# Chapter 5

# **Equilibrium of a Rigid Body**

# Chapter Objectives

- Develop the equations of equilibrium for a rigid body
- Concept of the free-body diagram for a rigid body
- Solve rigid-body equilibrium problems using the equations of equilibrium.

# Chapter Outline

1. Conditions for Rigid Equilibrium
2. Free-Body Diagrams
3. Equations of  Equilibrium in Two-Dimensions
4. Two and Three-Force Members
5. Free Body Diagrams
6. Equations of Equilibrium in Three-Dimensions
7. Constraints and Statical  Determinacy

# 5.1 Conditions for Rigid-Body Equilibrium

- **Assumptions**

  All bodies considered in this chapter are assumed to be rigid.

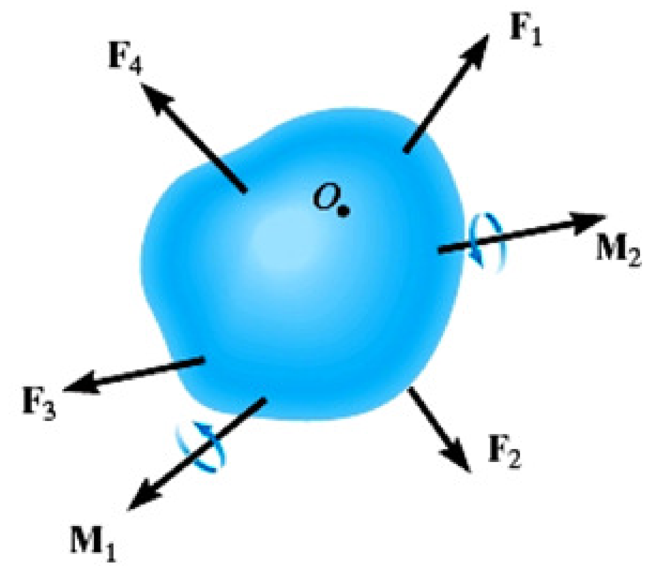
- **Rigid Body**

- A rigid body will not deform under the applied load.

- The direction of the applied forces and their moment arms with respect to a fixed reference remain unchanged before and after the body is loaded

- Consider a rigid body which is subjected to a system of ***external*** forces and couple moments.

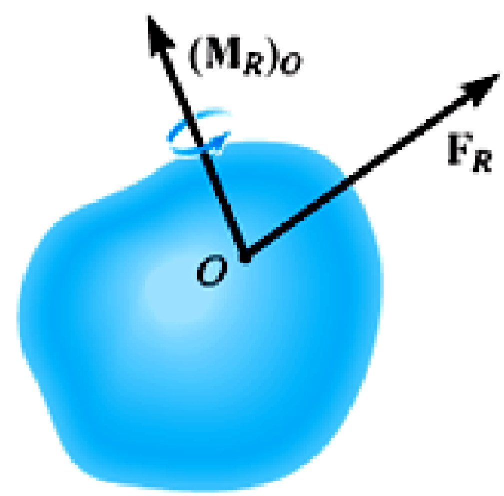
  (*Internal forces are not shown as they occur in equal but opposite collinear pairs which cancel out each other*)




- The above force and couple moment system can be reduced to an equivalent resultant force and resultant couple moment at any arbitrary point $O$ on or off the body.

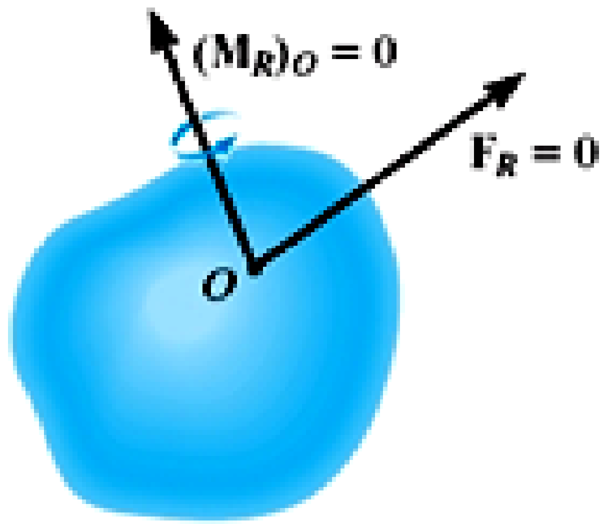
$$\mathbf{F}_R = \sum \mathbf{F}$$

$$(\mathbf{M}_R)_O = \sum \mathbf{M}_O$$



- A rigid body is said to be in equilibrium if both the resultant force and resultant couple moment about any point $O$ are equal to zero, i.e.,

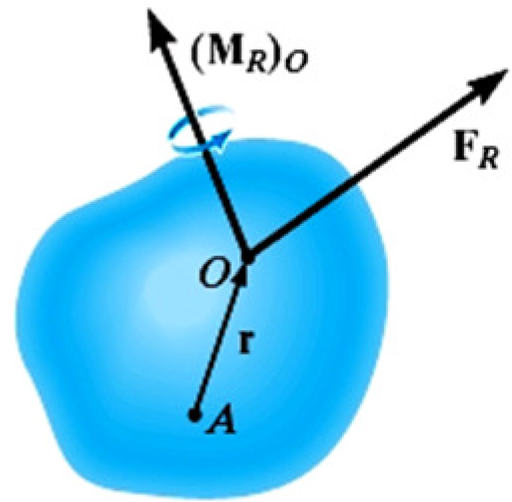
$$\boxed{\begin{aligned} \mathbf{F}_R &= \sum \mathbf{F} = \mathbf{0} \\ (\mathbf{M}_R)_O &= \sum \mathbf{M}_O = \mathbf{0} \end{aligned}} \qquad (1)$$




- These equations are the necessary and sufficient conditions for equilibrium.

## Proof

- Summing the moments in the equivalent resultant force & couple moment system about some other point, such as *A*, we have




$$\sum \mathbf{M}_A = \mathbf{r} \times \mathbf{F}_R + (\mathbf{M}_R)_O$$

- For equilibrium, the sum of the couple moments must be zero

$$\sum \mathbf{M}_A = \mathbf{r} \times \mathbf{F}_R + (\mathbf{M}_R)_O = \mathbf{0}$$

- Since $\mathbf{r} \neq \mathbf{0}$, the above equation is satisfied only if

$$\mathbf{F}_R = \mathbf{0}$$
$$(\mathbf{M}_R)_O = \mathbf{0}$$

which are the equilibrium conditions given in Eq. (1).

# EQUILIBRIUM IN TWO-DIMENSIONS

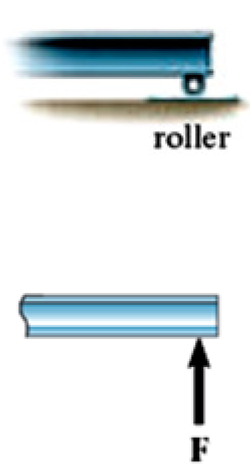
## 5.2 Free Body Diagrams

- External and internal forces can act on a rigid body.
- In performing equilibrium analysis, only external forces that act on the rigid body need be shown on the free-body diagram.
- External forces are :
  - *reactive forces* at the supports or at the point of contact with other bodies.
  - weight of the body
  - applied loading

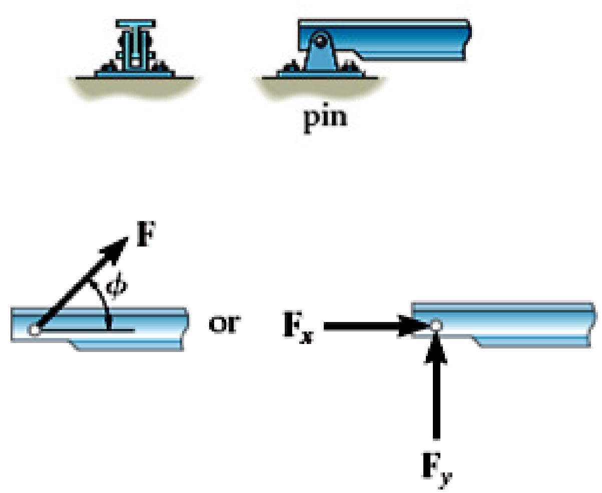
## ❑ Support Reactions

- If a support prevents the translation of a body in a given direction, then a force is developed on the body in that direction.
- If rotation is prevented, a couple moment is exerted on the body.

**Roller support**




**Pin support**




**Fixed support**

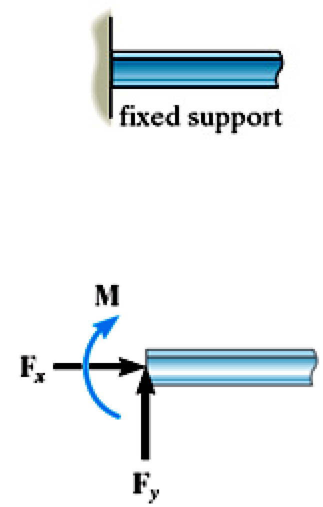


**TABLE 5–1 Supports for Rigid Bodies Subjected to Two-Dimensional Force Systems**

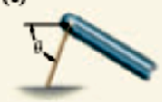
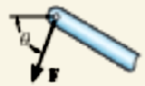
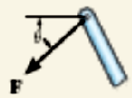
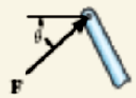
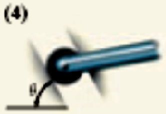
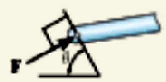
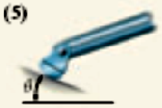
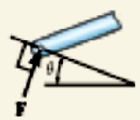
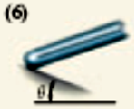
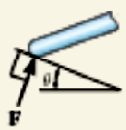
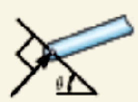
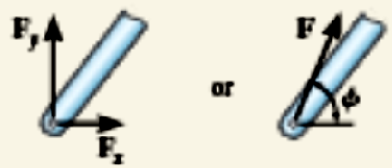
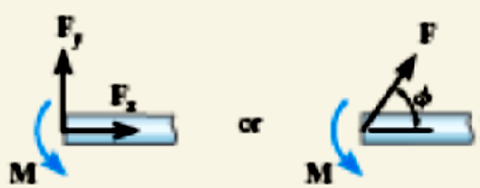
| Types of Connection | Reaction | Number of Unknowns |
|---|---|---|
| (1)  cable |  | One unknown. The reaction is a tension force which acts away from the member in the direction of the cable. |
| (2)  weightless link |  or  | One unknown. The reaction is a force which acts along the axis of the link. |
| (3)  roller |  | One unknown. The reaction is a force which acts perpendicular to the surface at the point of contact. |
| (4)  roller or pin in confined smooth slot |  or  | One unknown. The reaction is a force which acts perpendicular to the slot. |
| (5)  rocker |  | One unknown. The reaction is a force which acts perpendicular to the surface at the point of contact. |
| (6)  smooth contacting surface |  | One unknown. The reaction is a force which acts perpendicular to the surface at the point of contact. |
| (7)  member pin connected to collar on smooth rod |  or  | One unknown. The reaction is a force which acts perpendicular to the rod. |

**TABLE 5–1 Continued**

| Types of Connection | Reaction | Number of Unknowns |
|---|---|---|
| (8)   smooth pin or hinge |   | Two unknowns. The reactions are two components of force, or the magnitude and direction $\phi$ of the resultant force. Note that $\phi$ and $\theta$ are not necessarily equal [usually not, unless the rod shown is a link as in (2)]. |
| (9)  member fixed connected to collar on smooth rod |   | Two unknowns. The reactions are the couple moment and the force which acts perpendicular to the rod. |
| (10)  fixed support |   | Three unknowns. The reactions are the couple moment and the two force components, or the couple moment and the magnitude and direction $\phi$ of the resultant force. |

## Internal Forces

- Internal forces that act between adjacent particles in a body occur in collinear pairs.

- Since each pair of these forces have the same magnitude & act in opposite direction, they cancel each other, and thus will not create an external effect on the body

- Hence, internal forces should not be included in the free-body diagram if the entire body is to be considered.

## Weight and Center of Gravity

- Each particle in a body within a gravitational field has a specified weight
- This system of forces can be represented by a single resultant force, known as weight **W** of the body.
- The location of application of **W** is known as the *center of gravity.*

## Procedure for Drawing a FBD

- **Draw Outlined Shape**
  - Imagine the body to be isolated or cut "free" from its surroundings by drawing its outlined shape.
- **Show All Forces and Couple Moments**
  - Identify all external forces and couple moments that act on the body

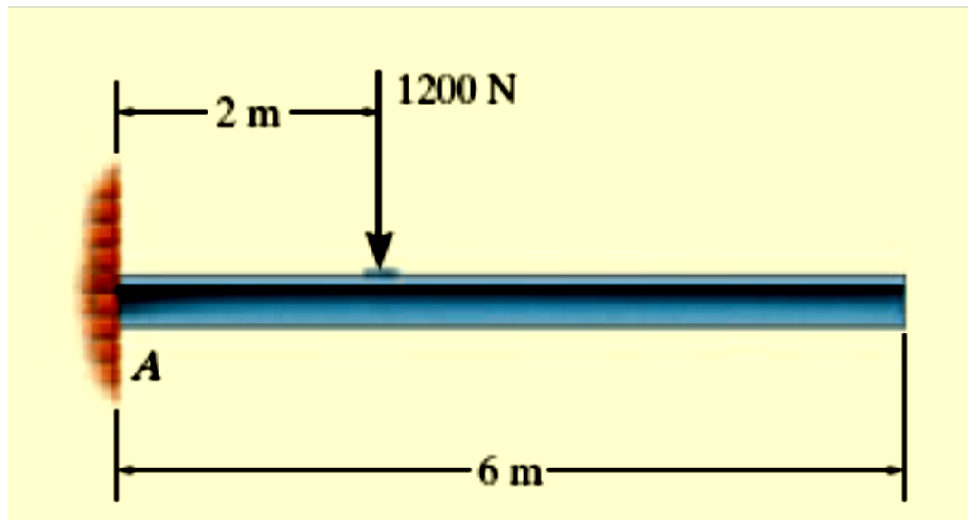
    (1) Applied loading

    (2) Reactive forces: at the supports or at the point of contact with other bodies
- **Identify Each Loading and Give Dimensions**
  - Label known forces with proper magnitude and direction
  - Represent magnitude and directions of unknown forces with letters.
  - Indicate dimensions for calculation of forces

# Example 5.1

**Given :**

The uniform beam shown in the figure has a mass of 100kg.




**Find :**

Draw the free-body diagram of the beam.

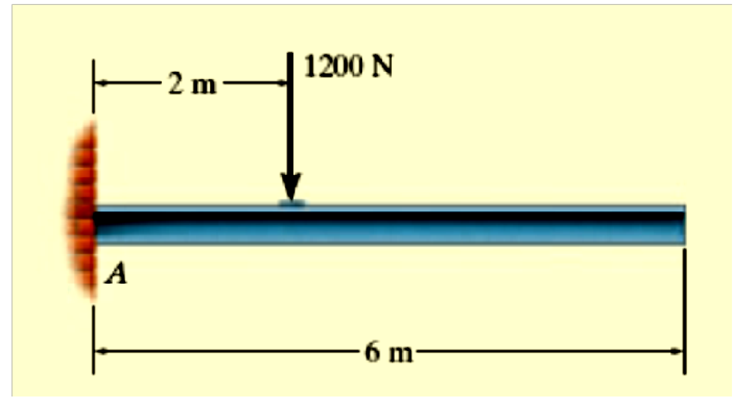
# Solution

## Known external forces

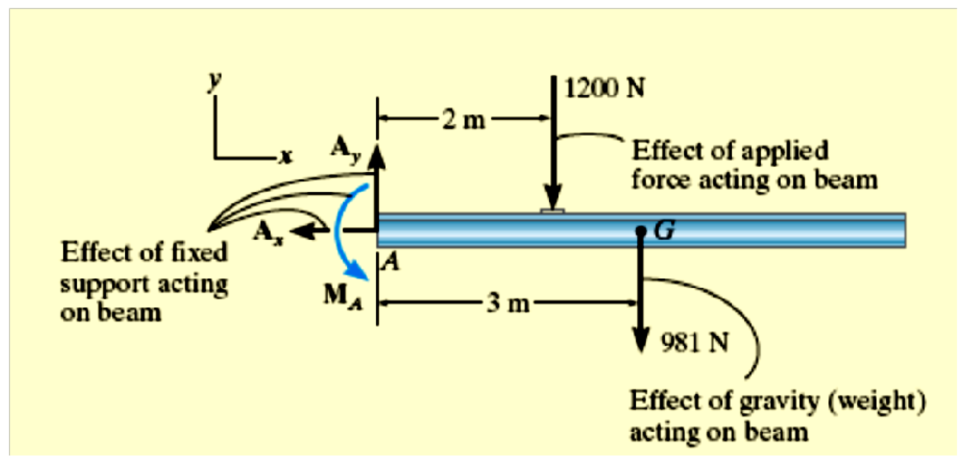
- Applied loading: 1200 N
- Weight of the uniform beam :

$$W = 100(9.81) = 981\text{N}$$

(it acts through the beam's center of gravity, which is 3m from *A*).




**Free-Body Diagram**




## Unknown external forces

- End *A* is a fixed support.
  - A force must be developed to prevent translation.
  - A couple moment must be developed prevent rotation.

# Example 5.2

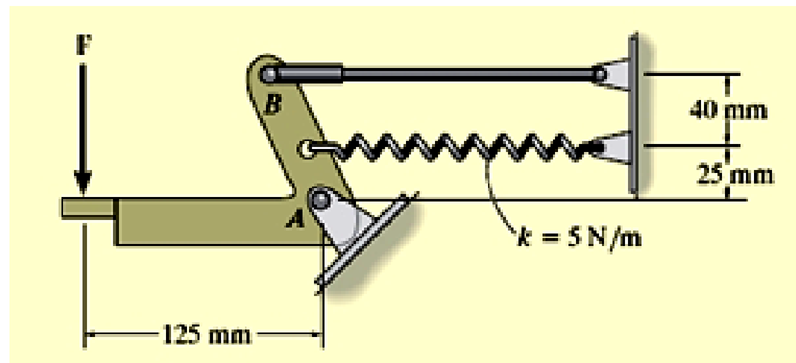
## Given :

- The operator applies a vertical force to the pedal of a foot lever as shown in the figure.

(a)

- The spring is stretched 40mm.

- The force in the short link at *B* is 100N.



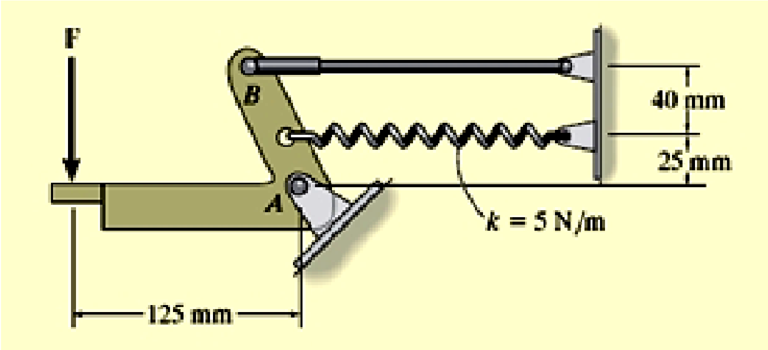

## Find:

Draw the free-body diagram of the foot lever.

# Solution

## Known external forces

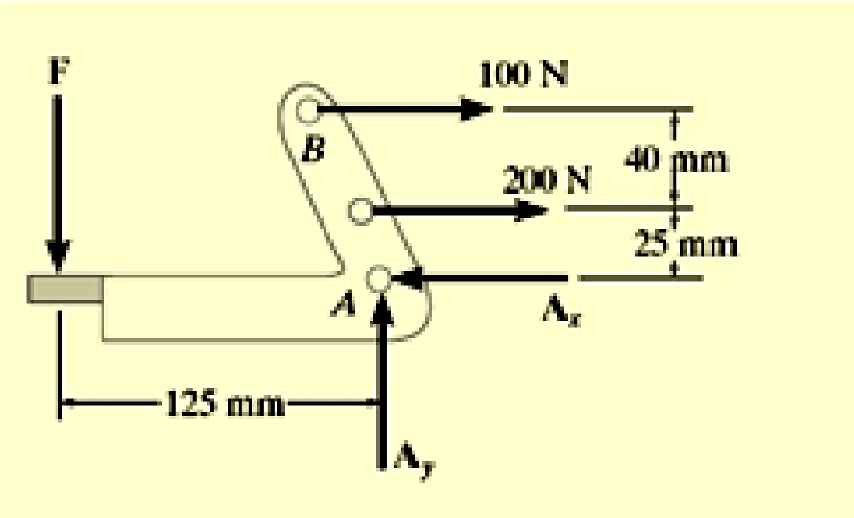
- Force at $B$ : 100N.
- Spring force:
  $F_s = ks = (5\text{ N/m})\ (40\times10^{-3}\text{ m})$
  $= 0.2\text{ N}$
- Applied force: F.




**Free-Body Diagram**

## Unknown external forces

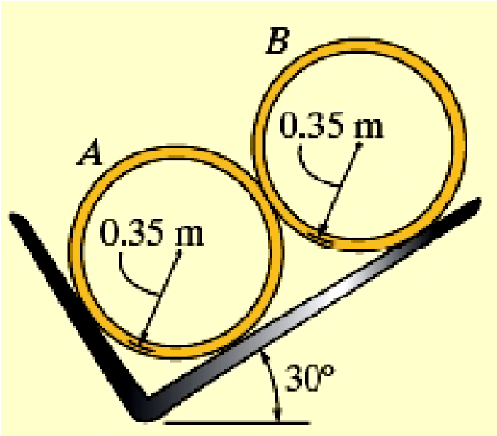
- Pin support at $A$.
  - Reaction has $x$ & $y$ components



# Example 5.2

## Given :

Two smooth pipes, each having a mass of 300 kg, are supported by the forked tines of the tractor as shown in the figure.



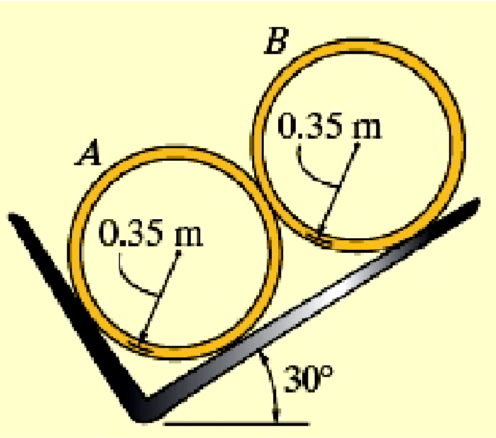

## Find:

Draw the free-body diagram for each pipe and both pipes together.

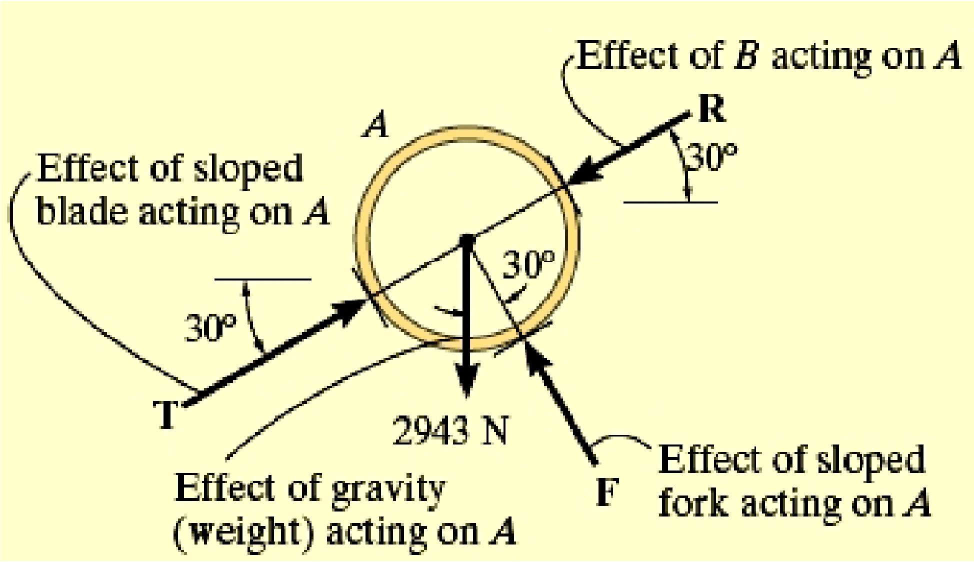
# Solution

## Free-body diagram for each pipe

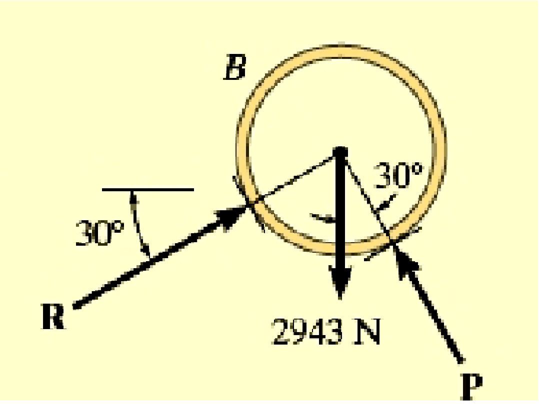
- The reactive forces **T**, **F**, **R** act in a direction normal to the tangent at their surfaces of contact



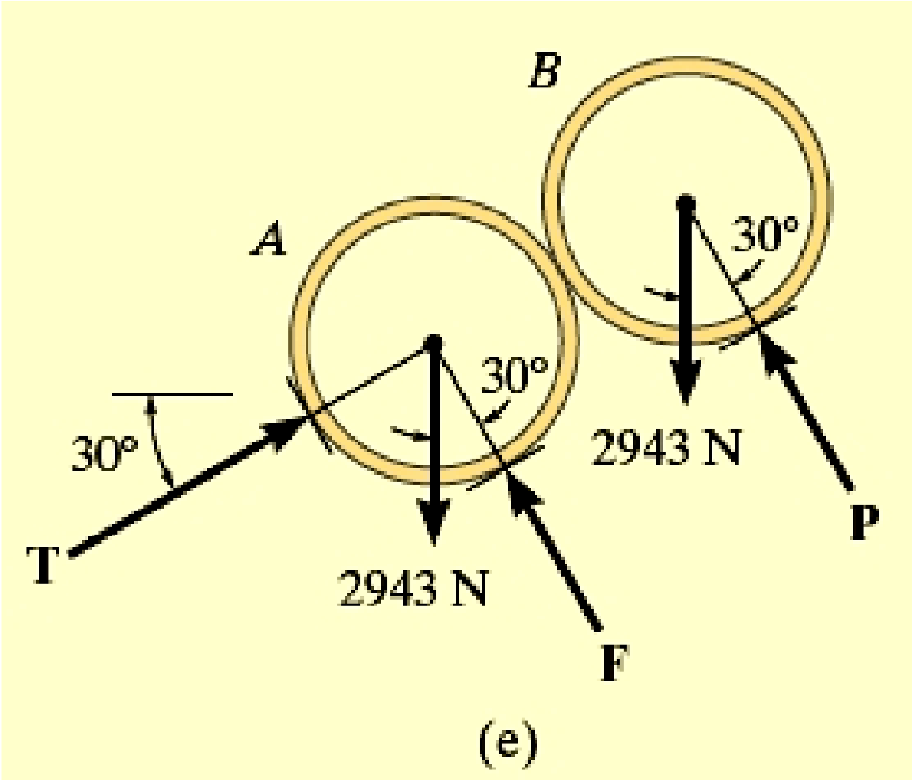

### FBD for pipe *A*




### FBD for pipe *B*



## Free-body diagram for both pipes considered as a whole




**Note** :

- The contact force **R** which acts between *A* & *B* is considered as an ***internal force*** when the pipes are considered as a unit and hense is not show n in the free-body diagram.
- It represents a pair of equal but opposite collinear forces which cancel each other.

# 5.3 Equations of Equilibrium in Two-Dimensions

## ❑ Coplanar Equilibrium Equations

- When a rigid body is subjected to a system of forces that lies in the $x$–$y$ plane, the conditions for 2D equilibrium are

$$\boxed{\begin{aligned} \sum F_x &= 0 \\ \sum F_y &= 0 \\ \sum M_O &= 0 \end{aligned}} \qquad (2)$$

- $\sum F_x$ and $\sum F_y$ represent sums of $x$ and $y$ components of all the forces.
- $\sum M_O$ represents the sum of the couple moments and moments of the force components about the $z$ axis which is perpendicular to the the $x$–$y$ plane and passes through the arbitrary point $O$.

## ❑ Alternative Sets of Equilibrium Equations

**(I) One force equilibrium, 2 moment equilibrium equations**

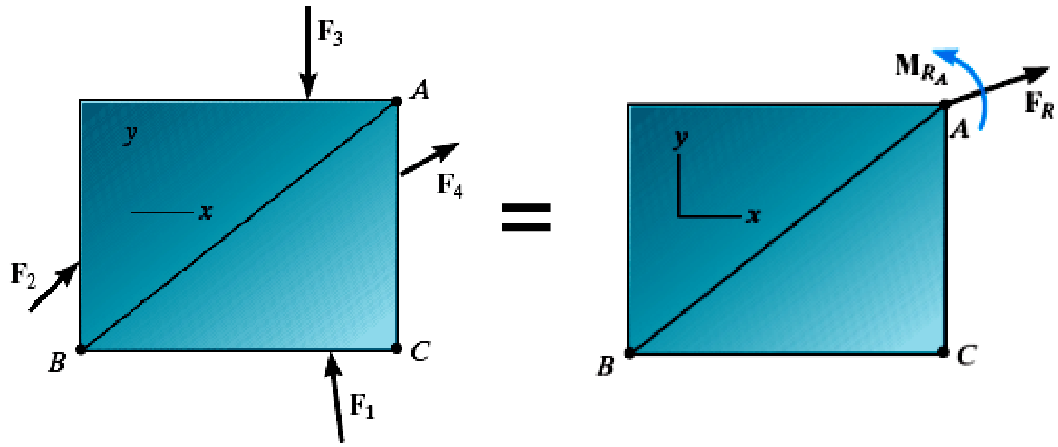
$$\boxed{\begin{aligned} \sum F_x &= 0 \\ \sum M_A &= 0 \\ \sum M_B &= 0 \end{aligned}} \qquad (3)$$

**Note :**

When using these equations, it is required that a line passing through points $A$ and $B$ is ***not parallel*** to the $y$ axis.

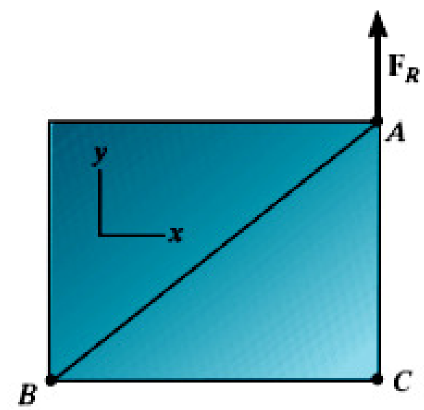
## Proof

- Reduce the system of forces to an equivalent resultant force and resultant couple moment at $A$.




- If $\sum F_x = 0$ then $F_R = F_y$

  If $\sum M_A = 0$ then $M_{RA} = 0$ $\Longrightarrow$




- If $\sum M_B = 0$ then $F_R = 0$

- Since $F_R = 0$ & $M_{RA} = 0$, the system is in equilibrium.

## (II) Three moment equilibrium equations

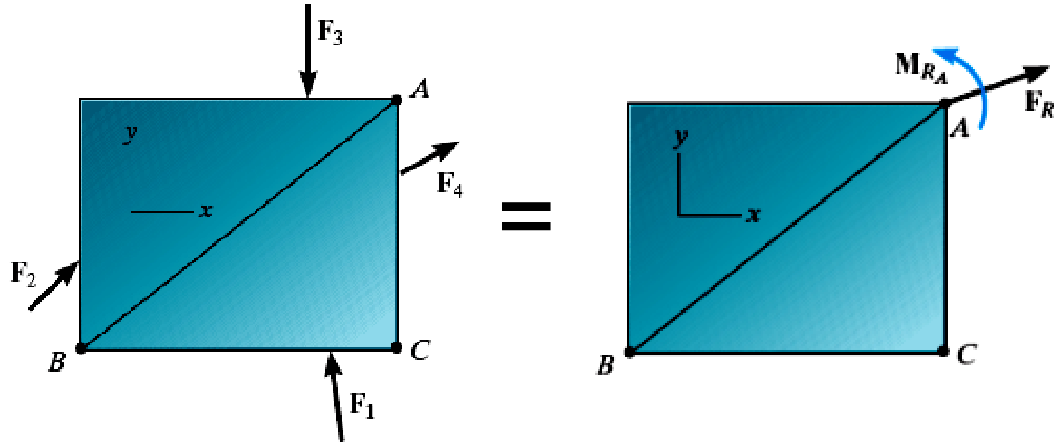
$$\boxed{\begin{aligned} \sum M_A &= 0 \\ \sum M_B &= 0 \\ \sum M_C &= 0 \end{aligned}} \qquad (4)$$

**Note :**

When using these equations, it is necessary that points $A$, $B$ and $C$ ***do not lie*** on the same line.

## Proof

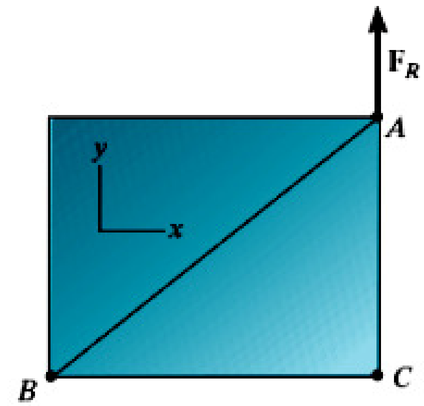
- Reduce the system of forces to an equivalent resultant force and resultant couple moment at $A$.




- If $\sum M_A = 0$ then $\mathbf{M}_{RA} = \mathbf{0}$

  If $\sum M_C = 0$ then $\mathbf{F}_R$ must pass through point $C$.

$\Longrightarrow$




- If $\sum M_B = 0$ then $\mathbf{F}_R = 0$

- Since $\mathbf{F}_R = 0$ & $\mathbf{M}_{RA} = \mathbf{0}$, the system is in equilibrium.

## ❑ Procedure for Analysis

### 1. Free-Body Diagram

- Establish the $x$, $y$ coordinate axes in any suitable orientation.
- Draw an outlined shape of the body.
- Show all the forces and couple moments acting on the body.
- Label known forces with proper magnitude and direction
- Represent magnitude and directions of unknown forces with letters. The sense of unknown force & couple moment can be assumed.
- Indicate the dimensions of the body necessary for computing the moments of forces.

## 2. Equations of Equilibrium

- Applying $\sum F_x = 0$ & $\sum F_y = 0$.

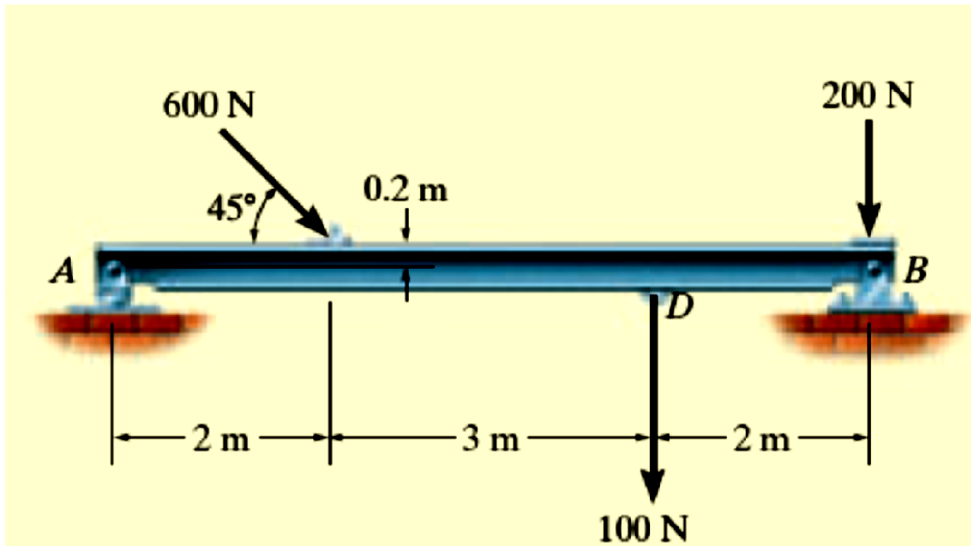
  *Orient the x and y axes along the lines that will provide the simplest resolution of the forces into their x and y components*

- Apply $\sum M_O = 0$ about a point *O* that lies at the intersection of the lines of action of 2 unknown forces.

  *In this way, the moments of these unknown forces are zero about O and a direct solution for the third unknown can be obtained.*

# Example 5.5

**Given :**



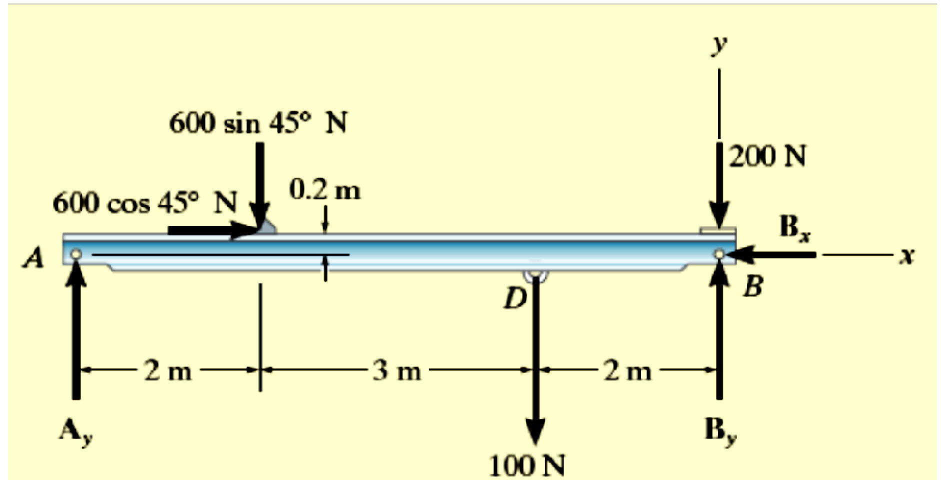

**Find :**

- Determine the horizontal and vertical components of reaction on the beam caused by the pin at *B* and the rocker at *A*.
- Neglect the weight of the beam.

# Solution

## Free-body diagram

- Resolve the 600-N force into its $x$ & $y$ components.
- Rocker at $A$: reaction is perpendicular to the contact surface.




- Pin at $B$: reaction has 2 components.

## Equations of Equilibrium

$$\overset{+}{\rightarrow} \ \Sigma F_x = 0: \quad 600 \cos 45^{o}\ \text{N} - B_x = 0 \qquad \text{(a)}$$

$$+\uparrow \ \Sigma F_y = 0: \quad A_y - 600 \sin 45^{o}\ \text{N} - 100\ \text{N} - 200 + B_y = 0 \qquad \text{(b)}$$

$$\circlearrowright + \Sigma M_B = 0: \quad (100\text{N})(2\text{m}) + (600 \sin 45^{o}\ \text{N})(5\ \text{m}) - (600 \cos 45^{o}\ \text{N})(0.2\ \text{m}) - A_y(7\ \text{m}) = 0 \qquad \text{(c)}$$

- From Eq. (a),

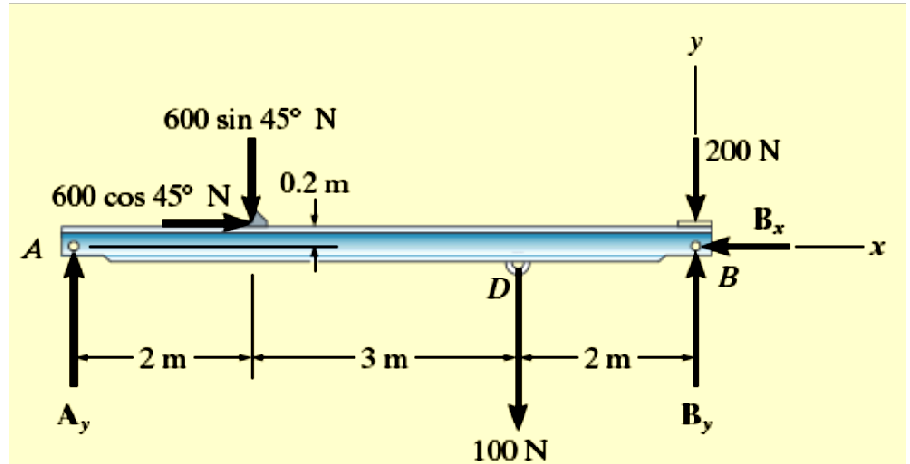
$$B_x = 424 \text{ N}$$

- From Eq. (c),

$$A_y = 319 \text{ N}$$

- Substituting $A_y$ into Eq. (b) yields,

$$B_y = 405 \text{ N}$$




**Note: Eq. (b) can be replaced by the moment equation $\Sigma M_A = 0$**

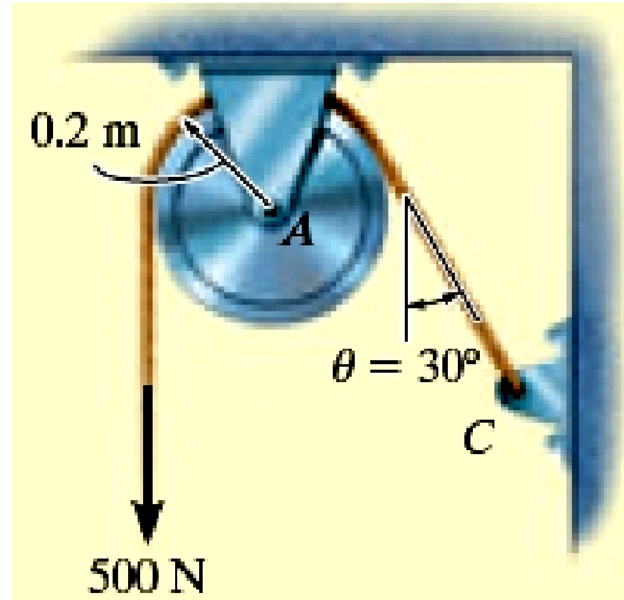
$$\circlearrowright + \; \Sigma M_A = 0: \quad -(600 \sin 45^\circ \text{ N})(2 \text{ m}) - (600 \cos 45^\circ \text{ N})(0.2 \text{ m}) - (100\text{N})(5\text{m}) - (200 \text{ N})(7 \text{ m}) + B_y(7\text{m}) = 0$$

$$B_y = 405 \text{ N}$$

# Example 5.6

## Given :

The cord shown in the figure supports a force of 500 N and wraps over the frictionless pulley.

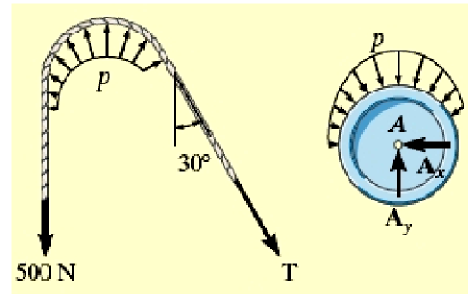



## Find :

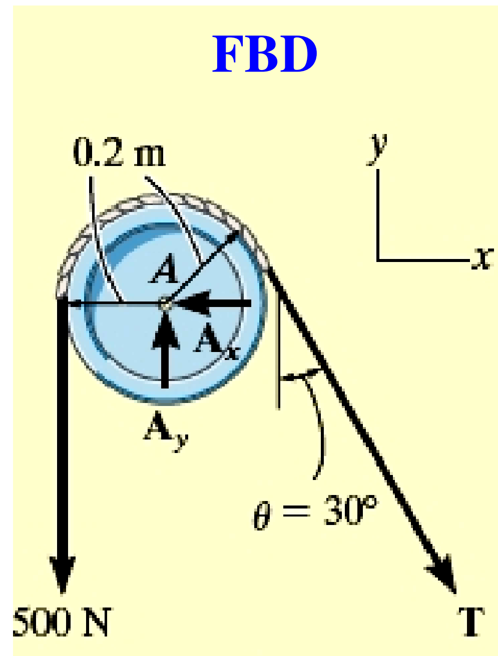
Determine the tension in the cord at *C* and the horizontal and vertical components of reaction at pin *A*.

# Solution

## Free-body diagram




- Consider the pulley & the cord as a whole so that the distributed load between the cord and the pulley becomes internal force to this system.
- Pin at $A$: reaction has $x$ & $y$ components.




## Equations of Equilibrium

$$\circlearrowright + \Sigma M_A = 0: \ (500\text{N})(0.2\text{m}) - T(0.2\text{ m}) = 0$$

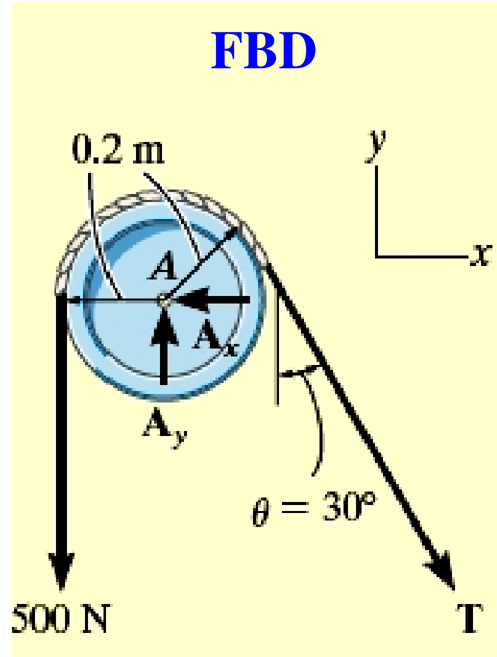
$$T = 500\text{ N}$$

$$\xrightarrow{+} \Sigma F_x = 0: \quad -A_x + T \sin 30^{o} = 0$$

$$A_x = T \sin 30^{o}$$

$$= (500 \text{ N}) \sin 30^{o}$$

$$= 250 \text{ N}$$




$$+\uparrow \Sigma F_y = 0: A_y - 500 \text{ N} - T \cos 30^{o} = 0$$

$$A_y = 500 \text{ N} + T \cos 30^{o}$$

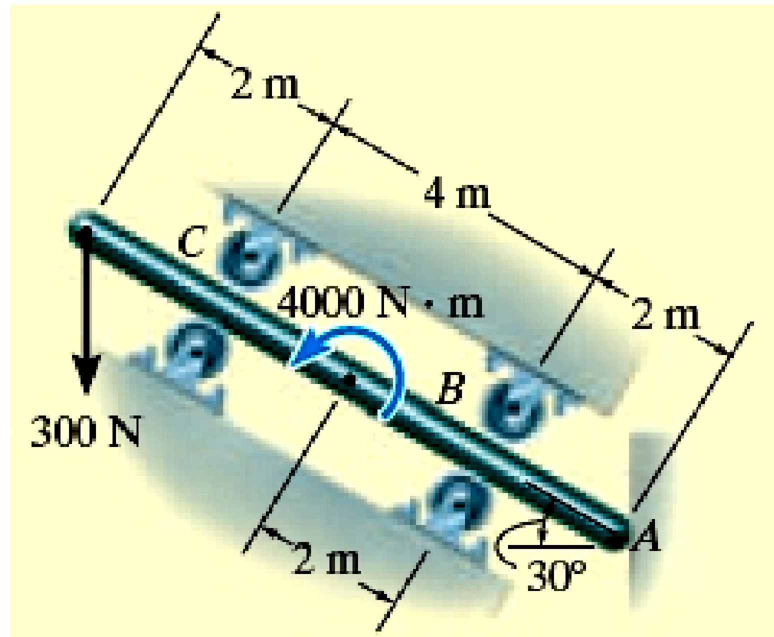
$$= 500 \text{ N} + (500 \text{ N}) \cos 30^{o}$$

$$= 933 \text{ N}$$

# Example 5.10

## Given :

- The uniform smooth rod shown in the figure is subjected to a force and couple moment.
- The rod is supported at *A* by a smooth wall and at *B* & *C* either at the top or bottom by rollers.




## Find :

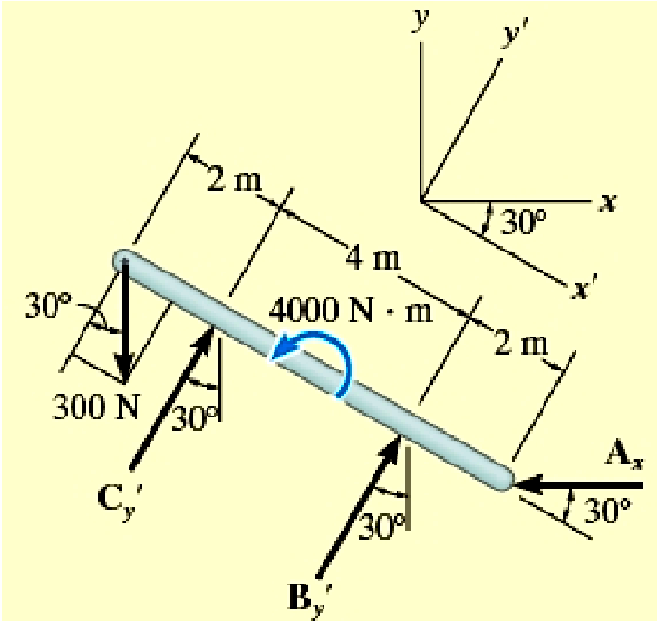
- Determine the reactions at the supports.
- Neglect the weight of the rod.

# Solution

## Free-body diagram

- Assume that only the rollers located at the bottom of the rod are used for support.

  For each roller at *B* & *C*, the reaction is perpendicular to the contact surface.




- The reaction at *A* is perpendicular to the wall as the wall is smooth.

## Equations of Equilibrium

$$\overset{+}{\rightarrow}\ \Sigma F_x = 0: \quad C_{y'} \sin 30^o + B_{y'} \sin 30^o - A_x = 0 \qquad (1)$$

$$+\uparrow\ \Sigma F_y = 0: \quad -300\text{ N} + C_{y'} \cos 30^o + B_{y'} \cos 30^o = 0 \qquad (2)$$

$$\circlearrowright +\ \Sigma M_A = 0: \quad -B_{y'}(2\text{ m}) + (4000\text{ N}\cdot\text{m}) - C_{y'}(6\text{m})$$
$$+ (300\text{ N})(8 \cos 30^o\text{ m}) = 0 \qquad (3)$$

- Solving Eqs. (2) & (3) simultaneously, we obtain

$$B_{y'} = -\,1000.0 \text{ N} = -\,1 \text{ kN}$$

$$C_{y'} = \;1346.4 \text{ N} = 1.35 \;\text{ kN}$$

**Note :** ***The negative sign of $B_{y'}$ indicates that the top roller at B serves as the support.***

- Substituting $B_{y'}$ and $C_{y'}$ $A_y$ into Eq. (1), we have

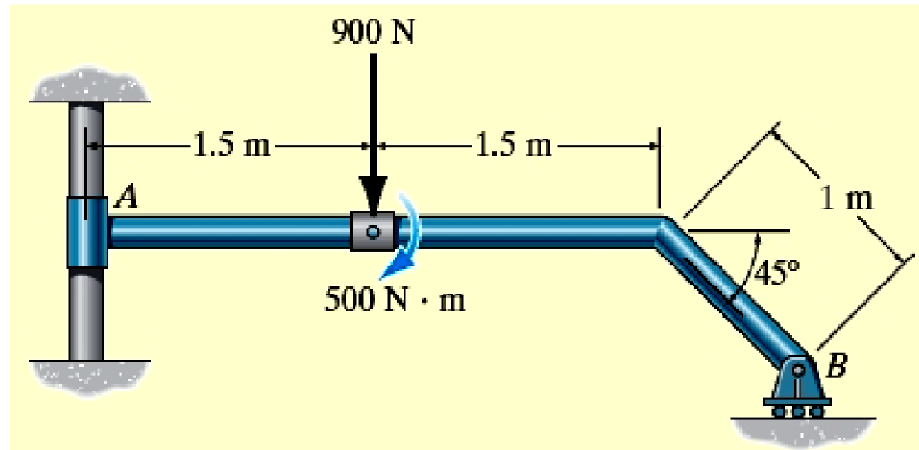
$$(1346.4 \text{ N}) \sin 30^{\circ} + (-\,1000.0 \text{ N}\,) \sin 30^{\circ} - A_x = 0$$

$$A_x = \;173 \text{ N}$$

# Example 5.12

## Given :

The collar at *A* shown in the figure is fixed to the member and slide vertically along the vertical shaft.

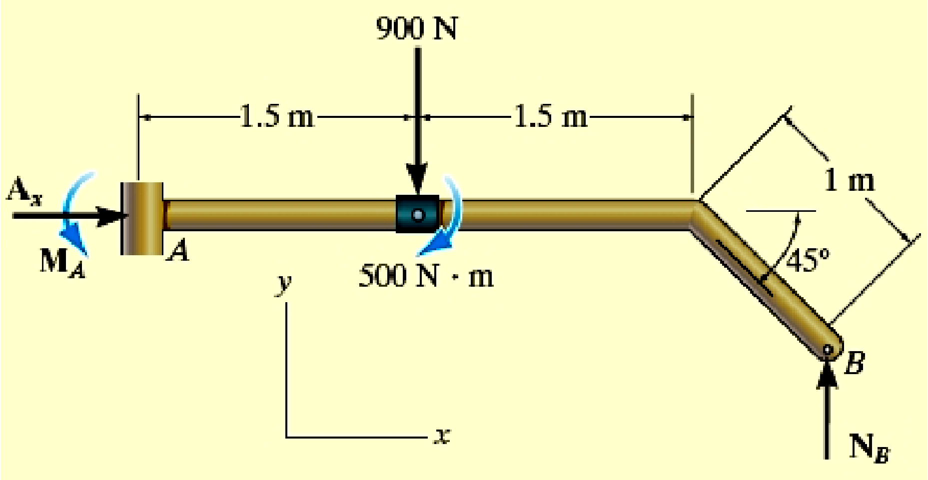



## Find :

Determine the support reactions on the member.

# Solution

## Free-body diagram



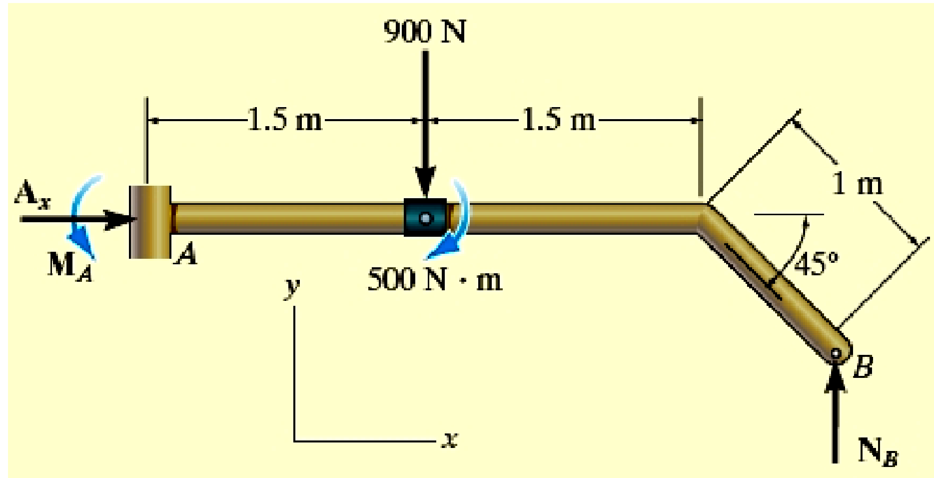

- Collar $A$:
  - The reaction at $A$ has only $x$ component since it can slide vertically.
  - A moment is developed at at $A$ as it is prevented from rotation.
- Roller at $B$: reaction is perpendicular to the contact surface.

## Equations of Equilibrium

$$\overset{+}{\rightarrow} \ \Sigma F_x = 0: \quad A_x = 0$$

$$+\uparrow \ \Sigma F_y = 0: \quad N_B - 900 \text{ N} = 0$$

$$\Rightarrow N_B = 900 \text{ N}$$




$+ \quad \Sigma M_A = 0:$

$$M_A - (900\text{ N})(1.5\text{ m}) - (500\text{ N}\cdot\text{m}) + N_B\,[3\text{ m} + (1\text{ m}\cos 45^{\circ})] = 0$$

$$M_A = (900\text{ N})(1.5\text{ m}) + (500\text{ N}\cdot\text{m}) - (900\text{N})\,[3\text{ m} + (1\text{ m}\cos 45^{\circ})]$$

$$= -\,1486\text{ N}\cdot\text{m}$$

$$= 1.49\text{ kN}\cdot\text{m}$$

**Note :** $M_A$ can also be determined by taking the moment about $B$

$\circlearrowleft + \ \Sigma M_B = 0$:

$$M_A + (900\text{ N})(1.5\text{ m} + 1\text{ m}\cos 45^o) - (500\text{ N·m}) - A_x\,(1\text{ m}\sin 45^o) = 0$$

$$M_A = -(900\text{ N})(1.5\text{ m} + 1\text{ m}\cos 45^o) + (500\text{ N·m}) - 0$$

$$= -1486\text{ N·m}$$

$$= 1.49\text{ kN·m}\ \circlearrowright$$