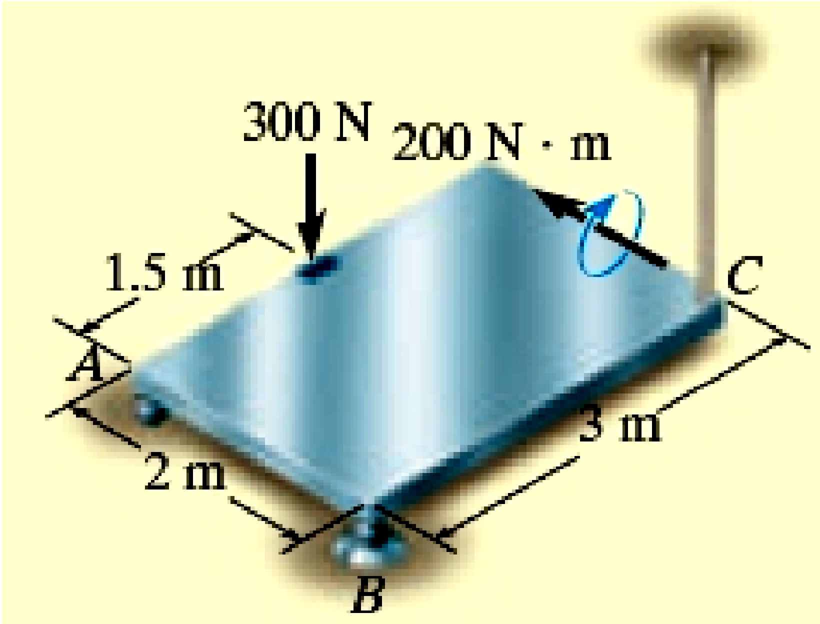
Example 5.15

Given :

- The homogenous plate has a mass of 100kg and is subjected to a force and couple moment along its edges.

- It is supported in the horizontal plane by means of a roller at *A*, a ball-and-socket joint at *B*, and a cord at *C*.

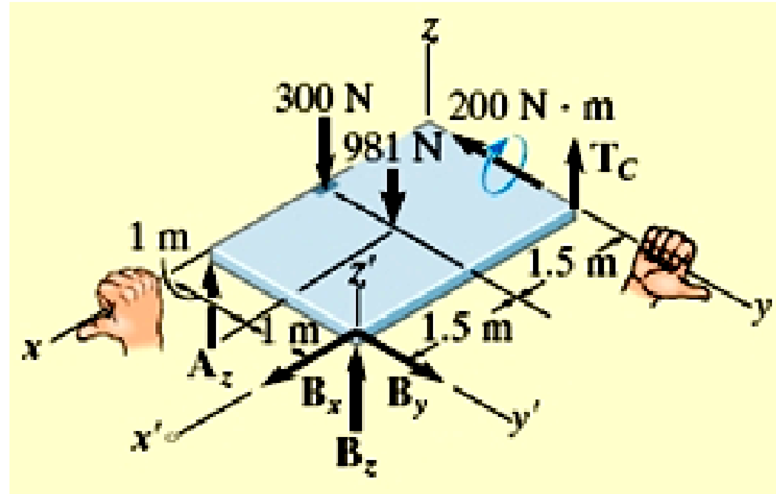


Find :

Determine the components of reactions at the supports.

Solution

Free-body diagram



- Roller at *A : reaction is perpendicular to the plate.*
- Ball-and-socket joint at *B :* reaction has *x, y, z* components.
- *C*: Tension force
- Weight at the centroid of the plate: $W = (100\text{ kg})(9.81\text{ m/s}^{-2}) = 981\text{ N}$

Equations of Equilibrium

$$\Sigma F_x = 0: \quad B_x = 0$$

$$\Sigma F_y = 0: \quad B_y = 0$$

$$\Sigma F_z = 0: \quad A_z + B_z + T_C - 300\text{ N} - 981\text{ N} = 0 \qquad (1)$$

$\Sigma M_x = 0$: $T_C(2\text{ m}) - (981\text{ N})(1\text{ m}) + B_z(2\text{ m}) = 0$ (2)

$\Sigma M_y = 0$:

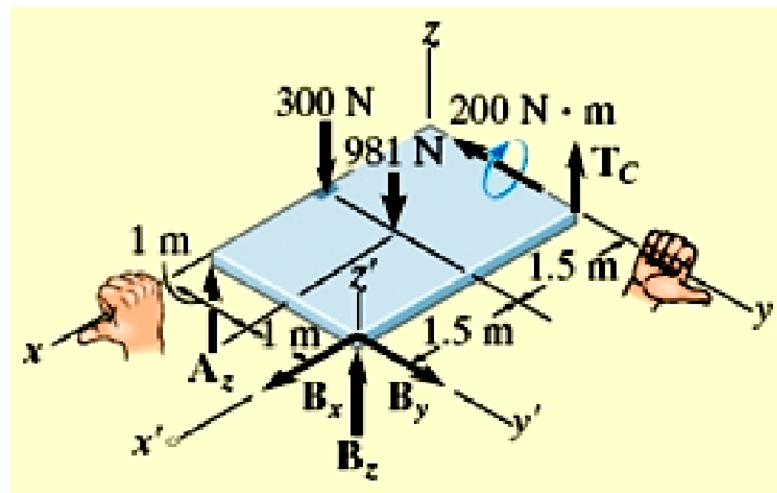
$300\text{ N}(1.5\text{m}) + (981\text{ N})(1.5\text{ m}) - B_z(3\text{ m}) - A_z(3\text{ m}) - 200\text{ N}\cdot\text{m} = 0$

$\Rightarrow \quad 1721.5\text{ N}\cdot\text{m} - B_z(3\text{ m}) - A_z(3\text{ m}) = 0$ (3)

- $2 \times$ Eq.(1) – Eq.(2):

$$2A_z - 1581 = 0$$

$$\Rightarrow A_z = 790.5\text{ N}$$



- Substituting A_z into Eq.(3), we get

$$\Rightarrow B_z = -216.7\text{ N}$$

- Substituting B_z into Eq.(2), we get

$$2T_C - 981\text{ N} - 2(216.7\text{ N}) = 0$$

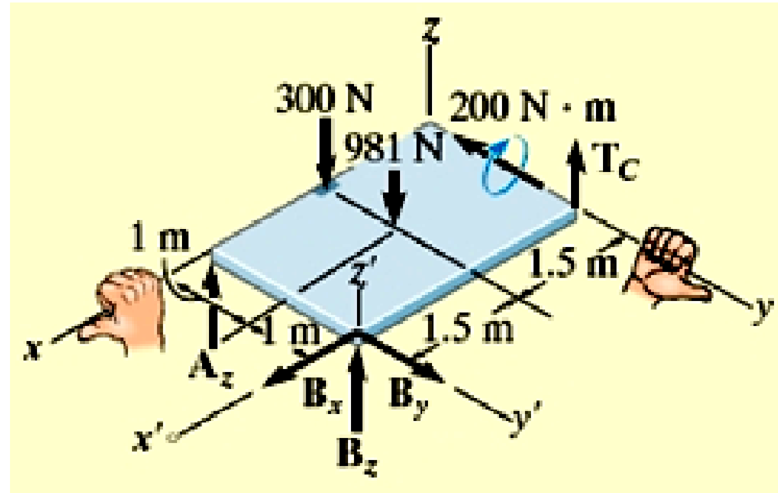
$$\Rightarrow T_C = 707.2\text{ N}$$

Note:

- Since $B_x = 0$ & $B_y = 0$, there is no moment about the z axis. Therefore, the condition $\Sigma M_z = 0$ is not used.

- As the supports cannot prevent the plate from turning about the z axis, it is said to be partially constrained.

Alternative Method



- This problem can also be solved by replacing Eqs.(2) & (3) with the following 2 equations.

$\Sigma M_{x'} = 0$: (981 N)(1 m)+ (300 N)(2 m) – A_z(2 m) = 0 (4)

$$\Rightarrow \quad A_z = 790.5 \text{ N}$$

$\Sigma M_{y'} = 0$:

–300 N(1.5m) –(981 N) (1.5 m)–200 N·m +T_C(3 m) = 0 (5)

$$\Rightarrow \quad T_C = 707.2 \text{ N}$$

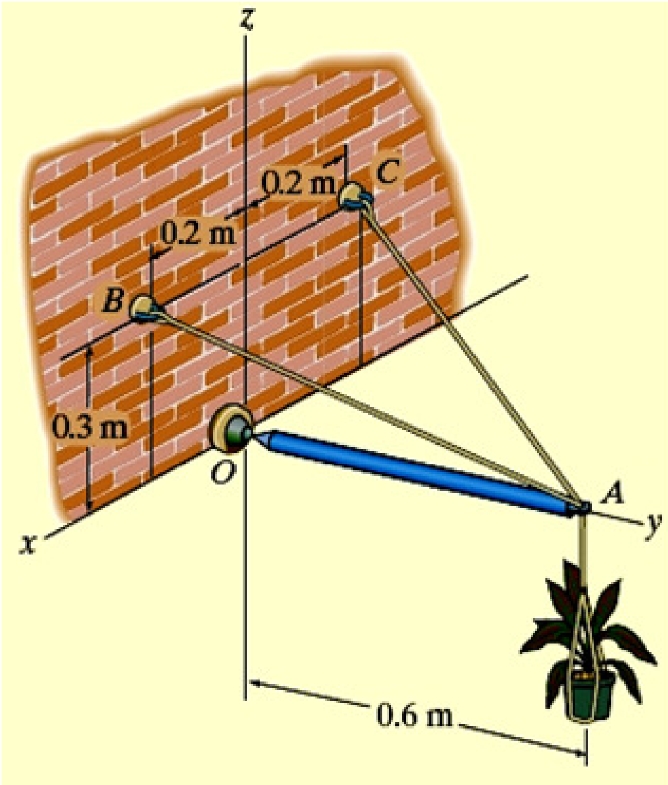
- Substituting A_z & T_C into Eq.(1), we get

$$B_z = 300 \text{ N} + 981 \text{ N} - A_z - T_C$$
$$\Rightarrow B_z = -216.7 \text{ N}$$

Example 5.17

Given :

- The boom is used to support the 375-N (= 37.5 kg) flowerpot.



Find :

Determine the tension developed in wires *AB* and *AC*.

Solution

Free-body diagram

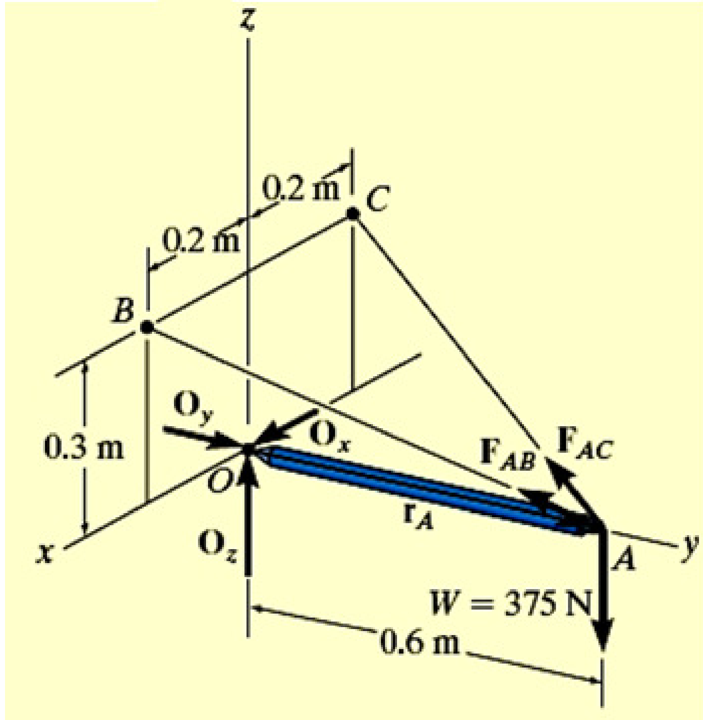
- *O : reaction force in x, y, z* directions
- B & *C*: Tension forces

Express all forces as Cartesian vectors

- Coordinates: *A* (0, 0.6, 0) m

  B (0.2, 0, 0.3) m

  C (– 0.2, 0, 0.3) m



- Position vectors:

$$\mathbf{r}_{AB} = (x_B - x_A)\mathbf{i} + (y_B - y_A)\mathbf{j} + (z_B - z_A)\mathbf{k}$$

$$= (0.2 - 0)\mathbf{i} + (0 - 0.6)\mathbf{j} + (0.3 - 0)\mathbf{k}$$

$$= \{0.2\mathbf{i} - 0.6\mathbf{j} + 0.3\mathbf{k}\}\ \text{m}$$

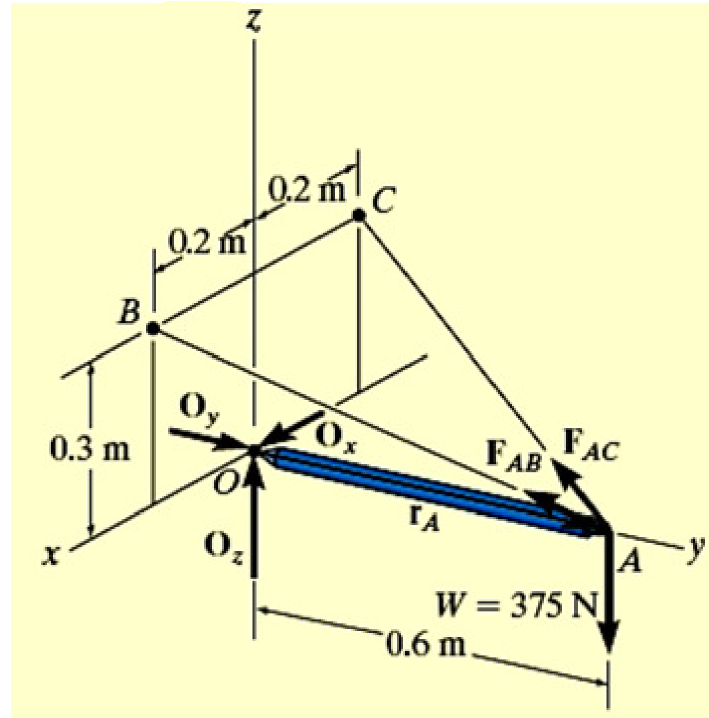
$$r_{AB} = \sqrt{(0.2)^2 + (-0.6)^2 + (0.3)^2} = 0.7\ \text{m}$$

$$\mathbf{r}_{CA} = (x_C - x_A)\mathbf{i} + (y_C - y_A)\mathbf{j} + (z_C - z_A)\mathbf{k}$$

$$= (-0.2 - 0)\mathbf{i} + (0 - 0.6)\mathbf{j} + (0.3 - 0)\mathbf{k}$$

$$= \{-0.2\mathbf{i} - 0.6\mathbf{j} + 0.3\mathbf{k}\}\ \text{m}$$

$$r_{AC} = \sqrt{(-0.2)^2 + (-0.6)^2 + (0.3)^2} = 0.7\ \text{m}$$



- Forces

$$\mathbf{F}_{AB} = F\,\mathbf{u}_{AB} = F_B\left(\frac{\mathbf{r}_{AB}}{r_{AB}}\right)$$

$$= F_{AB}\left(\frac{0.2\mathbf{i} - 0.6\mathbf{j} + 0.3\mathbf{k}}{0.7}\right)$$

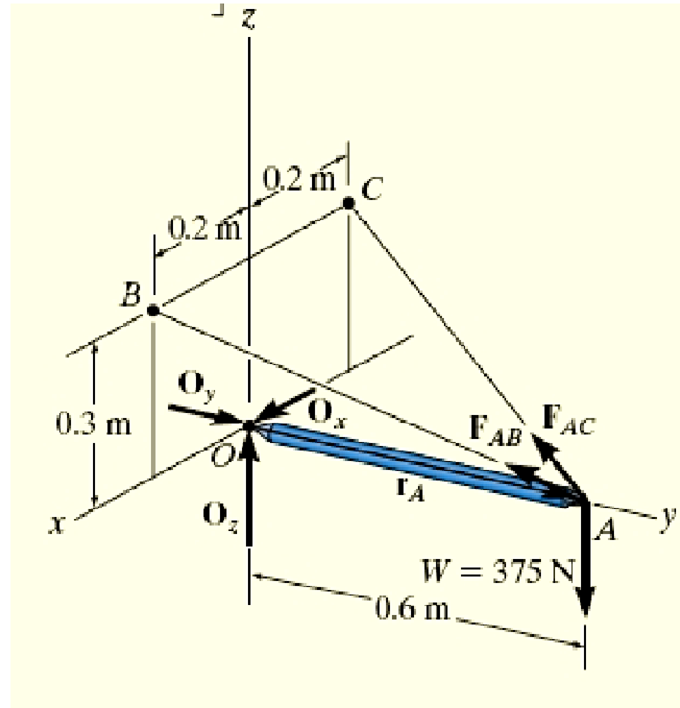
$$= \frac{2}{7}F_{AB}\mathbf{i} - \frac{6}{7}F_{AB}\mathbf{j} + \frac{3}{7}F_{AB}\mathbf{k}$$

$$\mathbf{F}_{AC} = F\,\mathbf{u}_{AC}$$

$$= F_{AC}\left(\frac{-0.2\mathbf{i}-0.6\mathbf{j}+0.3\mathbf{k}}{0.7}\right)$$

$$= -\frac{2}{7}F_{AC}\mathbf{i}-\frac{6}{7}F_{AC}\mathbf{j}+\frac{3}{7}F_{AC}\mathbf{k}$$

$$\mathbf{W} = -\,375\ \mathbf{k}$$



(b)

Equations of Equilibrium

$$\sum \mathbf{M}_O = \mathbf{0}: \qquad \mathbf{r}_A \times (\mathbf{F}_{AB} + \mathbf{F}_{AC} + \mathbf{W}) = \mathbf{0}$$

$$(0.6\mathbf{j})\times\left[\left(\frac{2}{7}F_{AB}\mathbf{i}-\frac{6}{7}F_{AB}\mathbf{j}+\frac{3}{7}F_{AB}\mathbf{k}\right)+\left(-\frac{2}{7}F_{AC}\mathbf{i}-\frac{6}{7}F_{AC}\mathbf{j}+\frac{3}{7}F_{AC}\mathbf{k}\right)+(-375\mathbf{k})\right]=\mathbf{0}$$

$$\left(\frac{1.8}{7}F_{AB}+\frac{1.8}{7}F_{AC}-225\right)\mathbf{i}+\left(-\frac{1.2}{7}F_{AB}+\frac{1.2}{7}F_{AC}\right)\mathbf{k}=\mathbf{0}$$

- Equating the respective **i**, **j**, **k** components, we have

i : $\Sigma F_x = 0$: $$\frac{1.8}{7} F_{AB} + \frac{1.8}{7} F_{AC} - 225 = 0 \qquad (1)$$

j: $\Sigma F_y = 0$: $$0 = 0$$

k: $\Sigma F_z = 0$: $$-\frac{1.2}{7} F_{AB} + \frac{1.2}{7} F_{AC} = 0 \qquad (2)$$

- From Eq. (2), $$F_{AC} = F_{AB} \qquad (3)$$

- Substituting Eq.(3) into Eq. (1), we have

$$\frac{3.6}{7} F_{AB} = 225$$

$$\Rightarrow F_{AB} = 437.5 \text{ N}$$

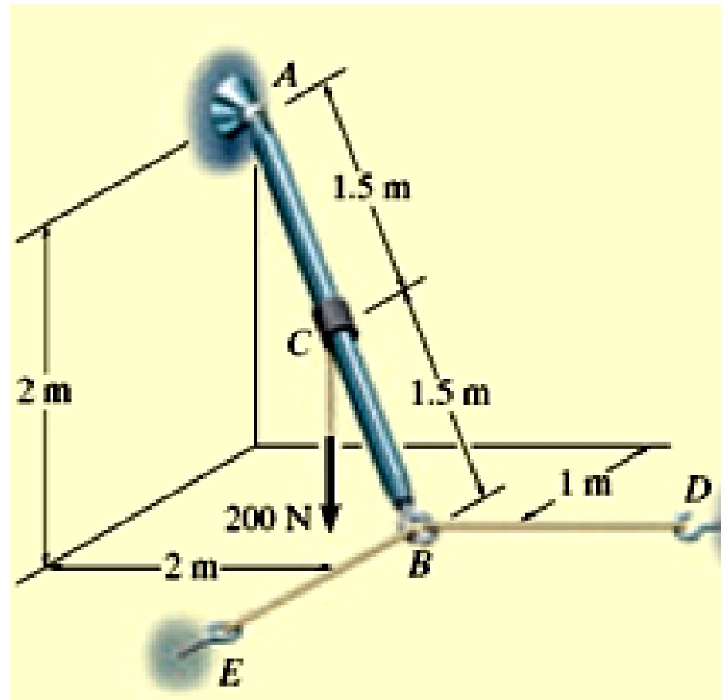
- Thus,

$$F_{AC} = F_{AB} = 437.5 \text{ N}$$

Example 5.18

Given :

- Rod *AB* shown in the figure is subjected to the 200-N force.



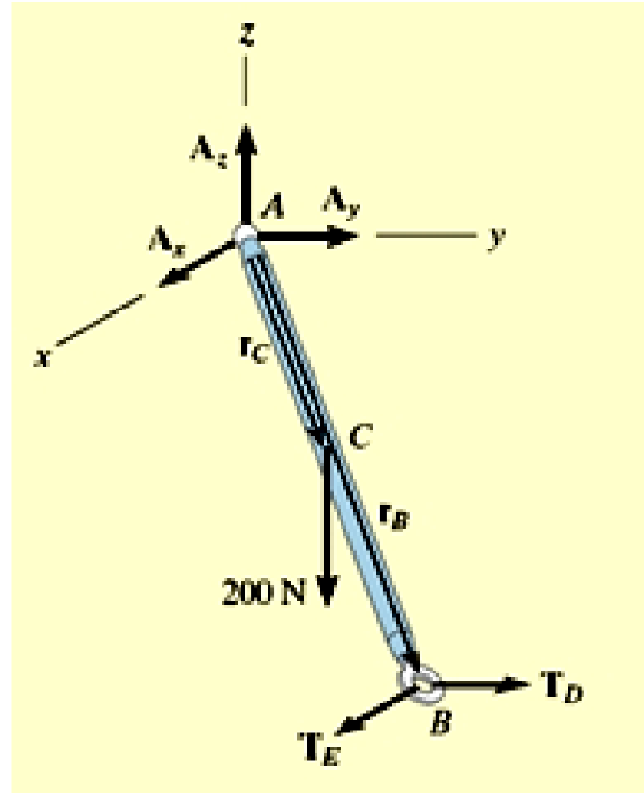
Find :

Determine the reactions at the ball-and-socket joint *A* and the tension in the cables *BC* and *BE*.

Solution

Free-body diagram

- Ball-and-socket joint at *A* : reaction $\mathbf{F}_A$ has *x, y, z* components.
- *B*: Tension force T_D & T_E
- *C* : Applied force F= 200 N



Express all forces as Cartesian vectors

$$\mathbf{F}_A = A_x\mathbf{i} + A_y\mathbf{j} + A_z\mathbf{k}$$

$$\mathbf{T}_E = T_E\mathbf{i}$$

$$\mathbf{T}_D = T_D\mathbf{j}$$

$$\mathbf{F} = \{-200\,\mathbf{k}\}\text{ N}$$

Force Equations of Equilibrium

$$\sum \mathbf{F} = \mathbf{0}: \qquad \mathbf{F}_A + \mathbf{T}_E + \mathbf{T}_D + \mathbf{F} = \mathbf{0}$$

$$(A_x\mathbf{i} + A_y\mathbf{j} + A_z\mathbf{k}) + T_E\mathbf{i} + T_D\mathbf{j} - 200\,\mathbf{k} = 0\mathbf{i} + 0\mathbf{j} + 0\,\mathbf{k}$$

$$(A_x + T_E)\mathbf{i} + (A_y + T_D)\mathbf{j} + (A_z - 200)\,\mathbf{k} = 0\mathbf{i} + 0\mathbf{j} + 0\,\mathbf{k}$$

- Equating the respective **i**, **j**, **k** components, we have

$\mathbf{i}: \quad \Sigma F_x = 0: \qquad A_x + T_E = 0 \qquad (1)$

$\mathbf{j}: \quad \Sigma F_y = 0: \qquad A_y + T_D = 0 \qquad (2)$

$\mathbf{k}: \quad \Sigma F_z = 0: \qquad A_z - 200 = 0 \qquad (3)$

Moment Equations of Equilibrium

- Coordinates: $A\ (0, 0, 0)$ m

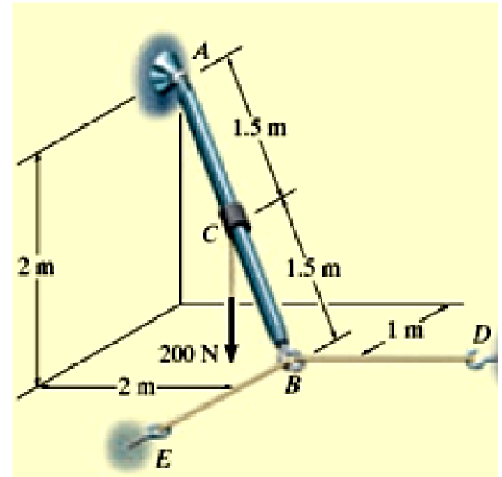
  $B\ (1, 2, -2)$ m

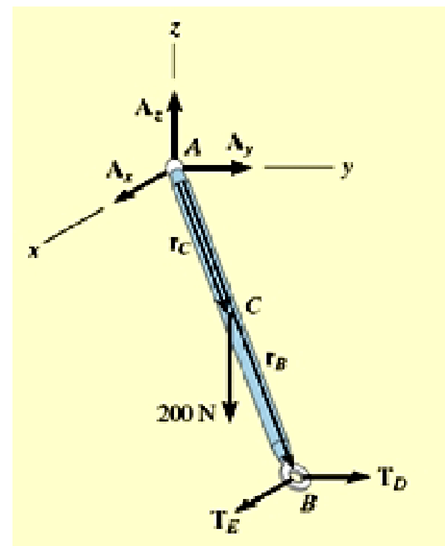
$$\mathbf{r}_B = \mathbf{r}_{AB} = (x_B - x_A)\mathbf{i} + (y_B - y_A)\mathbf{j} + (z_B - z_A)\mathbf{k}$$

$$= (1 - 0)\mathbf{i} + (2 - 0)\mathbf{j} + (-2 - 0)\mathbf{k}$$

$$= \{1\mathbf{i} + 2\mathbf{j} - 2\mathbf{k}\}\ \text{m}$$

$$\mathbf{r}_C = \mathbf{r}_{AC} = \tfrac{1}{2}\,\mathbf{r}_B = 0.5\mathbf{i} + 1\mathbf{j} - 1\mathbf{k}$$





- Summing moments about point A:

$$\sum \mathbf{M}_A = \mathbf{0}: \quad \mathbf{r}_C \times \mathbf{F} + \mathbf{r}_B \times (\mathbf{T}_E + \mathbf{T}_D) = \mathbf{0}$$

$$(0.5\mathbf{i} + 1\mathbf{j} - 1\mathbf{k}) \times (-200\,\mathbf{k}) + (1\mathbf{i} + 2\mathbf{j} - 2\mathbf{k}) \times (T_E\mathbf{i} + T_D\mathbf{j}) = \mathbf{0}$$

$$(2T_D - 200)\mathbf{i} + (-2T_E + 100)\,\mathbf{j} + (T_D - 2T_E)\mathbf{k} = 0\mathbf{i} + 0\mathbf{j} + 0\,\mathbf{k}$$

- Equating the respective **i**, **j**, **k** components, we have

$$\mathbf{i}: \Sigma F_x = 0: \qquad 2T_D - 200 = 0 \qquad (4)$$

$$\mathbf{j}: \Sigma F_y = 0: \qquad -2T_E + 100 = 0 \qquad (5)$$

$$\mathbf{k}: \Sigma F_z = 0: \qquad T_D - 2T_E = 0 \qquad (6)$$

- Solving Eqs.(1) ~ (6), we get

$$T_D = 100 \text{ N}$$

$$T_E = 50\text{N}$$

$$A_x = -50\text{N}$$

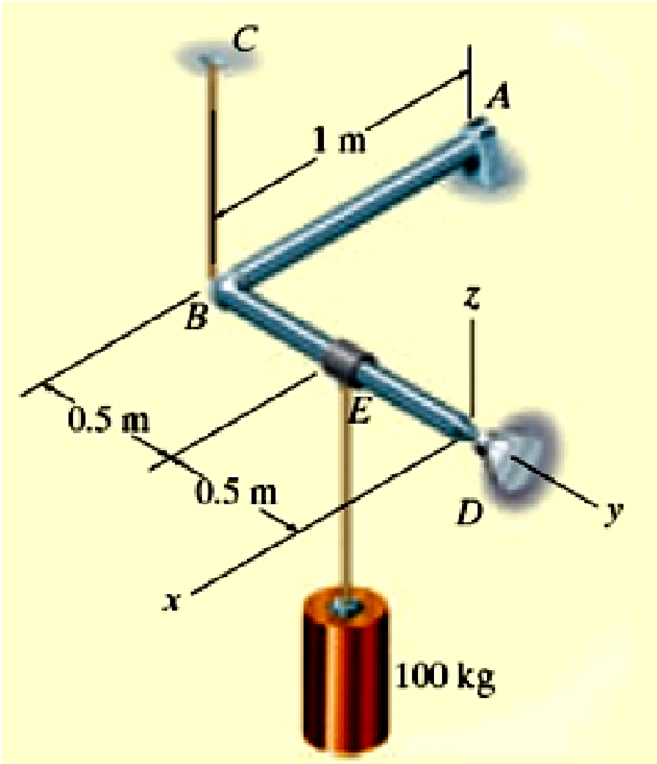
$$A_y = -100\text{N}$$

$$A_z = 200\text{N}$$

Example 5.19

Given :

- The bent rod in the figure is supported at *A* by journal bearing, at *D* by a ball-and-socket joint, and at *B* by means of cable *BC*.
- The bearing at *A* is capable of exerting force components only in the *z* and *y* directions since it is properly aligned on the shaft.



Find :

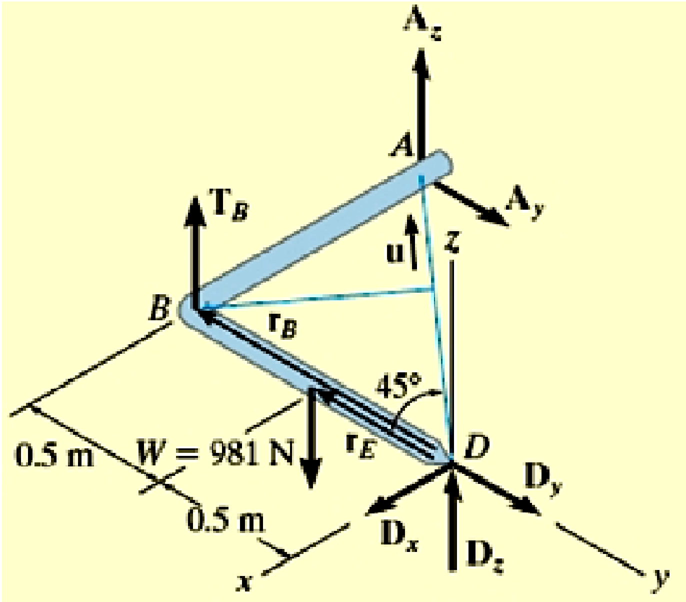
Determine the tension in cable *BC* using only one equilibrium equation.

Solution

Free-body diagram

- Journal bearing at A : *reaction has* y, z components only.
- Ball and socket joint at D : reaction has x, y, z components.
- B: Tension force T_B
- Weight at E:

  $W = (100 \text{ kg})(9.81 \text{ m/s}^{-2}) = 981 \text{ N}$



Equations of Equilibrium

- Since only T_B required, we may sum the moments about the axis that passes through points D and A as the moments produced by the reactive forces at A and D are zero about this axis.

Method I : Vector Analysis

- **Unit vector along the axis *DA*.**

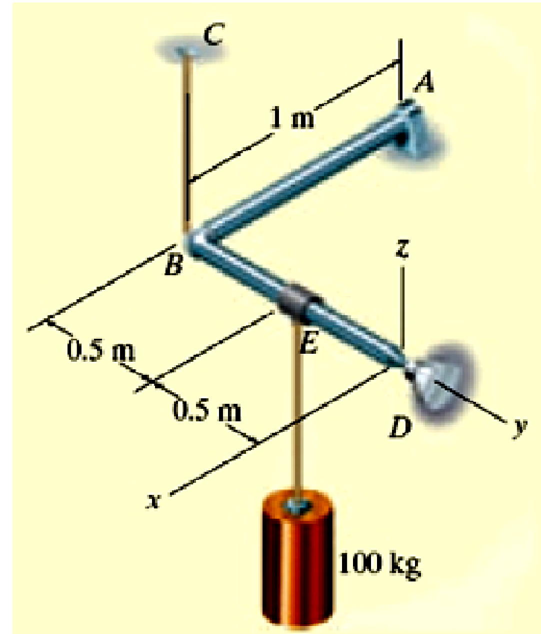
$$\mathbf{u} = \frac{\mathbf{r}_{DA}}{r_{DA}}$$

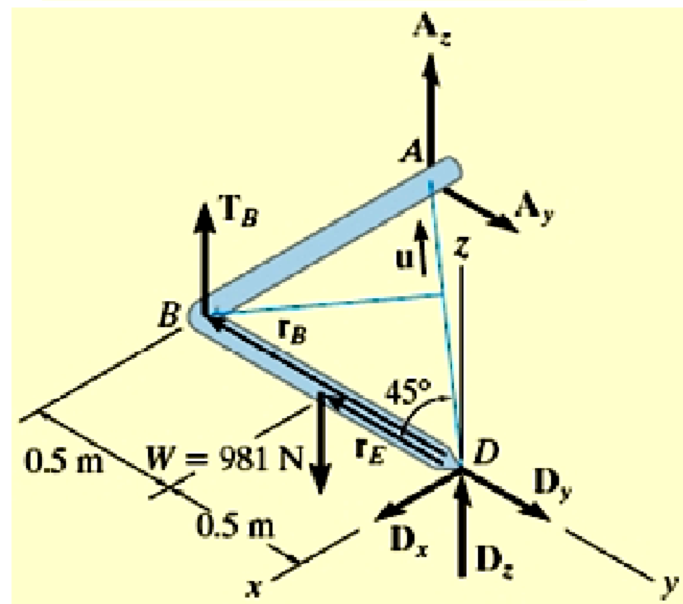
$$= \frac{(x_A - x_D)\mathbf{i} + (y_A - y_D)\mathbf{j} + (z_A - z_D)\mathbf{k}}{\sqrt{(x_A - x_D)^2 + (y_A - y_D)^2 + (z_A - z_D)^2}}$$

$$= \frac{(-1-0)\mathbf{i} + (-1-0)\mathbf{j} + (0-0)\mathbf{k}}{\sqrt{(-1-0)^2 + (-1-0)^2 + (0-0)^2}}$$

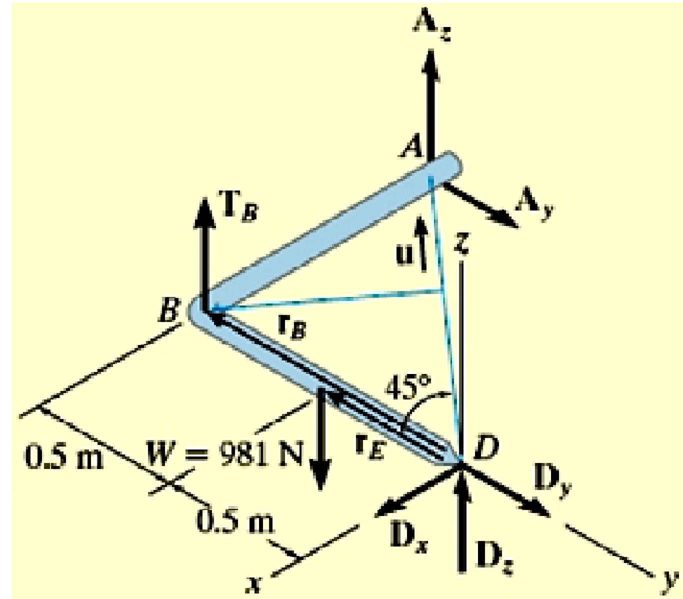
$$= -\frac{1}{\sqrt{2}}\mathbf{i} - \frac{1}{\sqrt{2}}\mathbf{j}$$

$$= -0.7071\,\mathbf{i} - 0.7071\,\mathbf{j}$$





- **Taking the moments about *DA***



$$\Sigma M_{DA} = \mathbf{u} \cdot \Sigma(\,\mathbf{r} \times \mathbf{F}\,) = 0$$

$$\mathbf{u} \cdot (\,\mathbf{r}_B \times \mathbf{T}_B + \mathbf{r}_E \times \mathbf{W}) = 0$$

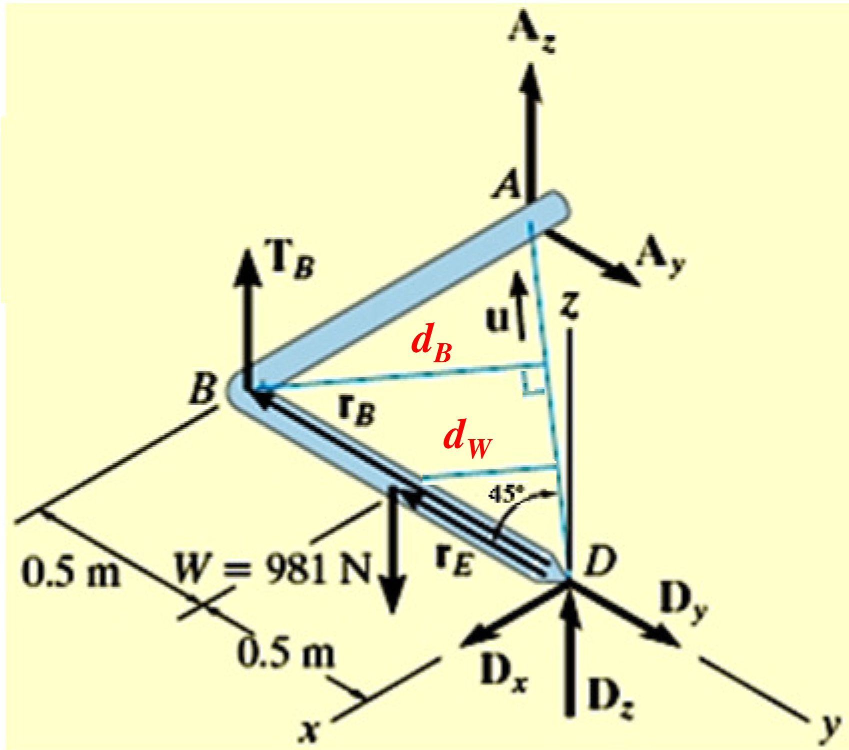
$$(-0.7071\,\mathbf{i} - 0.7071\,\mathbf{j}) \cdot [\ (-1.0\,\mathbf{j}) \times T_B\,\mathbf{k} + (-0.5\,\mathbf{j}) \times (-981\,\mathbf{k}\,) = 0$$

$$(-0.7071\,\mathbf{i} - 0.7071\,\mathbf{j}) \cdot [\,(-T_B - 490.5)\,\mathbf{i}\,] = 0$$

$$-0.7071\ (-T_B - 490.5) = 0$$

$$\Rightarrow\ T_B = 490.5\ \text{N}$$

Method II : Scalar Analysis



$$\circlearrowleft + \Sigma M_{DA} = 0: \quad -T_B\, d_B + W\, d_W = 0$$

$$-T_B(1 \text{ m} \sin 45^o) + (981 \text{ N})(0.5 \text{ m} \sin 45^o) = 0$$

$$\Rightarrow \quad T_B = 490.5 \text{ N}$$

5.7 Constraints and Statical Determinacy

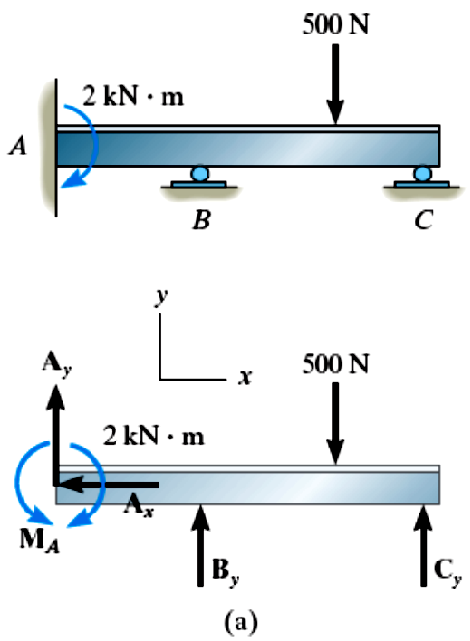
- To ensure the equilibrium of a rigid body, the body must also be properly held or constrained by its supports.

❑ Redundant Constraints

- A body is said to have redundant constraints if it has more supports than are necessary to hold it in equilibrium.
- A body with redundant constraints is ***statically indeterminate*** as there are more unknown loadings than equations of equilibrium.

- **Examples of statically indeterminate problems.**

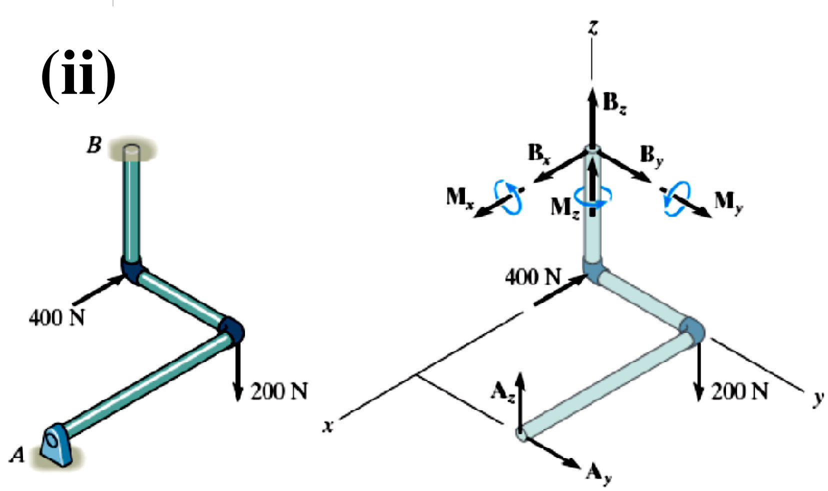
(i)



No. of equilibrium equations = 3

No. of unknowns = 5

(ii)



No. of equilibrium equations = 6

No. of unknowns = 8

Improper Constraints

- Having the same number of unknowns as the available equations of equilibrium does not always guarantee that a body will be stable when subjected to a particular loading.

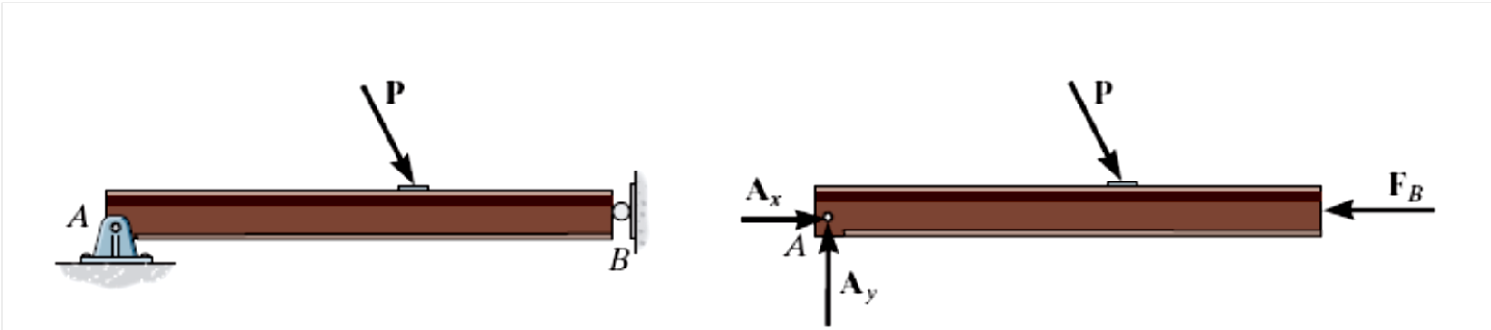
- Instability may occur in a rigid body if it is *improperly constrained* by its supports.

- A body is considered *improperly constrained* if

(1) all the reactive forces intersect at a common point (2-D case) or pass through a common axis (3-D case),

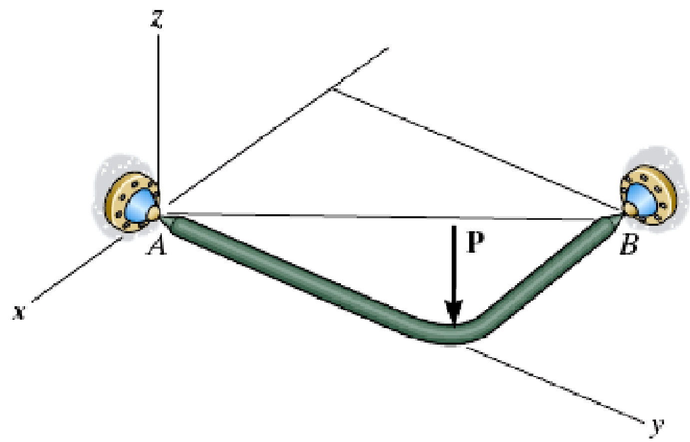
(2) all the reactive forces are parallel.

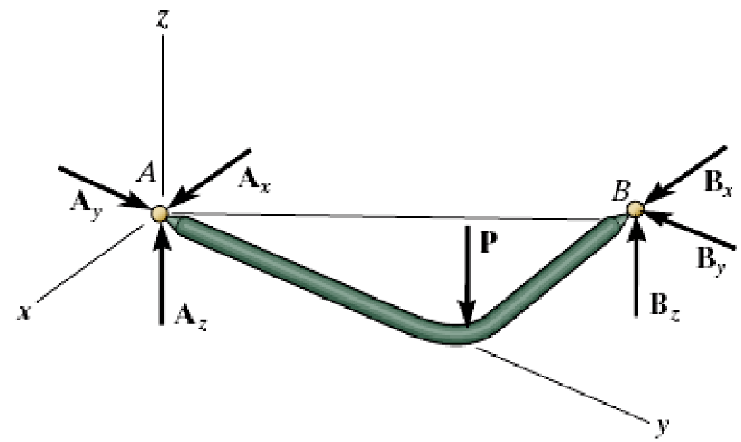
(a) 2-D Case



- The reactive forces A_x, A_y, and F_B are concurrent at point A. Therefore, the moments of these forces about A are zero.
- However, the presence of **P** causes $\sum M_A \neq 0$.
- Consequently, the beam will rotate about A.
- So, the beam is improperly constrained.

(b) 3-D Case





- The reactive forces at the ball-and socket supports A_x, A_y, A_z, B_x, B_y, and B_z, pass through the common axis *AB*.

  Therefore, the moments of these forces about *A* & *B* are all zero.

- However, the presence of **P** causes $\sum M_{AB} \neq 0$.
- Consequently, the member will rotate about the *AB* axis.
- So, the memeber is improperly constrained.