4.7 Simplification of a Force and Couple System

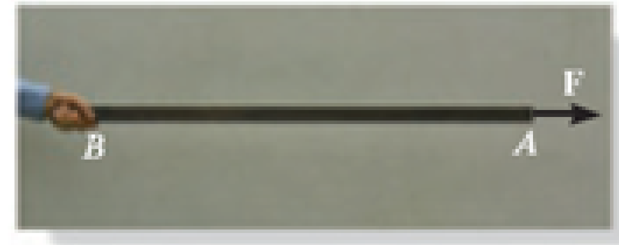
- A system of forces and couple moments acting on a body can be replaced by an ***equivalent system***.

- A system is *equivalent* if the *external effects* it produces on a body are the same as those caused by the original force and couple moment system.

- External effects of a system are :
  - the *translating and rotating motion* of the body if the body is free to move,
  - the *reactive forces* at the supports if the body is held fixed

Moving a force from one point to another along its line of action

- The stick is subjected to the force **F** at *A*.



(a)

- A pair of *equal but opposite* forces **F** & – **F** is applied at *B,* which is on the line of action at **F**.

(b)

- – **F** at *B* and **F** at *A* will cancel each other, leaving only **F** at *B*.

- Force **F** has been moved from *A* to *B* without modifying its *external effects* on the stick.

(c)

- **F** is a *sliding vector* and it can be applied at any point along its line of action **(*Principle of Transmissibility*)**.

Moving a force to another point that is not along its line of action

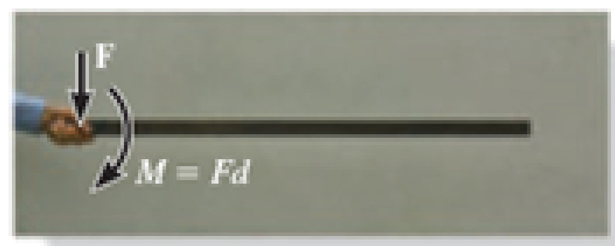
- The stick is subjected to the force **F** which is perpendicular to it at *A*.

(a)

- A pair of *equal but opposite* forces **F** & – **F** is applied at *B*.

(b)

- – **F** at *B* and **F** at *A* form a couple that produce the couple moment $M = Fd$.
- Force **F** has been moved from *A* to *B* without modifying its *external effects* on the stick.



(c)

System of Forces and Couple moments

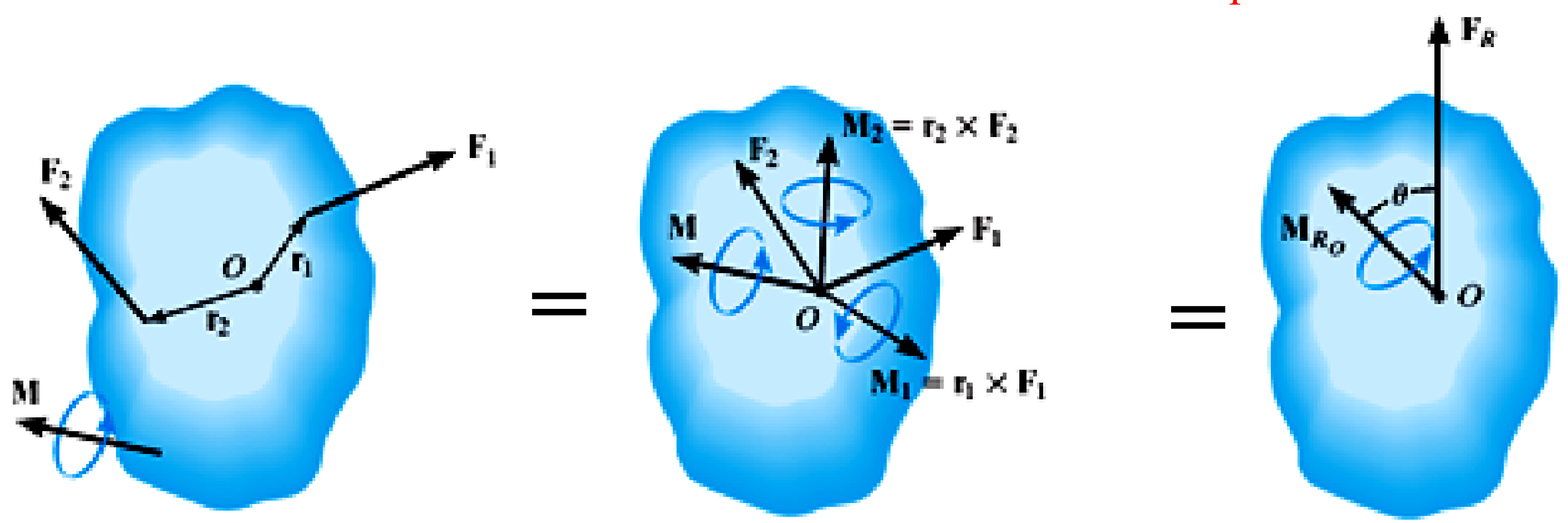
- A system of several forces and couple moments acting on a body can be reduced to
  - an equivalent *single* resultant force acting at a point *O*, &
  - a resultant couple moment

$$\mathbf{F_R} = \sum \mathbf{F}$$

$$(\mathbf{M}_R)_O = \sum \mathbf{M}_O + \sum \mathbf{M}$$

Sum of moments of all the forces about *O*

Sum of all the couple moments



- If the force system lies in the *x–y* plane and any couple moments are perpendicular to this plane, then the previous equations reduce to the following 3 scalar equations.

$$(F_R)_x = \sum F_x$$

$$(F_R)_y = \sum F_y$$

$$(M_R)_O = \sum M_O + \sum M$$

❑ Procedure for simplifying a force & couple system to an equivalent resultant force-couple system.

1. **Establish the coordinate axes** with the origin located at point *O* and the axes having a selected orientation.

2. **Force Summation**

   - For *coplanar* force system, resolve each force into its *x* & *y* components before summing the forces.
   - For 3-D problems, express each force as a Cartesian vector before summing the forces.

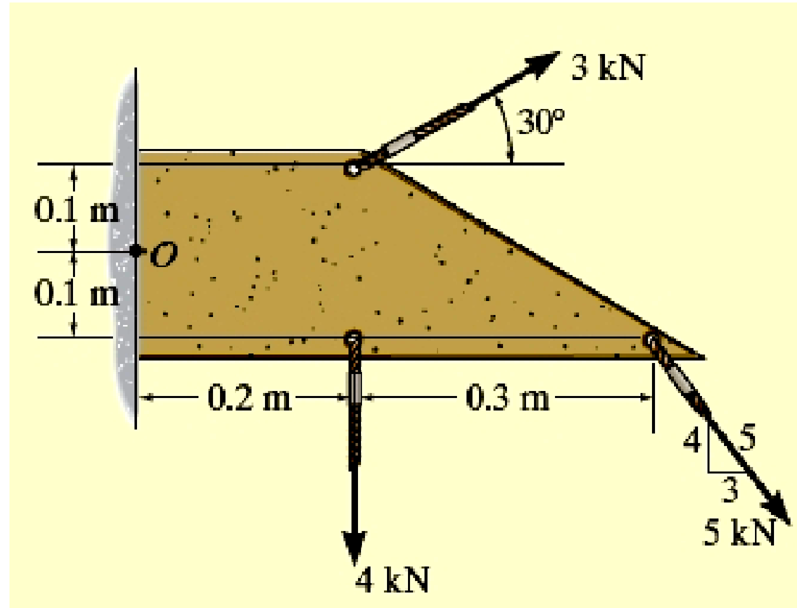
3. **Moment Summation**

   - For *coplanar* force system, use the principle of moments to determine the moments.
   - For 3-D problems, use the vector cross product to determine the moments

Example 4.14

Given :

A system of forces is given as shown.



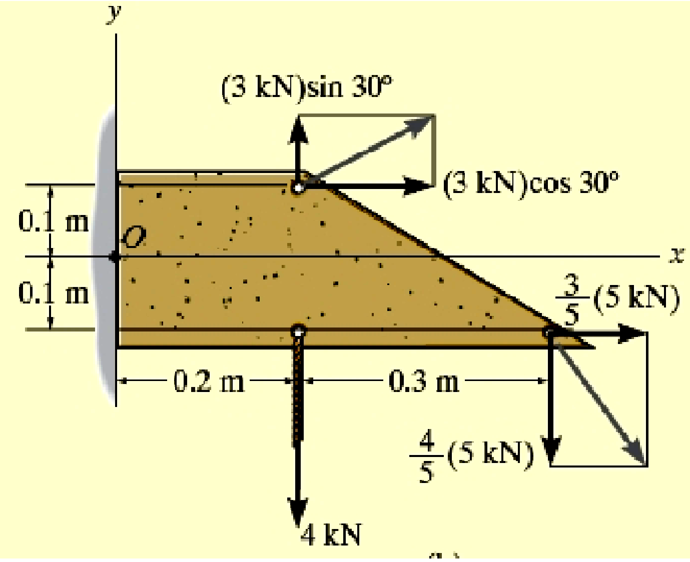
Find :

Replace the force and couple system by an equivalent resultant force and couple moment acting at point *O*.

Solution

❑ Force Summation

- Resolve each force into its *x* & *y* components.



$\overset{+}{\rightarrow}$ $(F_R)_x = \sum F_x$:

$(F_R)_x = (3\text{ kN})\cos 30^o + (5\text{ kN})(3/5)$

$= 5.598 \rightarrow$

$+\uparrow$ $(F_R)_y = \sum F_y$:

$(F_R)_y = (3\text{ kN})\sin 30^o - (5\text{ kN})(4/5) - (4\text{ kN})$

$= -6.50\text{ kN}$

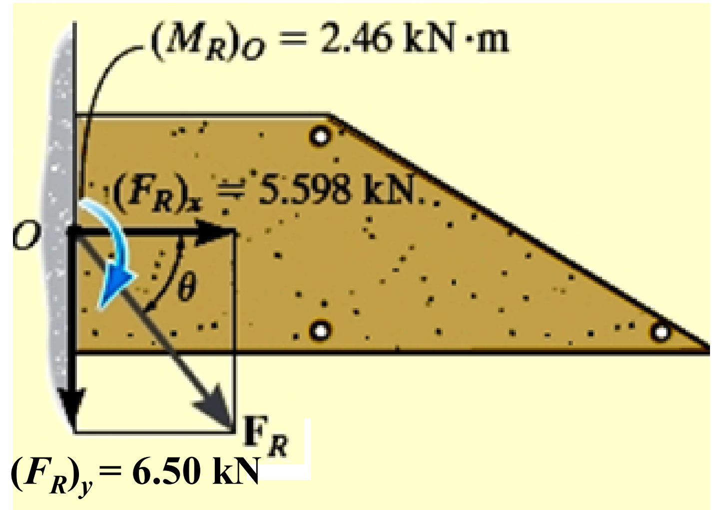
$= 6.50\text{ kN} \downarrow$

- Magnitude of the resultant force $\mathbf{F}_R$

$$F_R = \sqrt{[(F_R)_x]^2 + [(F_R)_y]^2}$$

$$= \sqrt{(5.598)^2 + (6.50)^2} \text{ kN}$$

$$= 8.58 \text{ kN}$$



- Direction of the resultant force $\mathbf{F}_R$

$$\theta = \tan^{-1}\left(\frac{(F_R)_y}{(F_R)_x}\right)$$

$$= \tan^{-1}\left(\frac{6.50 \text{ kN}}{5.598 \text{ kN}}\right)$$

$$= 49.3^\circ$$

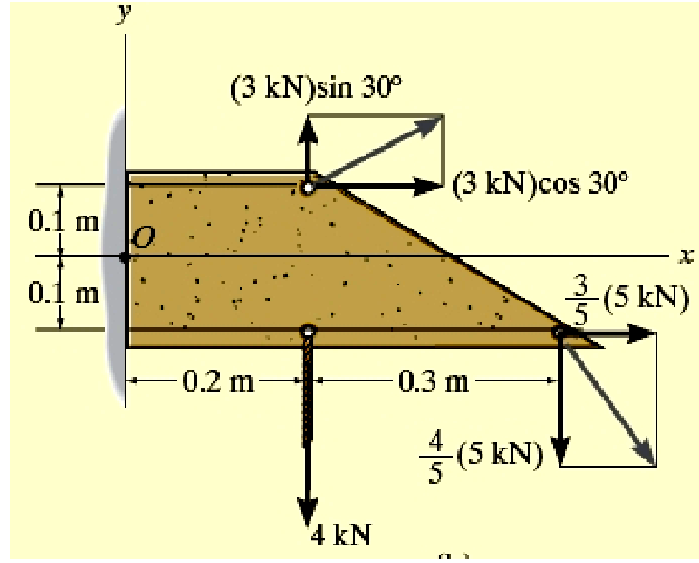
❑ Moment Summation

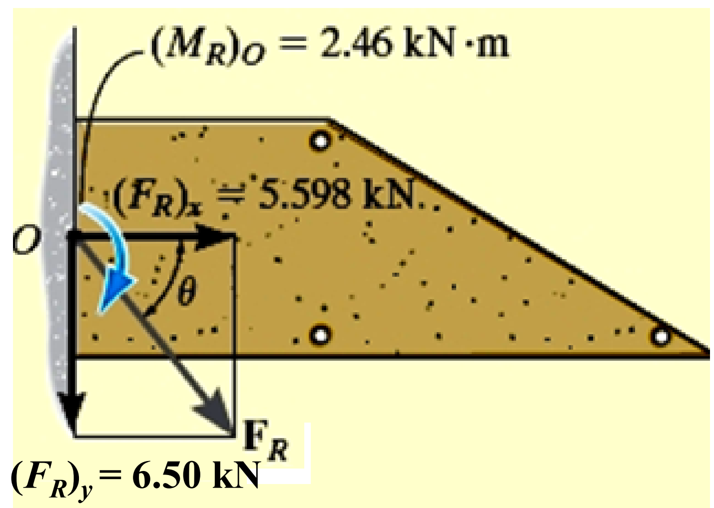
$\circlearrowleft + \quad (M_R)_O = \Sigma M_O :$

$$(M_R)_O = (3 \text{ kN} \sin 30^\circ)(0.2 \text{ m}) - (3 \text{ kN} \cos 30^\circ)(0.1 \text{ m}) + [5 \text{ kN}(3/5)](0.1 \text{ m}) - [5 \text{ kN}(4/5)](0.5 \text{ m}) - (4 \text{ kN})(0.2 \text{ m})$$

$$= -2.46 \text{ kN·m}$$

$$= 2.46 \text{ kN·m} \circlearrowright$$

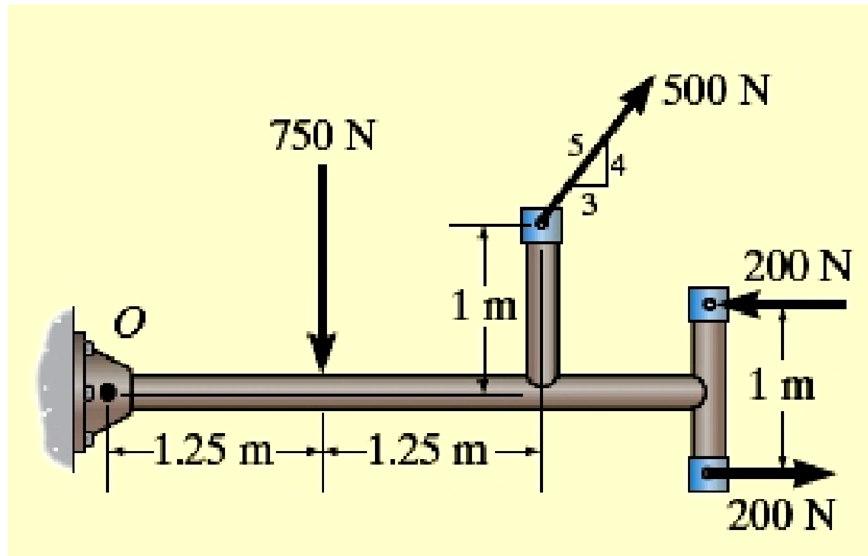




Example 4.15

Given :

A force and couple system is acting on the member as shown.



Find :

Replace the force and couple system by an equivalent resultant force and couple moment acting at point *O*.

Solution

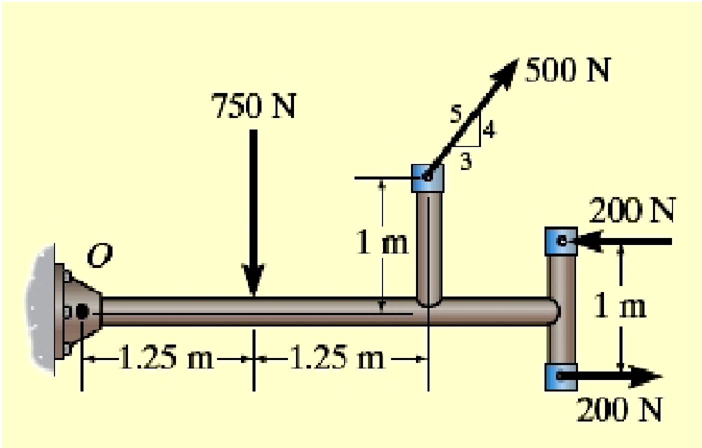
❑ Force Summation

- Resolve the 500-N force into its *x* & *y* components.
- The couple forces of 200 N need not be considered as their resultant is zero.

$\overset{+}{\rightarrow}$ $(F_R)_x = \sum F_x$:

$(F_R)_x = (500\text{ N})\,(3/5)$

$= 300\text{ N} \rightarrow$



$+\uparrow$ $(F_R)_y = \sum F_y$:

$(F_R)_y = (500\text{ N})\,(4/5) - 750\text{ N}$

$= -\,350\text{ N}$

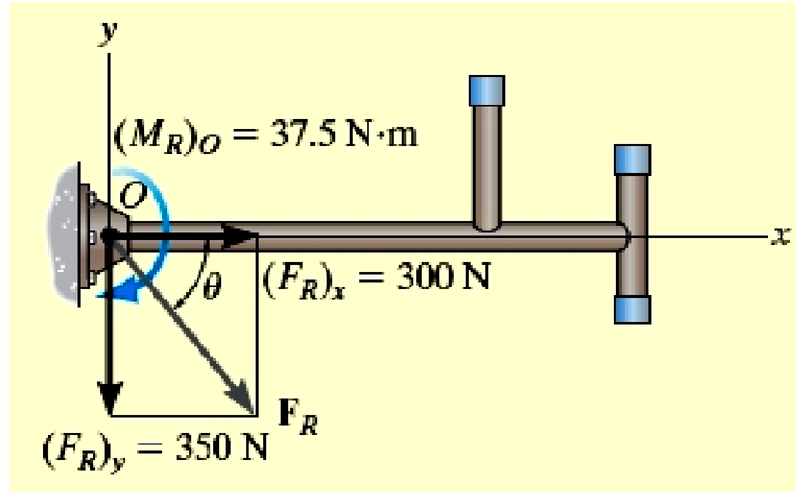
$= 350\text{ kN} \downarrow$

- Magnitude of the resultant force $\mathbf{F}_R$

$$F_R = \sqrt{\left[(F_R)_x\right]^2 + \left[(F_R)_y\right]^2}$$

$$= \sqrt{(300)^2 + (350)^2} \text{ N}$$

$$= 461 \text{ N}$$



- Direction of the resultant force $\mathbf{F}_R$

$$\theta = \tan^{-1}\left(\frac{(F_R)_y}{(F_R)_x}\right)$$

$$= \tan^{-1}\left(\frac{350 \text{ N}}{300 \text{ N}}\right)$$

$$= 49.4^\circ$$

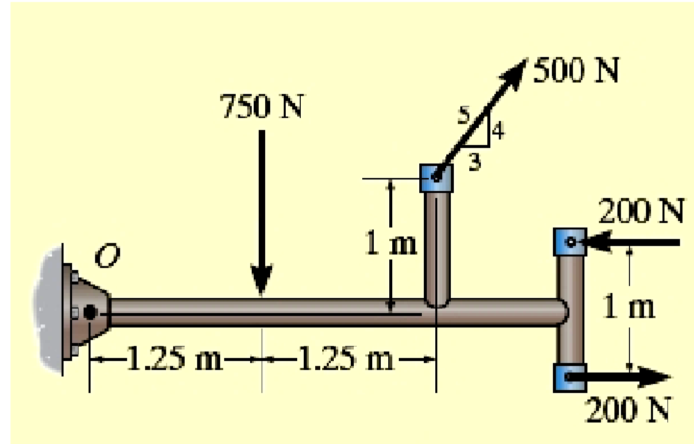
❑ Moment Summation

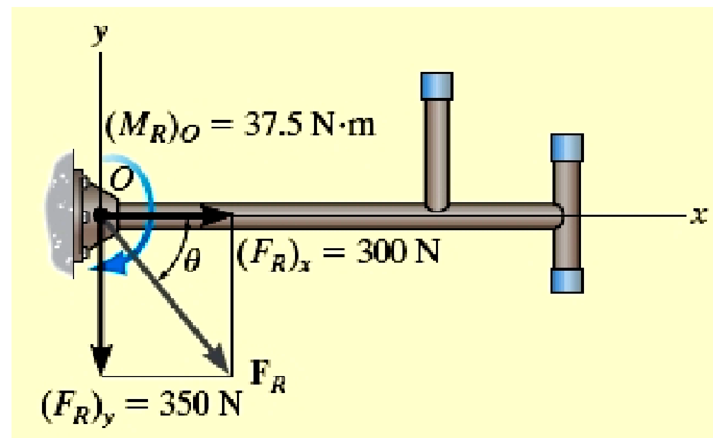
$\circlearrowright +\quad (M_R)_O = \Sigma M_O + \Sigma M_c :$

$$
\begin{aligned}
(M_R)_O = \{ & [(500\text{ N})(4/5)](2.5\text{ m}) \\
& - [(500\text{ N})(3/5)](1\text{ m}) \\
& - (750\text{ N})(1.25\text{ m})\} \\
& + 200\text{ N·m}
\end{aligned}
$$

$$= -37.5\text{ N·m}$$

$$= 37.5\text{ N·m} \circlearrowright$$

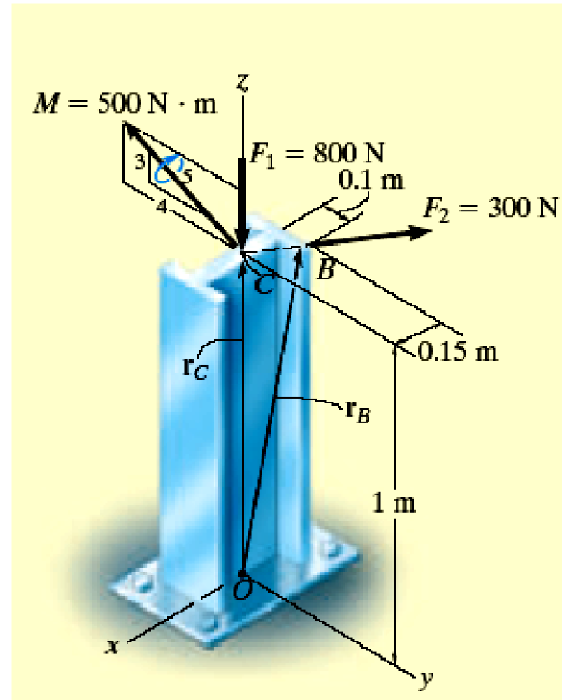




Example 4.16

Given :

A structural member is subjected to a couple moment **M** and forces $\mathbf{F}_1$ and $\mathbf{F}_2$ as shown in the figure.



Find :

Replace this system by an equivalent resultant force and couple moment acting at its base, point *O*.

Solution

- Express the forces and couple moments as Cartesian vectors.

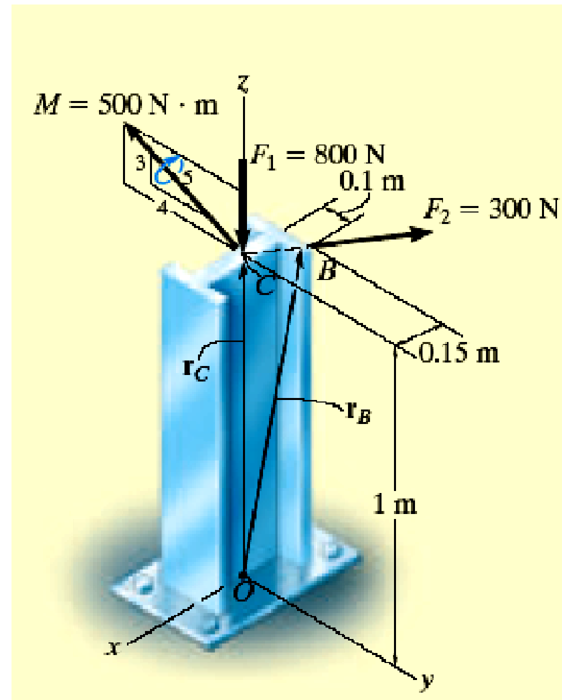
$$\mathbf{F}_1 = \{-800\,\mathbf{k}\}\ \text{N}$$

$$\mathbf{F}_2 = F_2\,\mathbf{u}_{CB}$$

$$= F_2\left(\frac{\mathbf{r}_{CB}}{r_{CB}}\right)$$

$$= (300\ \text{N})\left(\frac{-0.15\mathbf{i}+0.1\,\mathbf{j}}{\sqrt{(-0.15)^2+(0.1)^2}}\right)$$

$$= \{-249.6\mathbf{i}+166.4\mathbf{j}\}\ \text{N}$$



$$\mathbf{M} = \{-500(4/5)\,\mathbf{j} + 500(3/5)\,\mathbf{k}\}\ \text{N·m}$$

$$= \{-400\mathbf{j} + 300\,\mathbf{k}\}\ \text{N·m}$$

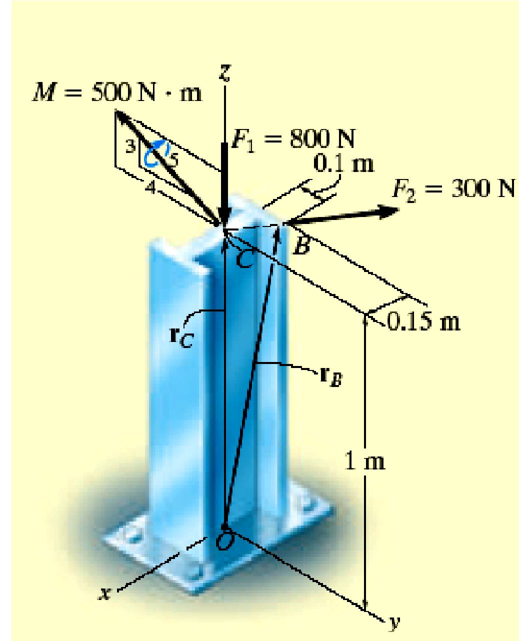
Force Summation

$$\mathbf{F}_R = \sum \mathbf{F}:$$

$$\mathbf{F}_R = \mathbf{F}_1 + \mathbf{F}_2$$

$$= \{-800\,\mathbf{k}\}\text{ N} + \{-249.6\mathbf{i} + 166.4\mathbf{j}\}\text{ N}$$

$$= \{-249.6\mathbf{i} + 166.4\mathbf{j} - 800\,\mathbf{k}\}\text{ N}$$



Moment Summation

$$(\mathbf{M}_R)_O = \Sigma\mathbf{M} + \Sigma\mathbf{M}_O:$$

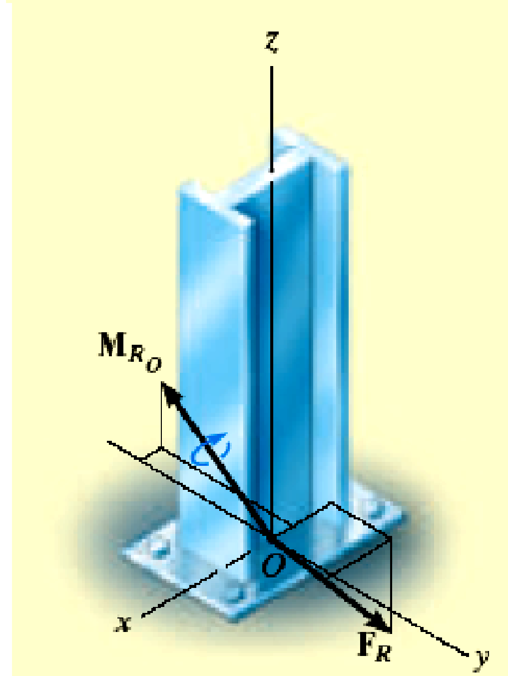
$$(\mathbf{M}_R)_O = \mathbf{M} + (\mathbf{r}_C \times \mathbf{F}_1) + (\mathbf{r}_B \times \mathbf{F}_2)$$

$$= (-400\mathbf{j} + 300\,\mathbf{k})$$

$$+ [1\mathbf{k} \times (-800\,\mathbf{k})]$$

$$+ \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -0.15 & 0.1 & 1 \\ -249.6 & 166.4 & 0 \end{vmatrix}$$

$$= \{-166\mathbf{i} - 650\mathbf{j} + 300\,\mathbf{k}\}\text{ N·m}$$



4.8 Further Simplification of a Force and Couple System

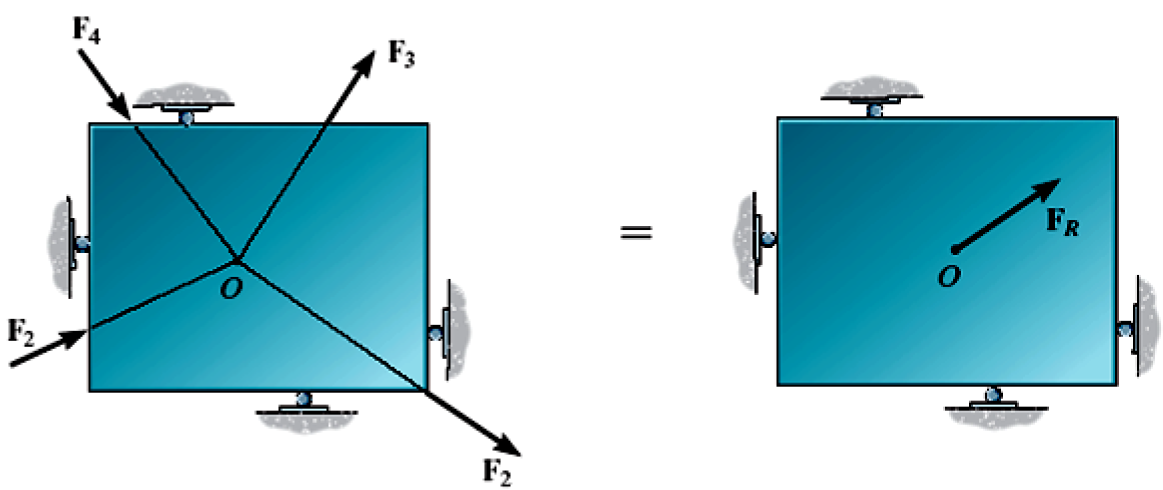
- An equivalent system of single resultant force $\mathbf{F}_R$ acting at a specified point, *O* and a resultant couple moment $(\mathbf{M}_R)_O$ for a force & couple system can be further reduced to *an equivalent single resultant* force if the lines of action of $\mathbf{F}_R$ and $(\mathbf{M}_R)_O$ are perpendicular to each other.

- This condition is satisfied by the following force systems:
  1. Concurrent Force System
  2. Coplanar Force System
  3. Parallel Force System

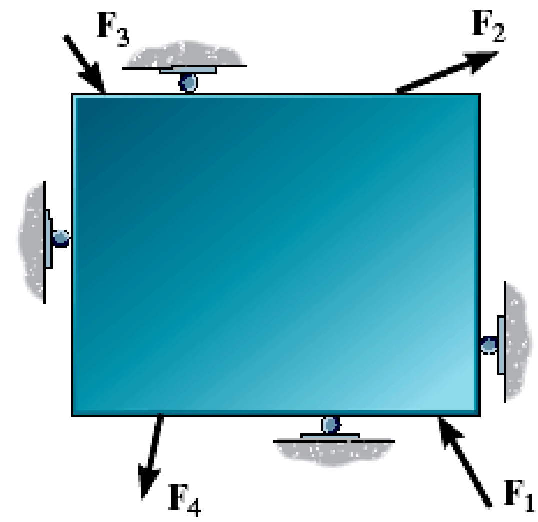
❑ Concurrent Force System

- A *concurrent force system* is one in which the lines of action of all the forces intersect at a common point *O*.
- Since the lines of action of all the forces passes through *O*, the force system produce no moment about *O*.
- The equivalent system can be represented by a single resultant force acting at *O*.

$$\mathbf{F}_R = \sum \mathbf{F}$$



❑ Coplanar Force System



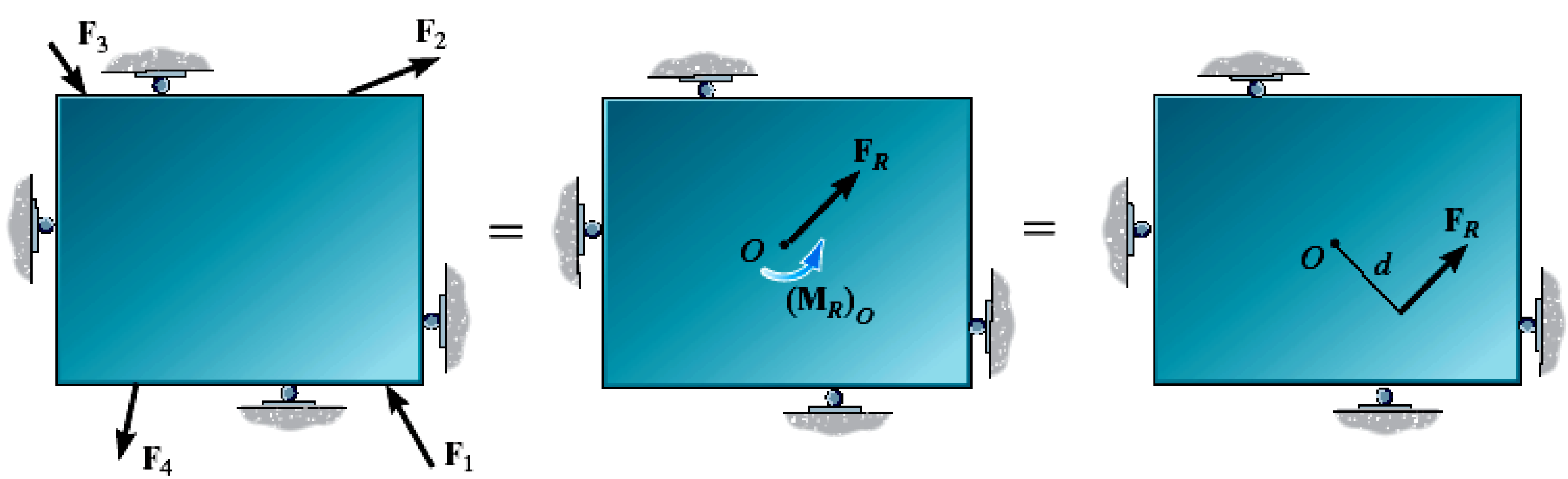
- In a coplanar force system, the lines of action of all the forces lie in the same plane.
- The resultant force $\mathbf{F}_R = \sum \mathbf{F}$ of this system also lies in this plane.
- The moment of each of the forces about any point *O* is directed perpendicular to this plane.
- The resultant moment $(\mathbf{M}_R)_O = \sum \mathbf{M}_O$ and resultant force $\mathbf{F}_R$ are mutually perpendicular.

- The resultant moment can be replaced by moving the resultant force $\mathbf{F}_R$ a perpendicular distance d away from point O.

- The distance d is given by

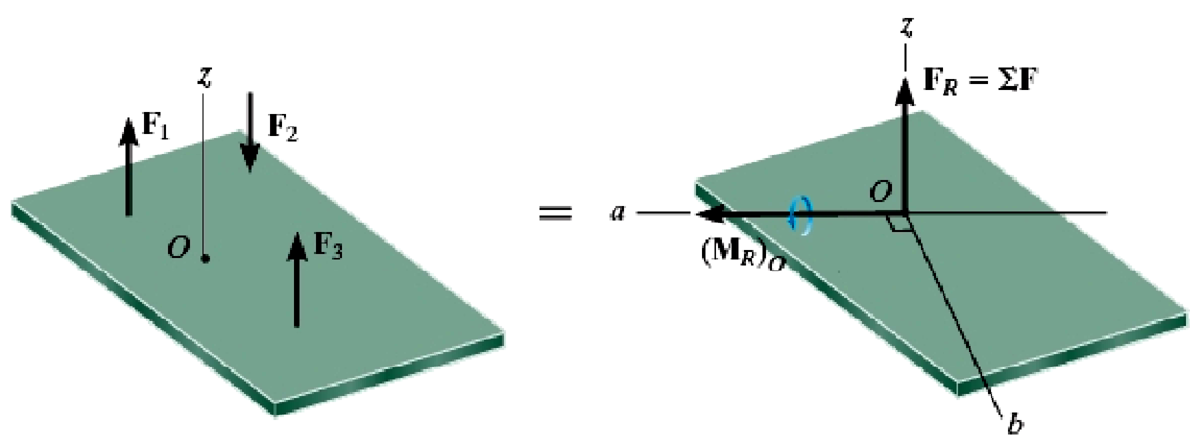
$$(M_R)_O = F_R d$$

$$d = \frac{(M_R)_O}{F_R}$$



❑ Parallel Force System

- The *parallel force system* shown in the figure consists of forces that are all parallel to the *z* axis
- The resultant force $\mathbf{F}_R$ at point *O* must also be parallel to the *z* axis.
- As the moment produced by each force lies in the plane of the plate, the resultant couple moment $(\mathbf{M}_R)_O$ also lies in this plane, along the moment axis *a*.
- The resultant couple moment $(\mathbf{M}_R)_O$ and resultant force $\mathbf{F}_R$ are mutually perpendicular.

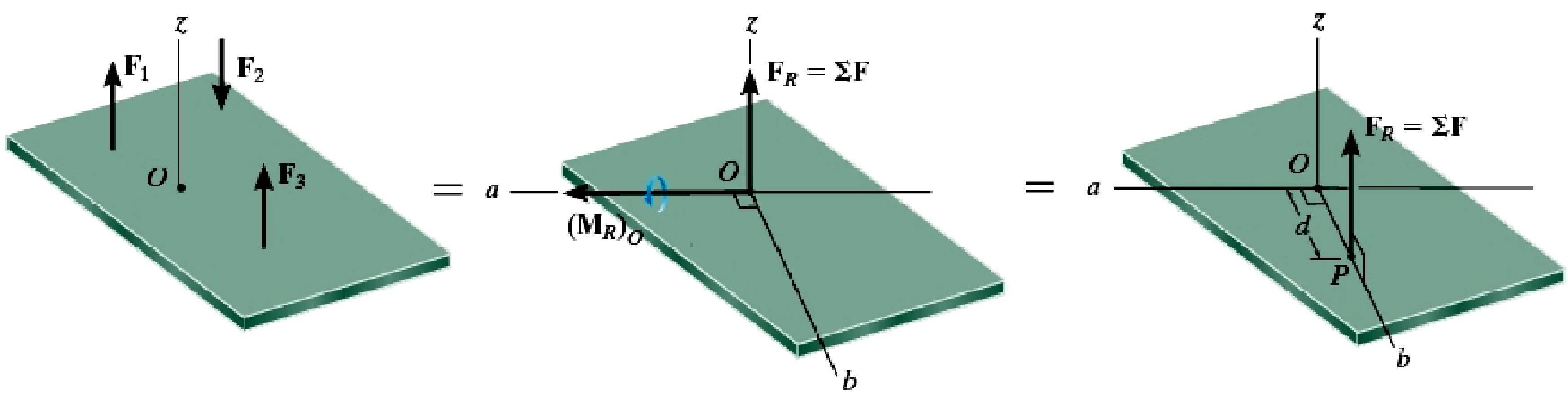


- The equivalent system can be further reduced to an equivalent single resultant force $\mathbf{F}_R$, acting through point P located on the perpendicular b axis.

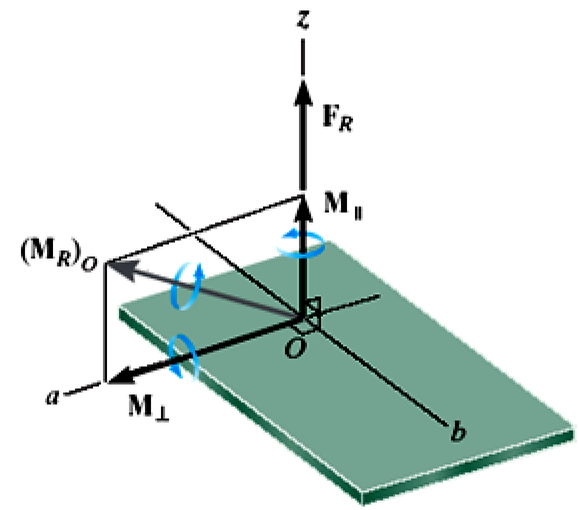
- The distance d is given by

$$(M_R)_O = F_R d$$

$$d = \frac{(M_R)_O}{F_R} = \frac{\Sigma M_O}{F_R}$$



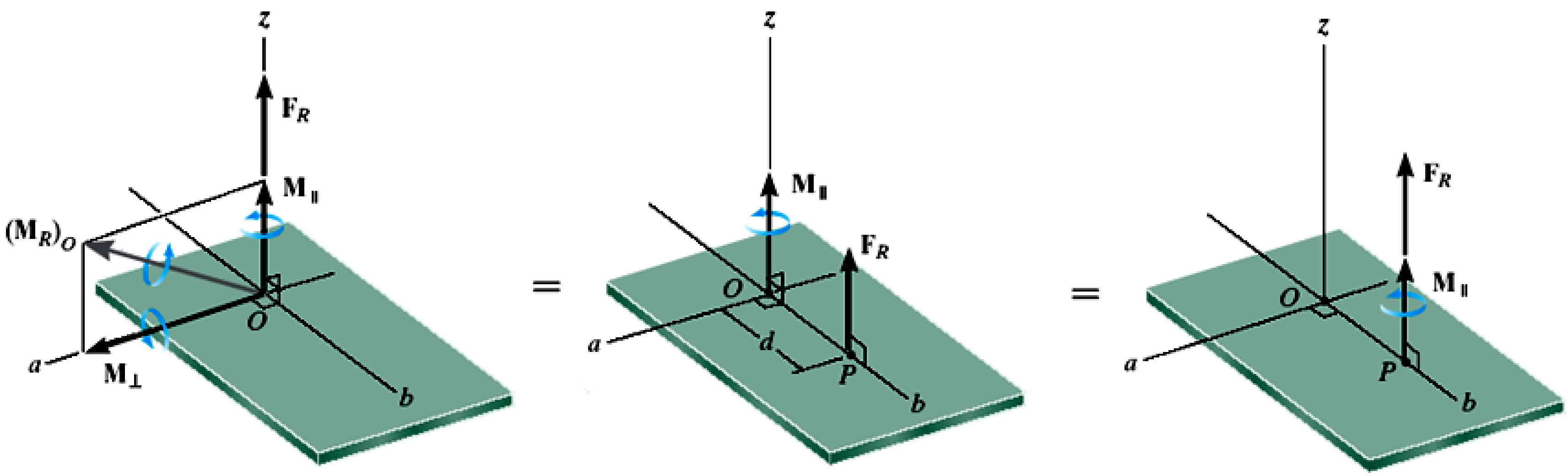
❑ Reduction to a Wrench



- In general, a 3-D force and couple moment system will have an equivalent resultant force $\mathbf{F}_R$ acting at point O and a resultant couple moment $(\mathbf{M}_R)_O$ that are ***not perpendicular*** to one another.
- This kind of force system cannot be further reduced to an equivalent single resultant force.
- However, the resultant couple moment $(\mathbf{M}_R)_O$ can be resolved into components parallel and perpendicular to the line of action of $\mathbf{F}_R$.
- The perpendicular component $\mathbf{M}_\perp$ can be replaced by moving $\mathbf{F}_R$ to point P, a perpendicular distance d away from point O given by

$$d = \frac{M_\perp}{F_R}$$

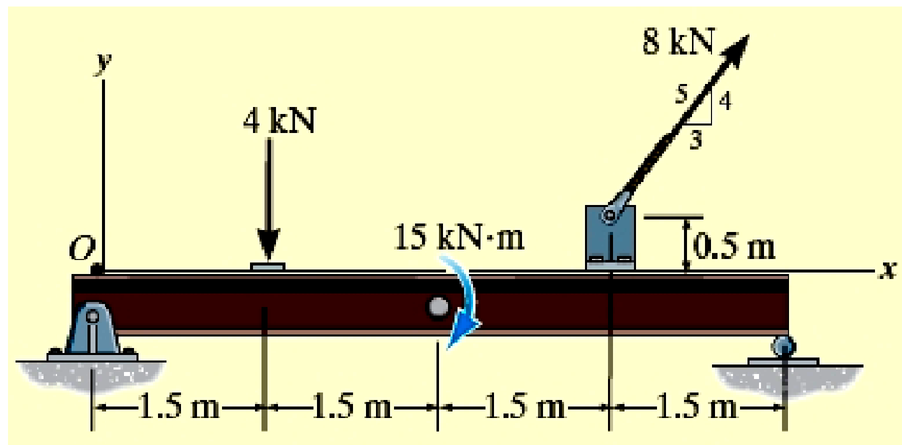
- Since the parallel component $\mathbf{M}_{\parallel}$ is a free vector, it can be moved to point *P*.
- This combination of a resultant force $\mathbf{F}_R$ and collinear couple moment $\mathbf{M}_{\parallel}$ is referred to as a ***wrench*** or a ***screw*** as it tends to translate and rotate the body about its axis.
- A ***wrench*** is the simplest system that can represent any general force and couple moment system acting on a body.



Example 4.17

Given :

The beam is subjected to the force and couple system as shown



Find :

Replace the force and couple system by an equivalent resultant force, and specify where its line of action intersects the beam, measured from point *O*.

Solution

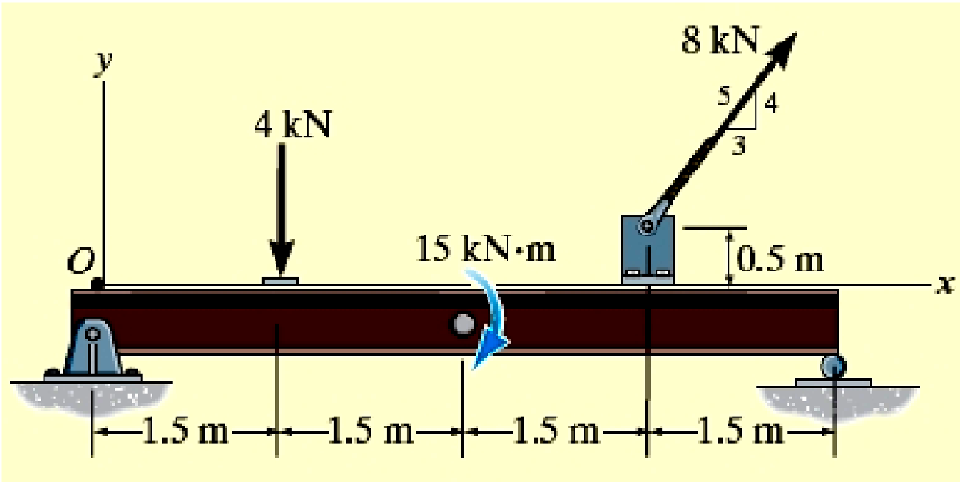
❑ Force Summation

- Resolve the 8-kN force into its *x* & *y* components.

$\overset{+}{\rightarrow}$ $(F_R)_x = \sum F_x$:

$(F_R)_x = 8\text{kN}(3/5)$

$= 4.80 \text{ kN} \rightarrow$



$+\uparrow$ $(F_R)_y = \sum F_y$:

$(F_R)_y = -4 \text{ kN} + 8 \text{ kN}(4/5)$

$= 2.40 \text{ kN} \uparrow$

- The magnitude of the resultant force $\mathbf{F}_R$ is

$$F_R = \sqrt{\left[(F_R)_x\right]^2 + \left[(F_R)_y\right]^2}$$

$$= \sqrt{(4.8)^2 + (2.4)^2}$$

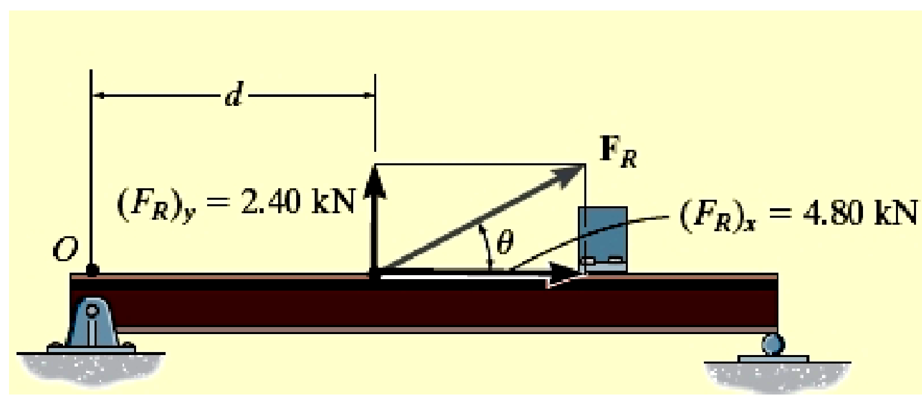
$$= 5.37 \text{ kN}$$

- Direction of the resultant force $\mathbf{F}_R$

$$\theta = \tan^{-1}\left(\frac{(F_R)_y}{(F_R)_x}\right)$$

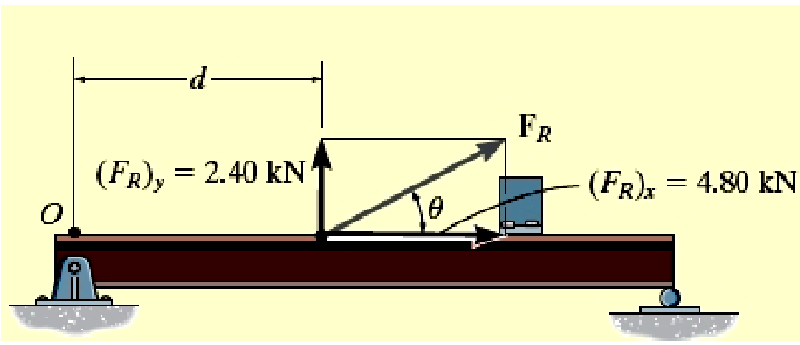
$$= \tan^{-1}\left(\frac{2.4 \text{ N}}{4.8 \text{ N}}\right)$$

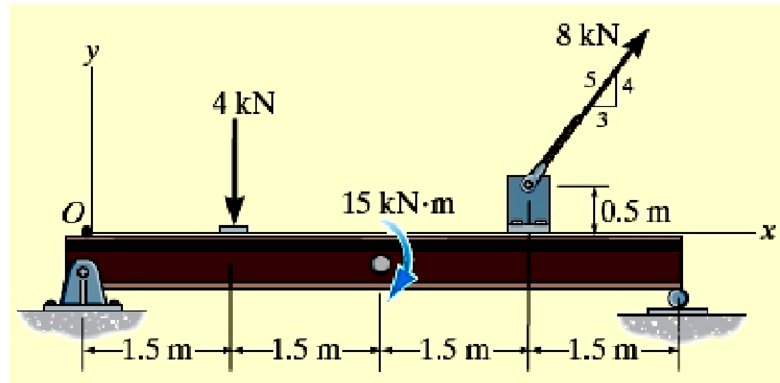
$$= 26.6^{\circ}$$



❑ Moment Summation

- Assuming the line of action of $\mathbf{F}_R$ intersects the beam at a distance d from O.





$$\circlearrowright + \quad (M_R)_O = \Sigma M_O:$$

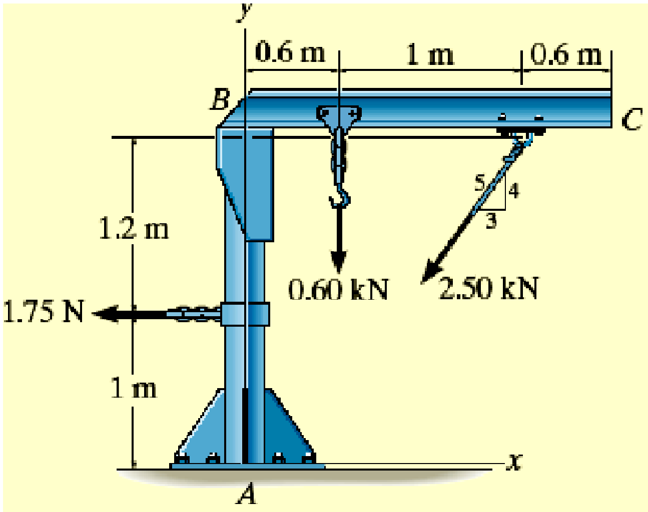
$$\begin{aligned}(2.4)(d)\text{ kN·m} = &- [(4\text{ kN})(1.5\text{ m})] \\ &- 15\text{ kN·m} \\ &- [8\text{ kN}(3/5)(0.5\text{ m})] \\ &+ [8\text{ kN}(4/5)(4.5\text{ m})]\end{aligned}$$

$$d = 2.25\text{ m}$$

Example 4.18

Given :

The jib crane is subjected to three coplanar forces.



Find :

Replace this loading by an equivalent resultant force and specify where the resultant's line of action intersects the column *AB* and boom BC.

Solution

❑ Force Summation

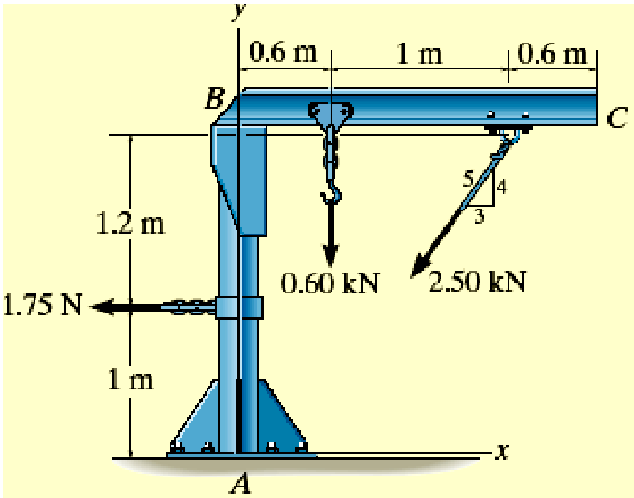
- Resolve the 2.50-kN force into its *x* & *y* components.

$$\overset{+}{\rightarrow} \quad (F_R)_x = \sum F_x:$$

$$(F_R)_x = -\,2.50\text{ kN}(3/5) - 1.75\text{ kN}$$

$$= -\,3.25\text{ kN}$$

$$= 3.25\text{ kN} \leftarrow$$



$$+\uparrow \quad (F_R)_y = \sum F_y:$$

$$(F_R)_y = -\,2.50\text{ kN}(4/5) - 0.60\text{ kN}$$

$$= -\,2.60\text{ kN}$$

$$= 2.60\text{ kN} \downarrow$$

- Magnitude of the resultant force $\mathbf{F}_R$

$$F_R = \sqrt{\left[(F_R)_x\right]^2 + \left[(F_R)_y\right]^2}$$

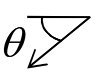
$$= \sqrt{(3.25)^2 + (2.60)^2}\ \text{kN}$$

$$= 4.16\ \text{kN}$$

- Direction of the resultant force $\mathbf{F}_R$

$$\theta = \tan^{-1}\left(\frac{(F_R)_y}{(F_R)_x}\right)$$

$$= \tan^{-1}\left(\frac{2.60\ \text{N}}{3.25\ \text{N}}\right)$$

$$= 38.7^{\circ}$$ θ 

❑ Moment Summation

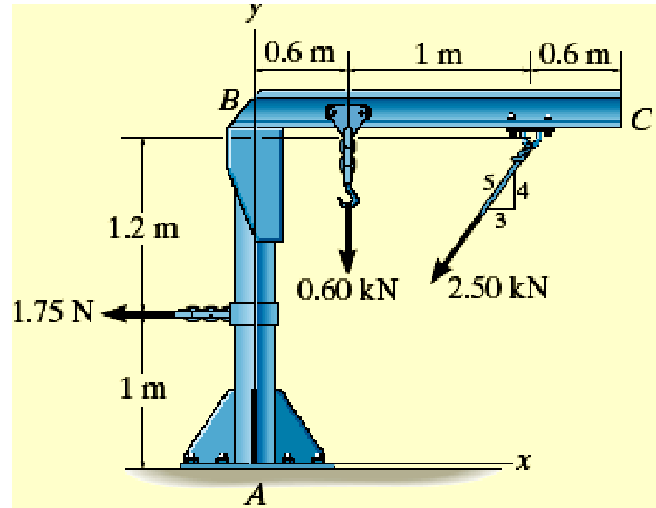
- Assuming the line of action of $\mathbf{F}_R$ intersects *AB* at a distance *y* from *A*.

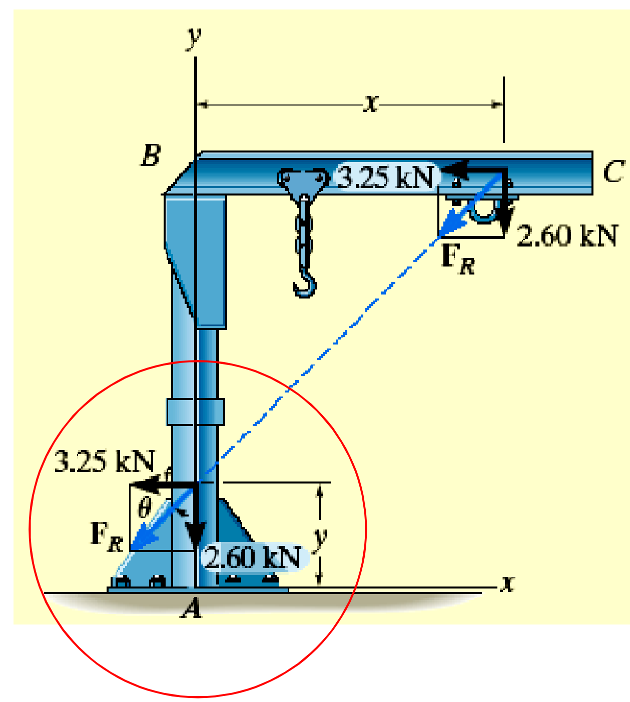
- Summing the moments about point *A*,

$$\circlearrowright + \quad (M_R)_A = \Sigma M_A:$$

$$\begin{aligned}(3.25\text{ kN})(y) &= [(1.75\text{ kN})(1\text{m})] \\ &\quad - [(0.60\text{ kN})(0.6\text{m})] \\ &\quad + [2.50\text{ kN}(3/5)(2.2\text{ m})] \\ &\quad - [2.50\text{ kN}(4/5)(1.2\text{ m})]\end{aligned}$$

$$y = 0.458\text{ m}$$



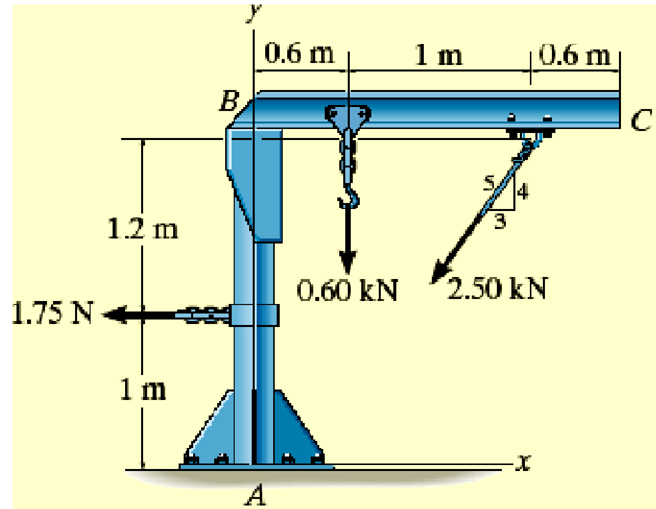


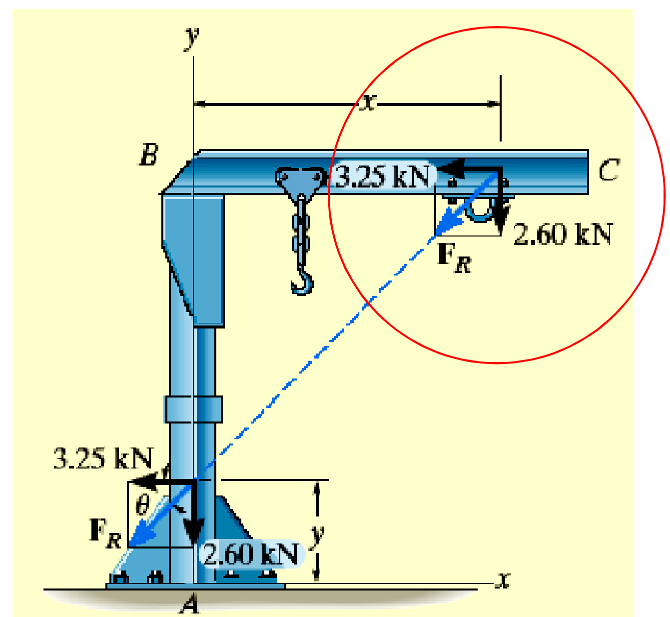
- By the principle of transmissibility, $\mathbf{F}_R$ can be placed at the point where it intersects BC, at a distance x from B.

$$\circlearrowleft + \quad (M_R)_A = \Sigma M_A:$$

$$(3.25 \text{ kN})(2.2) - (2.6\text{kN})(x)$$
$$= [(1.75 \text{ kN})(1\text{m})]$$
$$- [(0.60 \text{ kN})(0.6\text{m})$$
$$+ [2.50 \text{ kN}(3/5)(2.2 \text{ m})]$$
$$- [2.50 \text{ kN}(4/5)(1.2 \text{ m})]$$

$$x = 2.177 \text{ m}$$

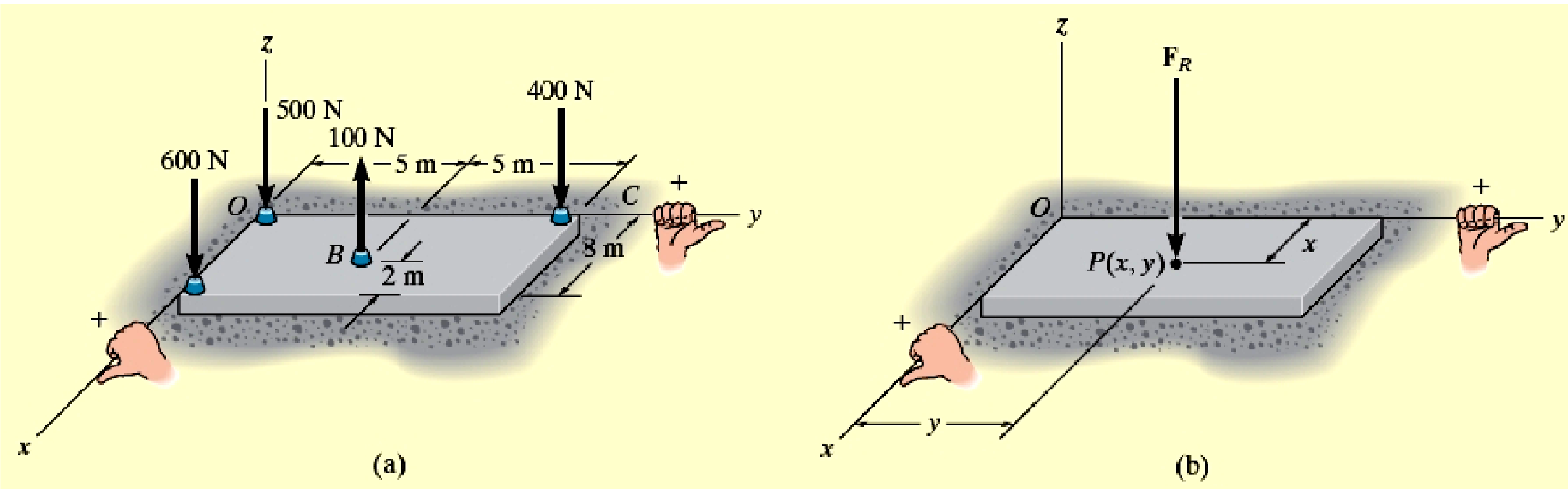




Example 4.19

Given :

The slab is subjected to four parallel forces.



Find :

Determine the magnitude and the direction of a resultant force equivalent to the given force system and locate its point of application on the slab.

Solution

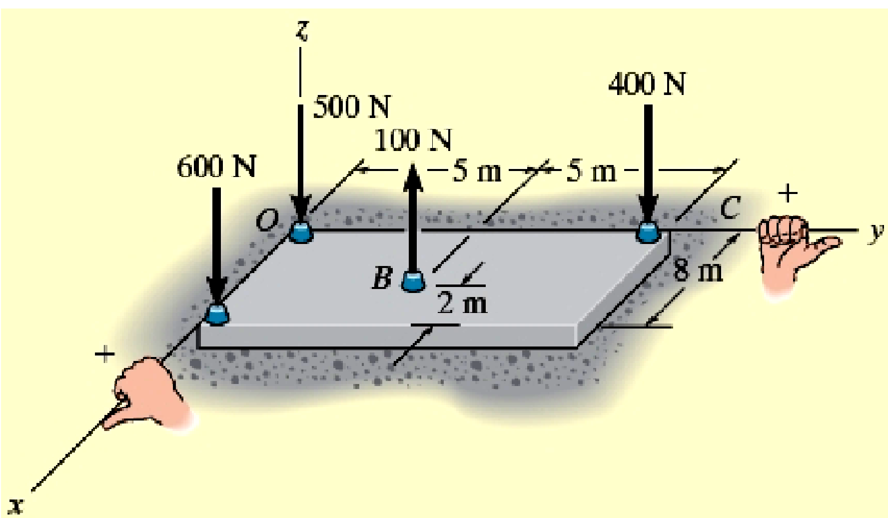
Force Summation

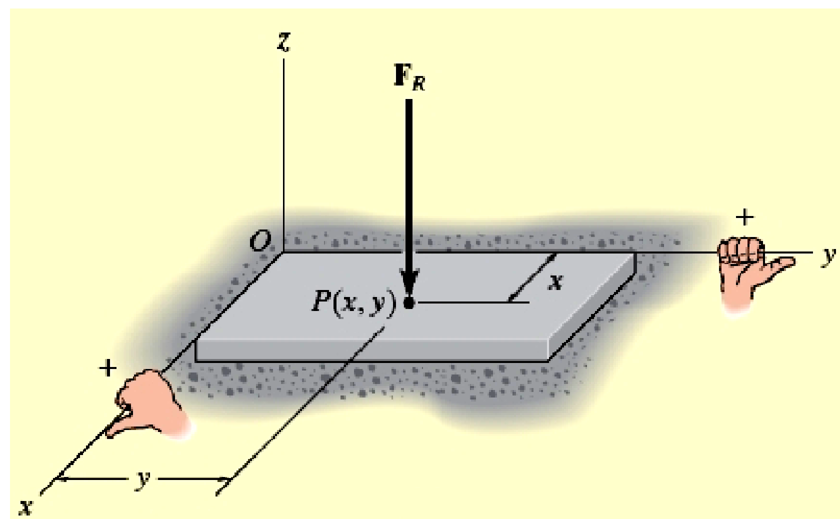
$$+\uparrow \; F_R = \sum F:$$

$$-F_R = -600\text{ N} + 100\text{ N} - 400\text{ N} - 500\text{ N}$$

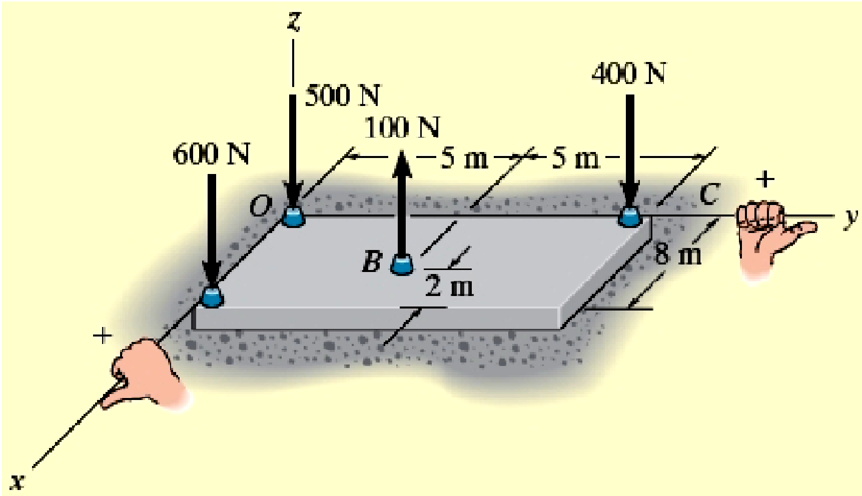
$$= -1400\text{N}$$

$$F_R = 1400\text{ N} \downarrow$$





❑ Moment Summation

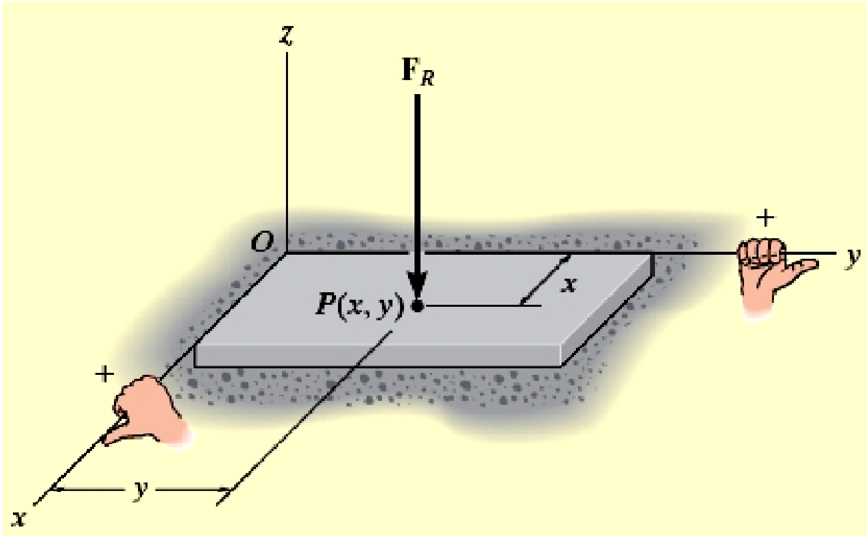


$$\circlearrowleft + \quad (M_R)_x = \Sigma M_x:$$

$$-F_R y = (100\text{ N})(5\text{m}) - (400\text{ N})(10\text{ m})$$

$$-(1400\text{N})\, y = -3500\text{ N·m}$$

$$\Rightarrow \; y = 2.50\text{ m}$$



$$\circlearrowleft + \quad (M_R)_y = \Sigma M_y:$$

$$F_R x = (600\text{ N})(8\text{m}) - (100\text{ N})(6\text{ m})$$

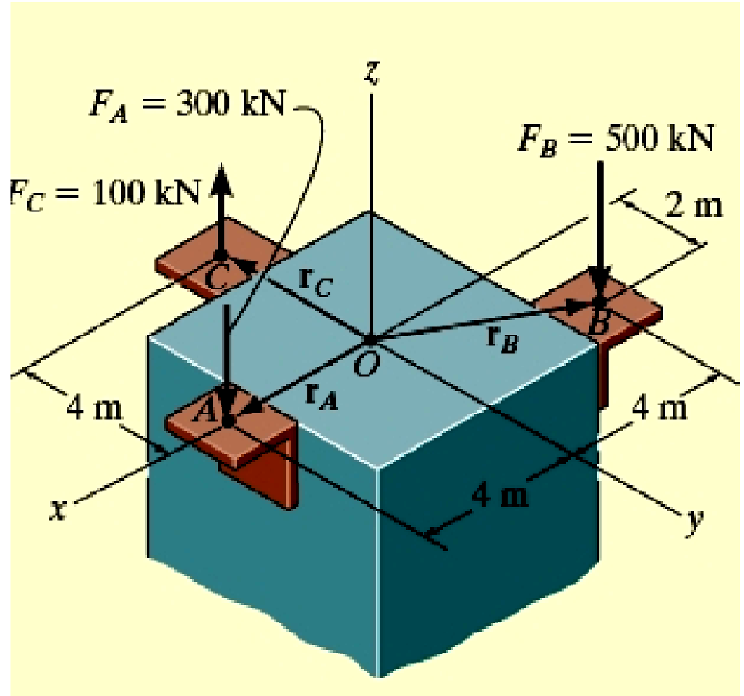
$$(1400\text{N})\, x = 4200\text{ N·m}$$

$$\Rightarrow \; x = 3\text{ m}$$

Example 4.20

Given :

The pedestal is subjected to three parallel forces.



Find :

Replace the force system by an equivalent resultant force and specify its point of application on the pedestal.

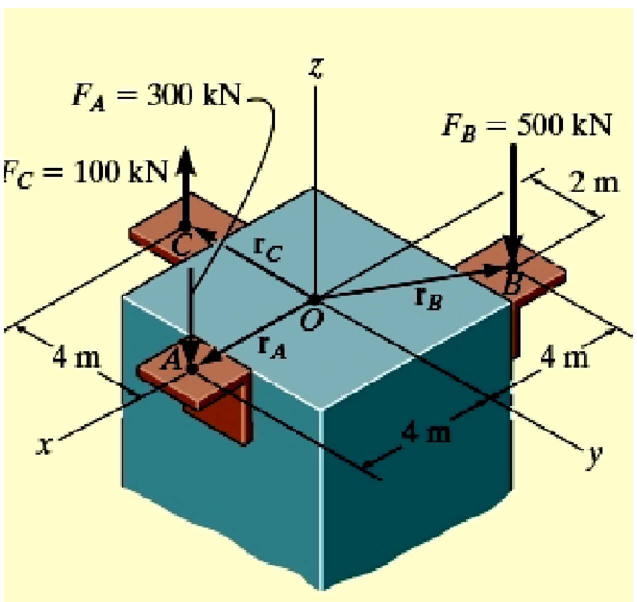
❑ Force Summation

$$\mathbf{F}_R = \sum \mathbf{F}:$$

$$\mathbf{F}_R = \mathbf{F}_A + \mathbf{F}_B + \mathbf{F}_C$$

$$= \{-300\,\mathbf{k}\}\ \text{kN} + \{-500\mathbf{k}\}\ \text{kN} + \{100\mathbf{k}\}\ \text{kN}$$

$$= \{-700\,\mathbf{k}\}\ \text{kN}$$



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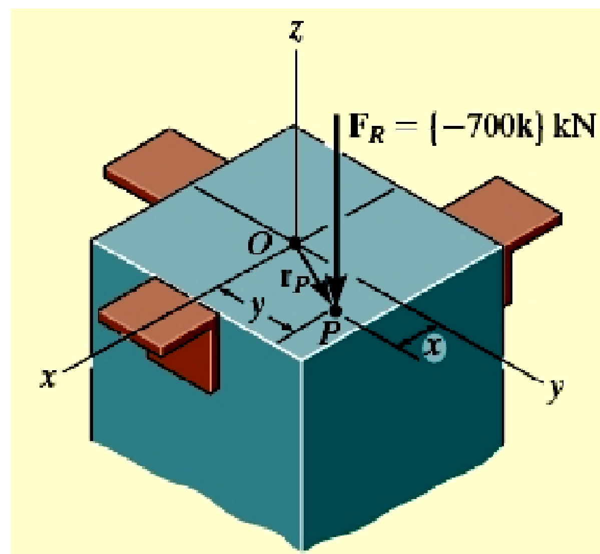
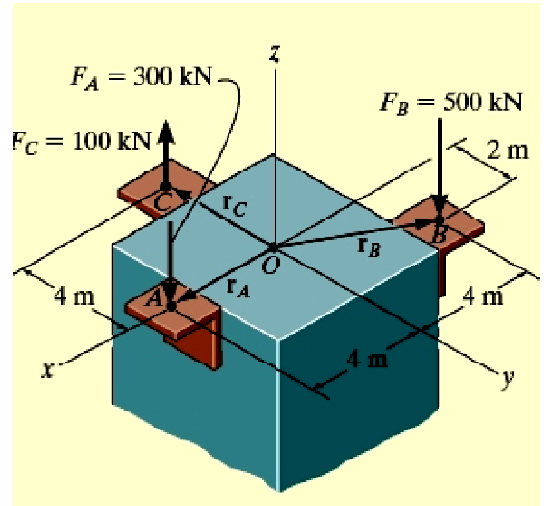
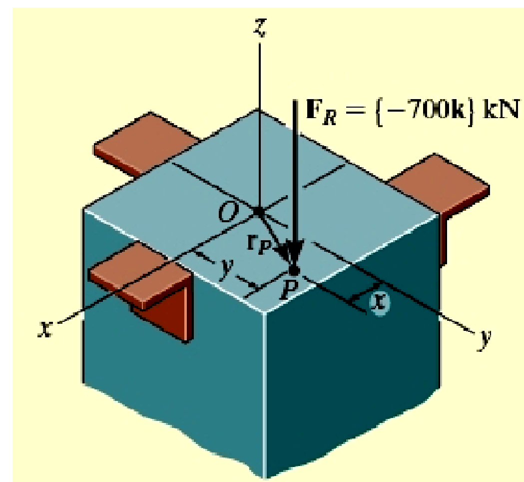


Moment Summation



$$(\mathbf{M}_R)_O = \Sigma \mathbf{M}_O:$$

$$\mathbf{r}_P \times \mathbf{F}_R = (\mathbf{r}_A \times \mathbf{F}_A) + (\mathbf{r}_B \times \mathbf{F}_B) + (\mathbf{r}_C \times \mathbf{F}_C)$$



$$(x\mathbf{i} + y\mathbf{j}) \times (-700\ \mathbf{k}) = (4\mathbf{i}) \times (-300\ \mathbf{k}) + (-4\mathbf{i} + 2\mathbf{j}) \times (-500\ \mathbf{k}) + (-4\mathbf{j}) \times (100\ \mathbf{k})$$

$$-700\,x\,(\mathbf{i} \times \mathbf{k}) - 700\,y\,(\mathbf{j} \times \mathbf{k}) = -1200\,(\mathbf{i} \times \mathbf{k}) + 2000\,(\mathbf{i} \times \mathbf{k}) - 1000\,(\mathbf{j} \times \mathbf{k}) - 400(\mathbf{j} \times \mathbf{k})$$

$$700\,x\mathbf{j} - 700y\mathbf{i} = 1200\mathbf{j} - 2000\mathbf{j} - 1000\mathbf{i} - 400\mathbf{i}$$

- Equating the respective **i**, **j** components, we have

i : $$-700y = -1400$$

$$y = 2 \text{ m}$$

j : $$700x = -800$$

$$x = -1.14 \text{ m}$$

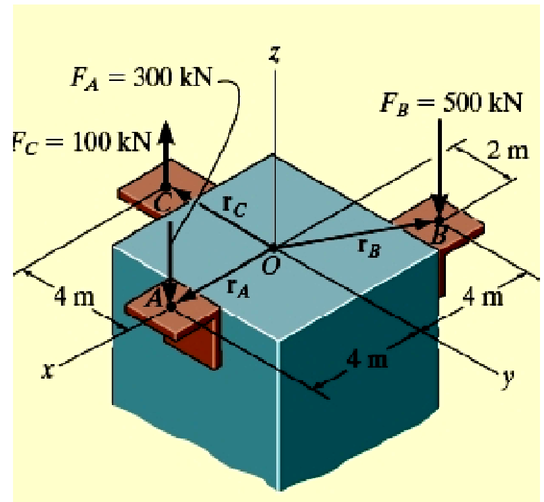
❑ Note

- The location of x and can also be determined by scalar analysis

$$\curvearrowright + \quad (M_R)_x = \Sigma M_x:$$

$$(-700\ \text{kN})\ y = (-100\ \text{kN})\ (4\text{m}) - (500\ \text{kN})\ (2\ \text{m})$$

$$\Rightarrow\ y = 2\ \text{m}$$



$$\curvearrowright + \quad (M_R)_y = \Sigma M_y:$$

$$(700\ \text{kN})\ x = (300\ \text{kN})\ (4\ \text{m}) - (500\ \text{kN})\ (4\ \text{m})$$

$$\Rightarrow\ x = -1.14\ \text{m}$$