Theory of Shear Strength

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Strength of different materials

- Steel
  - Tensile strength

- Concrete
  - Compressive strength

- Soil
  - Shear strength

Presence of pore water

Complex behavior
SOIL STRENGTH

DEFINITION
Shear strength of a soil is the maximum internal resistance to applied shearing forces.
The maximum or ultimate stress the material can sustain against the force of landslide, failure, etc.

APPLICATION
Soil Strength can be used for calculating:
- Bearing Capacity of Soil
- Slope Stability
- Lateral Pressure
Why it is important???????????
SHEAR STRENGTH OF SOIL

PARAMETER
- Cohesion (c)
- Internal Friction Angle (φ)

CONDITION
- Total (c and φ)
- Effective (c’ and φ’)

GENERAL EQUATION (COULOMB)
\[ \tau = c + \sigma_n \tan \phi \]
SOIL TYPES

- COHESIVE SOIL
  - Has cohesion (c)
  - Example: Clay, Silt

- COHESIONLESS Soil
  - Only has internal friction angle (\(\phi\)) ; c = 0
  - Example: Sand, Gravel
SHEAR STRENGTH PARAMETER

- **COHESION (C)**
  Sticking together of like materials.

- **INTERNAL FRICTION ANGLE (ϕ)**
  The stress-dependent component which is similar to sliding friction of two or more soil particles.
Factors controlling shear strength of soils

- **soil composition (basic soil material):** mineralogy, grain size and grain size distribution, shape of particles, pore fluid type and content, ions on grain and in pore fluid.

- **state (initial):** Define by the initial [void ratio](#), effective normal stress and shear stress (stress history). State can be describe by terms such as: loose, dense, overconsolidated, normally consolidated, stiff, soft, contractive, dilative, etc.

- **structure:** Refers to the arrangement of particles within the soil mass; the manner the particles are packed or distributed. Features such as layers, joints, fissures, slickensides, voids, pockets, cementation, etc, are part of the structure. Structure of soils is described by terms such as: undisturbed, disturbed, remolded, compacted, cemented; flocculent, honey-combed, single-grained; flocculated, deflocculated; stratified, layered, laminated; isotropic and anisotropic.

- **Loading conditions:** Effective, i.e., drained, and undrained; and type of loading, i.e., magnitude, rate (static, dynamic), and time history (monotonic, cyclic)).
Shear failure

Soils generally fail in shear

At failure, shear stress along the failure surface reaches the shear strength.
Shear failure

The soil grains slide over each other along the failure surface.

No crushing of individual grains.
Shear failure

At failure, shear stress along the failure surface ($\tau$) reaches the shear strength ($\tau_f$).
Mohr-Coulomb Failure Criterion

(in terms of total stresses)

\[ \tau_f = c + \sigma \tan \phi \]

\( \tau_f \) is the maximum shear stress the soil can take without failure, under normal stress of \( \sigma \).
Mohr-Coulomb Failure Criterion
(in terms of effective stresses)

\[ \tau_f = c' + \sigma' \tan \phi' \]

\[ \sigma' = \sigma - u \]
\[ u = \text{pore water pressure} \]

\( \tau_f \) is the maximum shear stress the soil can take without failure, under normal effective stress of \( \sigma' \).
Mohr-Coulomb Failure Criterion

Shear strength consists of two components: cohesive and frictional.

\[ \tau_f = C + \sigma_f \tan \phi \]
c and $\phi$ are measures of shear strength.

Higher the values, higher the shear strength.
Mohr Circles & Failure Envelope

Soil elements at different locations

X ~ failure
Y ~ stable
Initially, Mohr circle is a point. The soil element does not fail if the Mohr circle is contained within the envelope.
Mohr Circles & Failure Envelope

As loading progresses, Mohr circle becomes larger...

.. and finally failure occurs when Mohr circle touches the envelope
Shear Failure in Soils
Orientation of Failure Plane

Failure plane oriented at $45 + \phi/2$ to horizontal
Mohr circles in terms of $\sigma$ & $\sigma'$

$$\sigma_v$$

$$\sigma_h$$

$$\sigma_v'$$

$$\sigma_h'$$

$$u$$

$$\sigma_v$$

$$\sigma_h$$

$$u$$

Effective stresses

Total stresses
Envelopes in terms of $\sigma$ & $\sigma'$

Identical specimens initially subjected to different isotropic stresses ($\sigma_c$) and then loaded axially to failure.

Initially...

At failure,

$\sigma_3 = \sigma_c$; $\sigma_1 = \sigma_c + \Delta \sigma_f$

$\sigma_3' = \sigma_3 - u_f$; $\sigma_1' = \sigma_1 - u_f$

Failure

$c, \phi$

in terms of $\sigma$

$c', \phi'$

in terms of $\sigma'$
MOHR COULOMB CONCEPT

\[ \tau = c + \sigma \cdot \tan \phi \]

\[ \sigma_1 = \sigma_3 + \Delta \sigma \]
MOHR COULOMB CONCEPT

\[ \sigma_1 > \sigma_3 \]

(1) \[ \sigma_n = \frac{\sigma_1 + \sigma_3}{2} + \frac{\sigma_1 - \sigma_3}{2} \cdot \cos 2\theta \]

(2) \[ \tau = \frac{\sigma_1 - \sigma_3}{2} \cdot \sin 2\theta \]
(1) and (2) \[ \tau = c + \sigma_n \cdot \tan\phi \]

\[ \sigma_1 = \sigma_3 + \left( \frac{\sigma_3 \cdot \tan\phi + c}{0.5 \cdot \sin 2\theta - \cos^2\theta \cdot \tan\phi}\right) \]

The failure occurs when the value of \( \sigma_1 \) is minimum or the value of \((0.5 \cdot \sin 2\theta - \cos^2\theta \cdot \tan\phi)\) maximum

\[ \theta = 45^\circ + \frac{\phi}{2} \]

\[ \sigma_1 = \sigma_3 \cdot \tan^2\left(45^\circ + \frac{\phi}{2}\right) + 2c \cdot \tan\left(45^\circ + \frac{\phi}{2}\right) \]
Equations……

\[ \tau_f = c + \sigma \tan \phi \]

\[ \theta = 45 + \phi/2 \]

\[ \tau = \frac{\sigma_1 - \sigma_3}{2} \cdot \sin 2\theta \]

\[ \sigma_n = \frac{\sigma_1 + \sigma_3}{2} + \frac{\sigma_1 - \sigma_3}{2} \cdot \cos 2\theta \]

\[ \sigma_n = \sigma_3 + (\sigma_1 - \sigma_3)\cos^2\theta \]

\[ \sigma_1 = \sigma_3 \cdot \tan^2\left(45^\circ + \phi/2\right) + 2c \cdot \tan \left(45^\circ + \phi/2\right) \]
Example 1

A soil failed at $\sigma_3 = 100$ kN/m$^2$ and $\sigma_1 = 288$ kN/m$^2$. If the same soil is given $\sigma_3 = 200$ kN/m$^2$, what is the value of the new $\sigma_1$ when the failure is for:

i) Cohesive soil

ii) Cohesionless soil
Solution 1

Graphically

a. From the Mohr’s circle, 
   \[ \sigma_1 = 388 \text{ kN/m}^2 \]

b. From the Mohr’s circle, 
   \[ \sigma_1 = 580 \text{ kN/m}^2 \]
Solution 1

Analytically

i) Cohesive soil ($\phi = 0^\circ$)

$$\theta = 45 + 0/2 = 45^\circ$$

$$\tau = \frac{\sigma_1 - \sigma_3}{2} \cdot \sin 2\theta$$

$$\tau = \frac{288 - 100}{2} \cdot \sin 2(45) = 94 \text{ kPa}$$

$$\sigma_n = \sigma_3 + (\sigma_1 - \sigma_3) \cos^2 \theta$$

$$\sigma_n = 100 + (288 - 100) \cos^2 45 = 194 \text{ kPa}$$
Solution 1

Analytically

for soil, where \( \sigma_3 = 200 \text{ kPa} \)

\[
\sigma_1 = \sigma_3 \tan^2(45^\circ + \phi/2) + 2c\tan(45^\circ + \phi/2)
\]

\[
\sigma_1 = 200 \tan^2(45 - 0/2) + 2(94)\tan(45)
\]

= ???? kPa
Solution 1

Analytically

i) Cohesionless soil \((c = 0)\)

From graph:

\[ \tau_f = 80 \text{ kPa} \text{ and } \sigma_n = 145 \text{ kPa} \]

\[ \tau_f = c + \sigma \tan \phi \]

\[ \phi = \tan^{-1} \left( \frac{80}{145} \right) = 29^\circ \]

\[ \sigma_1 = \sigma_3 \cdot \tan^2 \left( 45^\circ + \frac{\phi}{2} \right) + 2c \cdot \tan \left( 45^\circ + \frac{\phi}{2} \right) \]

\[ \sigma_1 = 200 \tan^2 \left( 45^\circ - \frac{29}{2} \right) + 0 = 576 \text{ kPa} \]
Example 2

Given:
- $C = 86$ kPa
- $\phi = 17^\circ$
- $\sigma_3 = 70$ kPa
- $\sigma_1 = 346$ kPa

Determine angle of failure ($\theta$), shear stress at failure ($\tau_f$) and normal stress at failure ($\sigma_n$)
Solution 2

\[ \theta = 45 + \phi/2 \]
\[ \theta = 45 + 17/2 = 53.5^\circ \]

\[ \tau = \frac{\sigma_1 - \sigma_3}{2} \cdot \sin 2\theta \]
\[ \tau = (346 - 70) / 2 \cdot \sin 2(53.5) = 132 \text{ kPa} \]

\[ \sigma_n = \sigma_3 + (\sigma_1 - \sigma_3) \cdot \cos^2 \theta \]
\[ \sigma_n = 70 + (346 - 70) \cdot \cos^2 53.5 = 167 \text{ kPa} \]
Example 3

A proposed structure will cause the vertical stress to increase by 60 kN/m² at the 4m depth. Samples taken from a uniform deposit of granular soil are found to have a unit weight of 19.6 kN/m³ and an angle of internal friction of 35°. Assume that the weight of the structure also causes the shearing stress to increase to 52 kN/m² on a horizontal plane at this depth. Does this shearing stress exceed the shearing strength of the soil? (consider also when the water table rise to the ground surface)

Solution 3

\[ \text{Solution} \]

The total vertical pressure due to the structure and soil overburden is:

\[ 60 \text{ kN/m}^2 + 78.4 \text{ kN/m}^2 \text{ overburden pressure} = 138.4 \text{ kN/m}^2 \]

The shearing strength that can be developed by the soil at this depth is:

\[ \tau = (138.4 \text{ kN/m}^2)(\tan 35°) = 96.9 \text{ kN/m}^2 \]

This would indicate that the shear strength of the soil is greater than the imposed shear stress; therefore, a shear failure does not occur (96.9 kN/m² > 52 kN/m²).

If the water table rose to the ground surface, the effective soil overburden pressure would be reduced to about:

\[ \left(\frac{1}{2} \times 19.6 \text{ kN/m}^3\right)(4 \text{ m}) = 39.2 \text{ kN/m}^2 \]

[This value represents the effective vertical stress, i.e., \( \bar{\sigma} = \sigma_i - u = \gamma_{\text{sub}}Z = (\gamma_i - \gamma_w)Z \).]

The total vertical stress would be:

\[ 39.2 \text{ kN/m}^2 + 60 \text{ kN/m}^2 = 99.2 \text{ kN/m}^2 \]

The shear strength available is:

\[ \tau = (99.2 \text{ kN/m}^2)(\tan 35°) = 69.46 \text{ kN/m}^2 \]

This is still greater than the shear stress resulting from the loading conditions; that is, 69.46 kN/m² > 52 kN/m².
Example 4

c) A normally consolidated clay is consolidated under a stress of 150 kPa, then sheared undrained in an axial compression. The principle stress difference at failure is 100 kPa and the induced pore water pressure at failure is 88 kPa.

i) Determine the Mohr Coulomb strength parameters in terms of both total & effective stresses

ii) Compute the principle stress ratio for both total & effective stresses

iii) Determine the theoretical angle of failure plane in the specimen
Solution 4

To use these equations, we need $\sigma_{1f}$, $\sigma_{1f}'$, $\sigma_{3f}$, and $\sigma_{3f}'$. We know $\sigma_{3f} = 150 \text{ kPa}$ and $(\sigma_1 - \sigma_3)_f = 100 \text{ kPa}$. Therefore

$$\sigma_{1f} = (\sigma_1 - \sigma_3)_f + \sigma_{3f} = 100 + 150 = 250 \text{ kPa}$$
$$\sigma_{1f}' = \sigma_{1f} - u_f = 250 - 88 = 162 \text{ kPa}$$
$$\sigma_{3f}' = \sigma_{3f} - u_f = 150 - 88 = 62 \text{ kPa}$$

From Eq. (11.13),

$$\phi' = \arcsin \frac{100}{224} = 26.5^\circ$$
$$\phi_f = \arcsin \frac{100}{400} = 14.5^\circ$$

For the graphical solution, we need to plot the total and effective Mohr circles, and to do this we need to calculate $\sigma_{1f}$, $\sigma_{1f}'$, and $\sigma_{3f}'$. The centers of the circles are at $(200, 0)$ for total stresses and at $(112, 0)$ for effective stresses. The graphical solution including the failure envelopes is shown in Fig. Ex. 12.9.

The stress ratios at failure are

$$\frac{\sigma_1}{\sigma_3} = \frac{162}{62} = 2.61$$
$$\frac{\sigma_1}{\sigma_3} = \frac{250}{150} = 1.67$$

Another way to get these values would be to use Eq. (11.14).

$$\frac{\sigma_1'}{\sigma_3'} = \frac{1 + \sin 26.5^\circ}{1 - \sin 26.5^\circ} = \frac{1.45}{0.55} = 2.61$$
$$\frac{\sigma_1}{\sigma_3} = \frac{1 + \sin 14.5^\circ}{1 - \sin 14.5^\circ} = \frac{1.25}{0.75} = 1.67$$

d. Use Eq. (11.10), in terms of effective stresses:

$$\alpha_f = 45^\circ + \frac{\phi'}{2} = 58^\circ \text{ from the horizontal}$$
Example 5

a) A consolidated undrained (CU) triaxial test was performed on a specimen of saturated clay with a chamber pressure, $\sigma_3 = 2.0 \text{ kg/cm}^2$. At failure,
$\sigma_1 - \sigma_3 = 2.8 \text{ kg/cm}^2$
$u = 1.8 \text{ kg/cm}^2$
$\theta = 57^\circ$ (angle of failure plane)

Calculate:

i) Normal stress, $\sigma$ on the failure surface

ii) Shear stress, $\tau$ on the failure surface

iii) Maximum shear stress on the specimen

iv) If $\phi = 24^\circ$, $c' = 0.80 \text{ kg/cm}^2$, show why the sample failed at $57^\circ$ instead of at the plane of maximum shear stress ($\theta = 45^\circ$)
### Solution 5

\[ \sigma_1 = 2.8 + 2.0 = 4.8 \text{ kg/cm}^2 \]
\[ \sigma_3 = 2.0 \text{ kg/cm}^2 \]

i) \[ \sigma_n = \frac{\sigma_1 + \sigma_3}{2} + \frac{\sigma_1 - \sigma_3}{2} \cos 2\theta \]
\[ = \left( \frac{4.8 + 2.0}{2} \right) + \left( \frac{4.8 - 2.0}{2} \right) \cos 114^\circ \]
\[ = 2.83 \text{ kg/cm}^2 \]

ii) \[ \tau = \frac{\sigma_1 - \sigma_3}{2} \sin 2\theta = \left( \frac{4.8 - 2.0}{2} \right) \sin 114^\circ = 1.27 \text{ kg/cm}^2 \]

iii) \[ \tau_{\text{max}} = \frac{\sigma_1 - \sigma_3}{2} \sin 2\theta = \left( \frac{4.8 - 2.0}{2} \right) \sin 90^\circ = 1.4 \text{ kg/cm}^2 \]

iv) \[ S_{\text{ef}} = c' + \sigma' \tan \phi = 0.8 + \left( 2.83 \right) \tan 34^\circ \]
\[ = 3.27 \text{ kg/cm}^2 \] (Similar to previous \( \tau \))

Now at the plane of maximum shear stress \( \theta = 45^\circ \)

\[ \sigma = \left( \frac{4.8 + 2.0}{2} \right) + \left( \frac{4.8 - 2.0}{2} \right) \cos 90^\circ = 3.4 \text{ kg/cm}^2 \]
\[ \sigma' = \sqrt{3.4^2 - 1.8^2} = 1.6 \text{ kg/cm}^2 \]
\[ S_{\text{ef}} = c' + \sigma' \tan \phi = 0.8 + \left( 1.6 \tan 29^\circ \right) = 1.51 \text{ kg/cm}^2 \]

::: The shear strength at 45° is larger than at 57°, therefore failure does not occur at 57°.