STRESS PATHS
CRITICAL STATE SOIL MECHANICS

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STRESS PATHS

- Diagrams that represent the successive states of stress during both consolidation and shearing stage
- Can be plotted for all types of loading (UU, CU & CD)
- Considered for different types of loading condition (i.e. drained & undrained) and either effective or total stress

\[ q = \sigma_1 - \sigma_3 \]
\[ p = \frac{1}{3} (\sigma_1 + 2\sigma_3) \]
\[ t = \frac{1}{2} (\sigma_1 + \sigma_3) \]
\[ s = \frac{1}{2} (\sigma_1 - \sigma_3) \]
Total stress paths for shear plotted on p-q diagrams & alternate modified Mohr-Coulomb diagrams – CU TEST

\[ q = \frac{\sigma_1 - \sigma_3}{2} \]

\[ p = \frac{\sigma_1 + \sigma_3}{2} \]

a. p-q Diagram

b. Alternate modified Mohr-Coulomb diagram
Effective stress paths for shear plotted on p-q diagrams – CU TEST

\[ q = \frac{\sigma_1 - \sigma_3}{2} \]
\[ p' = \frac{\sigma_1' + \sigma_3'}{2} \]

a. Normally consolidated soil

\[ q = \frac{\sigma_1 - \sigma_3}{2} \]
\[ p' = \frac{\sigma_1' + \sigma_3'}{2} \]

b. Heavily overconsolidated soil
Effective stress paths for shear plotted on alternate modified Mohr-Coulomb diagrams – CU TEST

1. Normally consolidated soil

2. Heavily overconsolidated soil
Shear strength parameters \((c \& \phi)\)

\[ q = \frac{\sigma_1 - \sigma_3}{2} \]

\[ p = \frac{\sigma_1 + \sigma_3}{2} \]

\[ \phi = \arcsin(\tan \psi) \]

\[ c = \frac{d}{\cos \phi} \]

**a. p-q diagram**

\[ \phi = \arcsin\left(\frac{\tan \psi'}{2 + \tan \psi'}\right) \]

\[ c = \frac{d'(1 - \sin \phi)}{2 \cos \phi} \]

**b. Alternate modified Mohr-Coulomb diagram**
As similar to undrained case, the failure point also falls into similar curve line as normal compression line (CNL).

The total stress path is along 45° degree line extending from horizontal to the failure envelope (CSS).

If the stress decreases before failure, the stress path will move back along the initial path.

* The graph is a straight line.

For $q$-$p'$ plane, the graph is a straight line that sloping to the horizontal at $\tan^{-1} 3y$.

Points $C_1$, $C_2$, $C_3$ represents the failure point after shored in drained void ratio at failure is less than 0.3.
Results for undrained test for normally consolidated clay as shown in table. Given $G_s = 2.65$. Calculate $p_f'$ and $v$ and plot:

i) $q$ vs $p'$

ii) $v$ vs $p'$

iii) Show NCL and CSL

Solution…

*Calculate $\sigma_1$, $\sigma_1'$, $\sigma_3'$ for each sample, then calculate $p_f'$ and $v$*
### Table

<table>
<thead>
<tr>
<th>Confining pressure ($\sigma_3$)</th>
<th>Axial stress at failure, $q$ ($\sigma_1 - \sigma_3$)</th>
<th>Pore water pressure at failure ($u_f$)</th>
<th>Pore water pressure at failure ($w_f$)</th>
<th>Moisture contents at failure ($e$)</th>
<th>Moisture contents at failure ($w$)</th>
<th>Moisture contents at failure ($v = 1 + e$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>103.4</td>
<td>68.3</td>
<td>50.3</td>
<td>25.1</td>
<td>171.7</td>
<td>121.4</td>
<td>53.1</td>
</tr>
<tr>
<td>206.9</td>
<td>119.3</td>
<td>113.8</td>
<td>23</td>
<td>326.2</td>
<td>212.4</td>
<td>93.1</td>
</tr>
<tr>
<td>310.3</td>
<td>172.4</td>
<td>171.7</td>
<td>21.5</td>
<td>482.7</td>
<td>311</td>
<td>138.6</td>
</tr>
<tr>
<td>413.7</td>
<td>224.8</td>
<td>227.5</td>
<td>20.3</td>
<td>638.5</td>
<td>411</td>
<td>186.2</td>
</tr>
<tr>
<td>827.4</td>
<td>468.9</td>
<td>458.5</td>
<td>18.5</td>
<td>1296.3</td>
<td>837.8</td>
<td>368.9</td>
</tr>
</tbody>
</table>

### Diagram

- **CSL = V vs $p'$**
- **NCL = V vs $\sigma_3$**
CRITICAL STATE SOIL MECHANICS (CSSM)

- Is a tool to estimate soil responses when complete characterisation of soil at site is limited (to predict soil’s response from changes in loading during and after construction).

- Incorporates volume changes in its failure criterion (Mohr-Coulomb only defines failure as the attainment of the maximum stress. Failure stress state only is not sufficient to guarantee failure).

- Is an attempt to get a correlation between the shear strength and the void ratio in terms of a model that can be applied to all types of soils.

- The state of soil sample is characterized by 3 parameters:
  - Effective mean stress $p'$
  - Deviatoric (shear stress) $q$, and
  - Specific volume $V$.

  The specific volume is defined as $V = 1 + \varepsilon$, where $\varepsilon$ is the void ratio.
Critical State

- The condition in which a soil has reached a critical void ratio (deformation occurs under constant stress and constant volume)

Critical Void Ratio

- The value of the void ratio for a particular state of compaction of a granular material below which denser material tends to increase in volume when sheared and above which looser material tends to decrease in volume (thus the material will neither expand nor contract when disturb)

Critical State Line (CSL)

- The graph of critical void ratio (or specific volume) plotted against the effective stress under which that void ratio is achieved.

- The CSL lies parallel to the virgin compression line (NCL) and slightly below it.
CRITICAL STATE SOIL MECHANICS (CSSM)

Isotropic Consolidation

- Samples that consolidated under hydrostatic pressure, before the samples are sheared until failure.

- Consist of 2 lines:
  - Virgin compression line / Normal compression line (NCL)
  - Swelling line (SL) (unloading and re-loading lines)

- Any point at line ABC represents the ‘normal consolidation’ while any point at line BD, or below the line ABC represents ‘overconsolidation’.
\( \lambda \) is the slope of NCL (AC)

\( \kappa \) is the slope of SL (BD)

\( N \): is the specific volume of NCL at \( p' = 0 \)

\( V_{\kappa} \): is the specific volume of SL at \( p' = 0 \)

For line AC

\[ V = N - \lambda \ln p' \]

For line BD

\[ V = V_{\kappa} - \kappa \ln p' \]
Example 2 (CSSM)

<table>
<thead>
<tr>
<th>$p'$ (kN/m²)</th>
<th>25</th>
<th>50</th>
<th>100</th>
<th>200</th>
<th>300</th>
<th>400</th>
<th>600</th>
<th>25</th>
</tr>
</thead>
<tbody>
<tr>
<td>Change in volume (ml)</td>
<td>0</td>
<td>0.67</td>
<td>1.39</td>
<td>2.33</td>
<td>4.75</td>
<td>6.54</td>
<td>8.92</td>
<td>5.69</td>
</tr>
</tbody>
</table>

Physical properties of the specimen at $p' = 25$ kN/m²:

$G_s = 2.72 \quad \gamma = 18.6$ kN/m³ \quad w = 34\% \quad Volume = 86.19 ml

(a) Plot the $v/ln p'$ curves and hence determine values for the parameters $\lambda$, $\kappa$, N and $\nu_K$.

Solution....

It is first necessary to establish the specific volume corresponding to $p' = 25$ kN/m².

Dry density,

$$\rho_d = \frac{18.6}{9.81(1 + 0.34)} = 1.415 \text{ Mg/m}^3$$

Specific volume,

$$v = \frac{2.72}{1.415} = 1.922$$

Now, volumetric strain,

$$\varepsilon_v = \frac{\Delta V}{V} = \frac{\Delta v}{v}$$

Therefore,

$$\Delta v = (1.922/86.19)\Delta V$$
Solution....

<table>
<thead>
<tr>
<th>mean normal stress, ( p' ) (kPa)</th>
<th>25</th>
<th>50</th>
<th>100</th>
<th>200</th>
<th>300</th>
<th>400</th>
<th>600</th>
<th>25</th>
<th>25</th>
</tr>
</thead>
<tbody>
<tr>
<td>change in volume, ( \Delta v ) (ml)</td>
<td>0</td>
<td>0.67</td>
<td>1.39</td>
<td>2.33</td>
<td>4.75</td>
<td>6.54</td>
<td>8.92</td>
<td>5.69</td>
<td>5.69</td>
</tr>
<tr>
<td>change in specific volume, ( \Delta \nu )</td>
<td>0.000</td>
<td>0.015</td>
<td>0.031</td>
<td>0.052</td>
<td>0.106</td>
<td>0.146</td>
<td>0.199</td>
<td>0.127</td>
<td>0.127</td>
</tr>
<tr>
<td>specific volume, ( \nu )</td>
<td>1.922</td>
<td>1.907</td>
<td>1.891</td>
<td>1.870</td>
<td>1.816</td>
<td>1.776</td>
<td>1.723</td>
<td>1.795</td>
<td>1.795</td>
</tr>
<tr>
<td>( \ln p' )</td>
<td>3.2</td>
<td>3.9</td>
<td>4.6</td>
<td>5.3</td>
<td>5.7</td>
<td>6.0</td>
<td>6.4</td>
<td>3.2</td>
<td>3.2</td>
</tr>
</tbody>
</table>

\( \nu \) corresponding to \( p' \)

\[ \lambda = 0.134 \]
\[ \kappa = 0.0227 \]

\( N \) = the intercept of the isotropic NCL at \( p' = 1.0 \text{ kN/m}^2 \)
\[ = 1.870 + 0.134 \times \ln 200 = 2.580 \]
\[ v_k \) = the intercept of the swelling-reloading curve at \( p' = 1.0 \text{ kN/m}^2 \)
\[ = 1.922 + 0.224 \times \ln 25 = 1.994 \]
The Critical State Line (CSL)

- If a soil is continuously sheared it will eventually reach a critical state in which further **shear strains can occur with no changes in effective stresses or volume**.

- When a soil is at the critical state:

\[
q = M p'
\]

\[
V = \Gamma - \lambda \ln p'
\]

M and \( \tau \) are constant for particular soil

\( M, N, \Gamma, \kappa, \lambda \) are soil constants

\( p', q, V \) (and \( V_c \)) vary during a test.
The Equation of Critical State Line (CSL)

\[ q = M p' \]
\[ V = \Gamma - \lambda \ln p' \]

The equation may be written as:

\[ \log_e p' = \frac{\Gamma - \nu}{\lambda} \]

or

\[ p' = \exp \left( \frac{\Gamma - \nu}{\lambda} \right) \]

Hence the CSL is the line that fulfills both equations:

\[ q = Me^{\frac{\Gamma - \nu}{\lambda}} \]

\[ n = \frac{6s \tan \phi_c'}{3 - 5s \tan \phi_c'} \]

\[ q_f = M p_f' = \frac{3M p_o'}{3 - M} \]
The Equation of Critical State Line – Undrained Test

- No volume change occurs, thus $\Delta V = 0$

- The void ratio at failure, $e_f$ is similar after consolidation

$$V_0 = V_f$$

The equation may be written as:

$$e_f = e_o = e_\Gamma - \lambda \ln p_f'$$

$$p' = \text{exponent} \frac{\Gamma - \nu}{\lambda}$$

$$q = M p' = M \text{exponent} \frac{\Gamma - \nu}{\lambda}$$
Example 3 (CSSM- Undrained test)

A sample of weald clay was consolidated in a triaxial cell with cell pressure of 200 kPa, then sheared in undrained condition. Determine the values of \( q \), \( p' \) and \( \nu \) at failure. Given \( M = 0.85 \), \( \tau = 2.09 \), \( N = 2.13 \), \( \lambda = 0.10 \)

Solution....
The Equation of Critical State Line – Drained Test

- It is known that the projection of drained path at q:p plane is a straight line which sloping to $\tan^{-1} 3$ to horizontal. Thus:

\[ q_f = M p_f' \]

\[ = q_f = 3(p'_f - p'_o) \]

\[ p'_f = \frac{3p'_o}{3 - M} \]

The equation may be written as:

\[ q_f = M p'_f = \frac{3M p'_o}{3 - M} \]

Also the specific volume at failure ($\nu_f$), could be calculated as:

\[ \nu_f = T - \lambda \log e \left( \frac{3p'_o}{3-M} \right) \]
Example 4 (CSSM- Drained test)

A clay sample was isotropically consolidated to a pressure of 350 kPa. The sample is then sheared in drained condition. Determine the values of $q$, $p'$ and $\nu$ at failure if the characteristics of the soil are:

$M = 0.89$, $\tau = 2.76$, $N = 2.87$, $\lambda = 0.16$

**Solution....**

\[
\begin{align*}
\text{Drained Test:} \\
\sigma' = M \sigma' &= 3 (p'_f - p'_o) \\
\text{i) } p'_f &= \frac{3}{3-M} \frac{p'_o}{\frac{3}{3-M}} \\
&= \frac{3(350)}{3-0.89} \\
&= 498 \text{ kPa} \\
\text{ii) } q_f &= M p'_f \\
&= 0.89 (498) \\
&= 443 \text{ kPa} \\
\text{iii) } \nu_f &= \frac{1}{2} - \frac{\lambda \log_e \left( \frac{3p_o}{3-M} \right)}{3M} \\
&= \frac{2.76 - 0.16 \log_e 498}{2.87} \\
&= 2.33$
\]
Summary of Input Parameters for Cam-Clay and Modified Cam-Clay Materials

Specification of Cam-Clay and Modified Cam-Clay models requires five material parameters. These parameters are outlined below.

1. \( \lambda \) – the slope of the normal compression (virgin consolidation) line and critical state line (CSL) in \( v - \ln p' \) space

2. \( \kappa \) – the slope of a swelling (reloading-unloading) line in \( v - \ln p' \) space

3. \( M \) – the slope of the CSL in \( q - p' \) space

4. \( \begin{cases} N & \text{the specific volume of the normal compression line at unit pressure} \\ \text{or} \\ \Gamma & \text{the specific volume of the CSL at unit pressure} \end{cases} \)

5. \( \begin{cases} \mu & \text{Poisson’s ratio} \\ \text{or} \\ G & \text{shear modulus} \end{cases} \)

The initial state of consolidation of such materials must also be specified. This is accomplished by indicating

\( \begin{cases} \text{OCR} & \text{the overconsolidation ratio: the ratio of the previous maximum mean stress to the current mean stress} \\ \text{or} \\ p_c & \text{the preconsolidation pressure} \end{cases} \)
A series of drained and undrained triaxial compression tests yielded the following results at point of failure:

<table>
<thead>
<tr>
<th>Test No. (D = drained, U = undrained)</th>
<th>D1</th>
<th>U1</th>
<th>D2</th>
<th>U2</th>
<th>D3</th>
<th>U3</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cell pressure, $\sigma_3$ (kPa)</td>
<td>120</td>
<td>120</td>
<td>200</td>
<td>200</td>
<td>400</td>
<td>400</td>
</tr>
<tr>
<td>Total axial stress, $\sigma_1$ (kPa)</td>
<td>284</td>
<td>194</td>
<td>493</td>
<td>320</td>
<td>979</td>
<td>645</td>
</tr>
<tr>
<td>Pore pressure at failure, $u_f$ (kPa)</td>
<td>0</td>
<td>69</td>
<td>0</td>
<td>117</td>
<td>0</td>
<td>230</td>
</tr>
<tr>
<td>Specific volume, $\nu_f$</td>
<td>1.80</td>
<td>1.97</td>
<td>1.70</td>
<td>1.86</td>
<td>1.54</td>
<td>1.72</td>
</tr>
</tbody>
</table>

Plot the critical state line and obtain the critical state parameters $M$, $\Gamma$ and $\lambda$.

Solution....

<table>
<thead>
<tr>
<th>test</th>
<th>D1</th>
<th>U1</th>
<th>D2</th>
<th>U2</th>
<th>D3</th>
<th>U3</th>
</tr>
</thead>
<tbody>
<tr>
<td>cell pressure, $\sigma_3$ (kPa)</td>
<td>120</td>
<td>120</td>
<td>200</td>
<td>200</td>
<td>400</td>
<td>400</td>
</tr>
<tr>
<td>total axial stress, $\sigma_1$ (kPa)</td>
<td>284</td>
<td>194</td>
<td>493</td>
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<td>979</td>
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<td>pore pressure at failure, $u_f$ (kPa)</td>
<td>0</td>
<td>69</td>
<td>0</td>
<td>117</td>
<td>0</td>
<td>230</td>
</tr>
<tr>
<td>specific volume, $\nu_f$</td>
<td>1.8</td>
<td>1.97</td>
<td>1.70</td>
<td>1.86</td>
<td>1.54</td>
<td>1.72</td>
</tr>
<tr>
<td>effective cell pressure, $\sigma_3'$ (kPa)</td>
<td>120</td>
<td>51</td>
<td>200</td>
<td>83</td>
<td>400</td>
<td>170</td>
</tr>
<tr>
<td>effective total axial stress, $\sigma_1'$ (kPa)</td>
<td>284</td>
<td>125</td>
<td>493</td>
<td>203</td>
<td>979</td>
<td>415</td>
</tr>
<tr>
<td>mean effective stress, $p'$</td>
<td>175</td>
<td>76</td>
<td>298</td>
<td>123</td>
<td>593</td>
<td>252</td>
</tr>
<tr>
<td>$\ln p'$</td>
<td>5.16</td>
<td>4.33</td>
<td>5.70</td>
<td>4.81</td>
<td>6.39</td>
<td>5.53</td>
</tr>
<tr>
<td>difference in stress, $q$ (kPa)</td>
<td>164</td>
<td>74</td>
<td>293</td>
<td>120</td>
<td>579</td>
<td>245</td>
</tr>
</tbody>
</table>
Solution....

Undrained supposedly curve

Drained test

$M = 1.298$

$\ln p' = 0.15$

$\tau = 1.84$

$\lambda = 0.15$
Determination of critical state soil parameters from - OEDOMETER TEST -

According to Cam-clay theory, one-dimensional loading also gives a “λ - line” in a (V, ln p’) plot.

Also, it is easy to show that:

\[ \lambda = \frac{C_e}{2.3} \]

\[ \kappa = \frac{C_s}{2.3} \]

where \( C_e \) and \( C_c \) are obtained by the standard interpretation of the oedometer test.

(NB: \( \ln (10) = 2.303 \))
Determination of critical state soil parameters from - OEDOMETER TEST-

“conventional” plot $e, \log(\sigma'_v)$

\[ \nu = 1 + e \]

$\log \sigma'_v$

$\sigma'_v \text{ max} \rightarrow p'_c$

CSSM plot $v, \ln p'$

$\lambda$

$\ln p'$
Determination of critical state soil parameters from - INDEX TEST-

These are done to establish the moisture contents corresponding to the Plastic Limit (PL) and Liquid Limit (LL) of the soil.

If we assume that the strength of the soil at the Plastic Limit is 100 times the strength at the Liquid Limit, then:

\[ \lambda = \frac{V_L - V_p}{\ln 100} = \frac{(W_L - W_p)G_s}{\ln 100} \]

or

\[ \lambda = \frac{PI \times G_s}{160} \]

where PI is % Plasticity Index.
THE END