Z}_{p^m}\times\mathbb{Z}_{q^n})) &= \sum_{v\in V'} e(v)\deg(v) + \sum_{v\in V\setminus V'} e(v)\deg(v)\\
&= \sum_{v\in V'} 3\deg(v) + \sum_{v\in V\setminus V'} 2\deg(v)\\
&= 3\sum_{v\in V'}\deg(v) + 2\sum_{v\in V\setminus V'}\deg(v)\\
&= \sum_{v\in V'}\deg(v) + 2\Big(\sum_{v\in V'}\deg(v) + \sum_{v\in V\setminus V'}\deg(v)\Big)\\
&= \sum_{v\in V'}\deg(v) + 2R_1^0\\
&= (m+n)(p^m - p^{m-1})(q^n - q^{n-1})\\
&+ 2\Big(mn(p^m - p^{m-1})(q^n - q^{n-1}) + m(p^m - p^{m-1})q^n + n(q^n - q^{n-1})p^m + p^m q^n\\
&\quad - p^{\lceil\frac{m-1}{2}\rceil}q^{\lceil\frac{n-1}{2}\rceil}\Big) - 4p^m q^n + 4\\
&= (2mn + m + n)(p^m - p^{m-1})(q^n - q^{n-1}) + 2m(p^m - p^{m-1})q^n\\
&\quad + 2n(q^n - q^{n-1})p^m - 2p^m q^n - 2p^{\lceil\frac{m-1}{2}\rceil}q^{\lceil\frac{n-1}{2}\rceil} + 4.
\end{aligned}$$

This completes the proof. □

To illustrate Theorem 3.13, the following example is presented.

Example 3.5. Given $\Gamma(\mathbb{Z}_{2^2}\times\mathbb{Z}_{3^2})$ as shown in Figure 5.

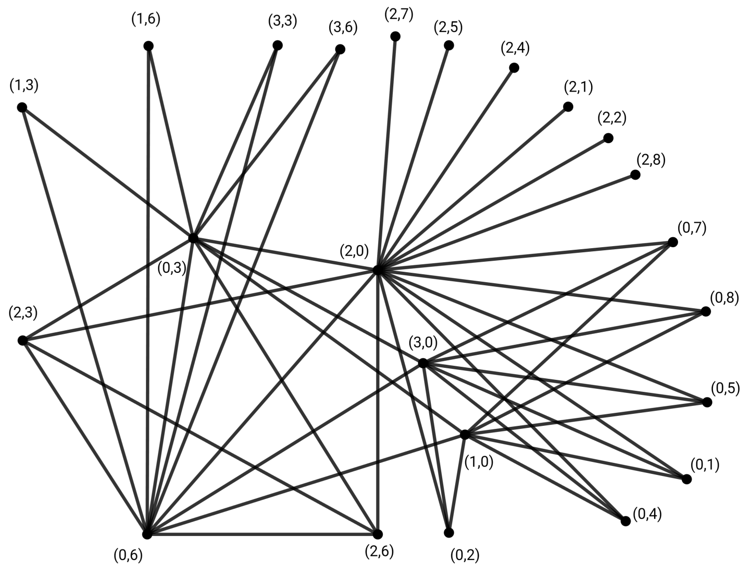

Figure 5. $\Gamma(\mathbb{Z}_{2^2} \times \mathbb{Z}_{3^2})$.

Based on this figure, we have

$$e((1,3)) = e((1,6)) = e((3,3)) = e((3,6)) = e((1,0)) = e((3,0)) = e((2,1)) = e((2,2)) = 3,$$
$$e((2,4)) = e((2,5)) = e((2,7)) = e((2,8)) = e((0,1)) = e((0,2)) = e((0,4)) = e((0,5)) = 3,$$
$$e((0,7)) = e((0,8)) = 3,$$

and the eccentricity of all other vertices is 2.

According to Definition 2.12, the connective eccentric index is

$$\begin{aligned}\xi^C(\Gamma(\mathbb{Z}_{2^2} \times \mathbb{Z}_{3^2})) &= \sum_{v \in V(\Gamma(\mathbb{Z}_{2^2} \times \mathbb{Z}_{3^2}))} e(v)\deg(v) \\ &= 3(2+2+2+2+8+8+1+1+1+1+1+1+3+3+3+3+3+3) \\ &\quad + 2(4+4+16+10+10) \\ &= 232.\end{aligned}$$

By Theorem 3.13, we can also compute it as

$$\begin{aligned}\xi^C(\Gamma(\mathbb{Z}_{2^2} \times \mathbb{Z}_{3^2})) &= (8+2+2)(2^2-2)(3^2-3) + 4(2^2-2)3^2 + 4(3^2-3)2^2 \\ &\quad + 2(2^2 3^2) - 2 \cdot 2^{\lceil \frac{2-1}{2} \rceil} 3^{\lceil \frac{2-1}{2} \rceil} - 4(2^2)(3^2) + 4 \\ &= 144 + 72 + 96 + 72 - 12 - 144 + 4 \\ &= 232.\end{aligned}$$

It can be seen that the same result is obtained both with the theorem and the definition.

The final discussion is the Schultz index which is derived from the degree and distance between any two distinct vertices. Since the Schultz index examined in this paper is limited to particular cases, a theorem regarding to the upper bound $Sc(\Gamma(\mathbb{Z}_{p^m} \times \mathbb{Z}_{q^n}))$ for any natural number m, n is presented.

Theorem 3.14. *If p and q are primes and m, n are natural numbers, then*

$$Sc(\Gamma(\mathbb{Z}_{p^m} \times \mathbb{Z}_{q^n})) \leq \frac{3}{2} R_1^0 (p^{m-1} q^{n-1}(p+q-1) - 2).$$

Proof. According to Theorem 3.1, $e(v) \leq 3$ for all $v \in V(\Gamma(\mathbb{Z}_{p^m} \times \mathbb{Z}_{q^n})$. By Definition 2.11

$$\begin{aligned}
Sc(\Gamma(\mathbb{Z}_{p^m} \times \mathbb{Z}_{q^n}) &= \frac{1}{2} \sum_{u,v \in V(\Gamma(\mathbb{Z}_{p^m} \times \mathbb{Z}_{q^n}))} (\deg(u) + \deg(v)) d(u,v) \\
&< \frac{3}{2} \sum_{u,v \in V(\Gamma(\mathbb{Z}_{p^m} \times \mathbb{Z}_{q^n}))} (\deg(u) + \deg(v)) \\
&= \frac{3}{2} (|\Gamma(\mathbb{Z}_{p^m} \times \mathbb{Z}_{q^n})| - 1) \sum_{u,v \in V(\Gamma(\mathbb{Z}_{p^m} \times \mathbb{Z}_{q^n}))} \deg(v) \\
&= \frac{3}{2} (|\Gamma(\mathbb{Z}_{p^m} \times \mathbb{Z}_{q^n})| - 1) {R_1}^0 \\
&= \frac{3}{2} {R_1}^0 (p^{m-1} q^{n-1}(p+q-1) - 2).
\end{aligned}$$

This completes the proof. □

In the particular case where $m = 2$ and $n = 1$,the Schultz index of $\Gamma(\mathbb{Z}_{p^m} \times \mathbb{Z}_{q^n})$ for $m = 1$ and $n = 2$ is given below.

Theorem 3.15. *If p and q are primes, then*

$$Sc(\Gamma(\mathbb{Z}_p \times \mathbb{Z}_{q^2})) = \frac{1}{2}(q-1)(8pq^3 - 20pq^2 + 8p^2q^2 - 2p^2q - 8pq - 6q^3 + 7q^2 + 10q + 4).$$

Proof. Note that

$$V(\Gamma(\mathbb{Z}_p \times \mathbb{Z}_{q^2})) = V_{0,1} \cup V_{0,2} \cup V_{1,0} \cup V_{1,1}.$$

Consider any $u, v \in V(\Gamma(\mathbb{Z}_p \times \mathbb{Z}_{q^2}))$ with $u \neq v$. When examining the cases in which each partition contains at least two elements, there are ten possible scenarios as follows:

(1) $u, v \in V_{0,2}$

$$\begin{aligned}
\sum_{u,v \in V_{0,2}} (\deg(u) + \deg(v)) d(u,v) &= \sum_{u,v \in V_{0,2}} 2(2q^2 - 2) \\
&= \binom{|V_{0,2}|}{2} (4q^2 - 4) \\
&= (p-1)(p-2)(2q^2 - 2).
\end{aligned}$$

(2) $u, v \in V_{1,0}$

$$\begin{aligned}
\sum_{u,v \in V_{1,0}} (\deg(u) + \deg(v)) d(u,v) &= \sum_{u,v \in V_{1,0}} 2(2p - 2) \\
&= \binom{|V_{1,0}|}{2} (2p - 2) \\
&= 2q(p-1)(q-1)(q^2 - q - 1).
\end{aligned}$$

(3) $u, v \in V_{1,1}$

$$\begin{aligned}\sum_{u,v\in V_{1,1}} (\deg(u) + \deg(v))d(u, v) &= \sum_{u,v\in V_{1,1}} 2(pq - 2)\\ &= \binom{|V_{1,1}|}{2}(2pq - 4)\\ &= (q - 1)(q - 2)(pq - 2).\end{aligned}$$

(4) $u, v \in V_{0,1}$

$$\begin{aligned}\sum_{u,v\in V_{0,1}} (\deg(u) + \deg(v))d(u, v) &= \sum_{u,v\in V_{0,1}} 2(2q - 2)\\ &= \binom{|V_{0,1}|}{2}(2q - 2)\\ &= 2(p - 1)(q - 1)^2(pq - p - q).\end{aligned}$$

(5) $u \in V_{0,2}$ and $v \in V_{1,0}$

$$\begin{aligned}\sum_{u\in V_{0,2},v\in V_{1,0}} ((\deg(u) + \deg(v))d(u, v) &= \sum_{u\in V_{0,2},v\in V_{1,0}} (q^2 - 1 + p - 1)\\ &= |V_{0,2}||V_{1,0}|(p + q^2 - 2)\\ &= (p - 1)q(q - 1)(p + q^2 - 2).\end{aligned}$$

(6) $u \in V_{0,2}$ and $v \in V_{1,1}$

$$\begin{aligned}\sum_{u\in V_{0,2},v\in V_{1,1}} ((\deg(u) + \deg(v))d(u, v) &= \sum_{u\in V_{0,2},v\in V_{1,1}} (q^2 - 1 + pq - 2)\\ &= |V_{0,2}||V_{1,1}|(q^2 - 1 + pq - 2)\\ &= (p - 1)(q - 1)(pq + q^2 - 3).\end{aligned}$$

(7) $u \in V_{0,2}$ and $v \in V_{0,1}$

$$\begin{aligned}\sum_{u\in V_{0,2},v\in V_{0,1}} ((\deg(u) + \deg(v))d(u, v) &= \sum_{u\in V_{0,2},v\in V_{0,1}} 2(q^2 - 1 + pq - 2)\\ &= |V_{0,2}||V_{0,1}|2(q^2 - 1 + q - 1)\\ &= 2(p - 1)^2(q - 1)(q^2 + q - 2).\end{aligned}$$

(8) $u \in V_{1,0}$ and $v \in V_{1,1}$

$$\begin{aligned}\sum_{u\in V_{1,0},v\in V_{1,1}} ((\deg(u) + \deg(v))d(u, v) &= \sum_{u\in V_{1,0},v\in V_{1,1}} 2(p - 1 + pq - 2)\\ &= |V_{1,0}||V_{1,1}|2(p - 1 + pq - 2)\\ &= 2q(q - 1)^2(pq + p - 3).\end{aligned}$$

(9) $u \in V_{1,0}$ and $v \in V_{0,1}$

$$\begin{aligned}\sum_{u\in V_{1,0},v\in V_{0,1}} (\deg(u)+\deg(v))d(u,v) &= \sum_{u\in V_{1,0},v\in V_{0,1}} 3(p-1+q-1)\\ &= |V_{1,0}||V_{0,1}|3(p-1+q-1)\\ &= 3(p-1)q(q-1)^2(p+q-2).\end{aligned}$$

(10) $u \in V_{1,0}$ and $v \in V_{0,1}$

$$\begin{aligned}\sum_{u\in V_{1,1},v\in V_{0,1}} (\deg(u)+\deg(v))d(u,v) &= \sum_{u\in V_{1,1},v\in V_{0,1}} (pq+q-3)\\ &= |V_{1,1}||V_{0,1}|(pq-2+q-1)\\ &= (p-1)(q-1)^2(pq+q-3).\end{aligned}$$

When p or q equals 2, some partitions contain only one element, making point 1, point 3, or point 4 equal to zero. Substituting these values into the formulas still yields zero, so the expressions in points 1–10 hold in general.

Therefore, the Schultz index can be formulated along these lines.

$$\begin{aligned}2Sc(\Gamma(\mathbb{Z}_p\times\mathbb{Z}_{q^2})) &= \sum_{u,v\in V(\Gamma(\mathbb{Z}_p\times\mathbb{Z}_{q^2})),u\neq v} (\deg(u)+\deg(v))d(u,v)\\ &= (p-1)(p-2)(2q^2-2)+2q(p-1)(q-1)(q^2-q-1)+(q-1)(q-2)\\ &\quad (pq-2)+2(p-1)(q-1)^2(pq-p-q)+(p-1)q(q-1)(p+q^2-2)\\ &\quad +(p-1)(q-1)(pq+q^2-3)+2(p-1)^2(q-1)(q^2+q-2)\\ &\quad +2q(q-1)^2(pq+p-3)+3(p-1)q(q-1)^2(p+q-2)\\ &\quad +(p-1)(q-1)^2(pq+q-3)\\ &= (q-1)(8pq^3-20pq^2+8p^2q^2-2p^2q-8pq-6q^3+7q^2+10q+4).\end{aligned}$$

Hence,

$$Sc(\Gamma(\mathbb{Z}_p\times\mathbb{Z}_{q^2})) = \frac{1}{2}(q-1)(8pq^3-20pq^2+8p^2q^2-2p^q-8pq-6q^3+7q^2+10q+4).$$

This completes the proof. □

An example is provided to enhance the understanding of the theorem above.

Example 3.6. Given $\Gamma(\mathbb{Z}_3\times\mathbb{Z}_{2^2})$ for $p=3$ and $q=2$.

From Figure 6, we have degree and distance between any two distinct vertices.

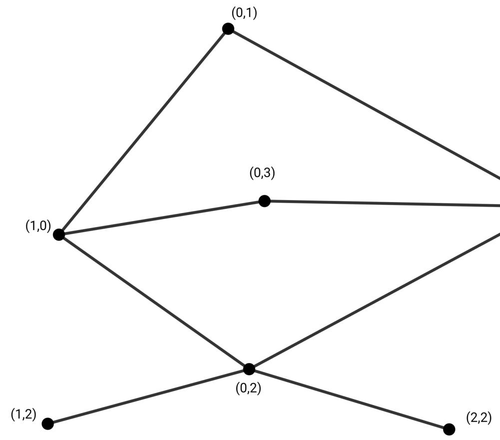

Figure 6. $\Gamma(\mathbb{Z}_3 \times \mathbb{Z}_{2^2})$.

By Definition 2.11, the Schultz index of $\Gamma(\mathbb{Z}_3 \times \mathbb{Z}_{2^2})$ can be computed as follows:

$$\begin{aligned} Sc(\Gamma(\mathbb{Z}_3 \times \mathbb{Z}_{2^2})) &= \sum_{u,v \in V(\Gamma(\mathbb{Z}_3 \times \mathbb{Z}_{2^2})), u \neq v} (\deg(u) + \deg(v))d(u,v) \\ &= (2+3) + (2+4)2 + (2+1)3 + (2+1)3 + (2+3) + (2+2)2 \\ &\quad + (3+2) + (3+3)2 + (3+4) + (3+1)2 + (3+1)2 + (4+1) \\ &\quad + (4+1) + (4+2)2 + (4+3) + (1+1)2 + (1+2)3 + (1+3)2 \\ &\quad + (1+2)3 + (1+3)2 + (2+3) \\ &= 80. \end{aligned}$$

According to Theorem 3.15,

$$Sc(\Gamma(\mathbb{Z}_3 \times \mathbb{Z}_{2^2})) = \frac{1}{2}(2-1)(192 - 240 + 288 - 36 - 48 - 48 + 28 + 20 + 4) = 80.$$

The same results are obtained whether using the definition or the theorem.

4. Conclusions

In this research, we determined several structural properties of $\Gamma(\mathbb{Z}_{p^m} \times \mathbb{Z}_{q^n})$, including its order, size, vertex degrees, diameter, minimal separators, and vertex connectivity, showing that

$$\kappa(\Gamma(\mathbb{Z}p^m \times \mathbb{Z}q^n)) = \min\{p-1, q-1\}.$$

We obtained general formulas for the general zeroth-order Randić index, the eccentric connectivity index, and the Schultz index of this graph. These topological indices may be applied to predict physical properties of compounds isomorphic to the given graph, although such isomers are not addressed in this study.

Author contributions

Nurhabibah: conceptualization, methodology, formal analysis, and writing–original draft; Abdul Gazir Syarifudin: methodology, validation, formal analysis, writing–review and editing; Intan Muchtadi-Alamsyah: supervision, conceptualization, validation, writing–review and editing; Erma Suwastika: investigation, resources, writing–review and editing; Nor Haniza Sarmin: formal analysis, validation, writing–review and editing; Nur Idayu Alimon: investigation, visualization, validation; Ghazali Semil @ Ismail: formal analysis, investigation, writing–review and editing. All authors have read and agreed to the published version of the manuscript.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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