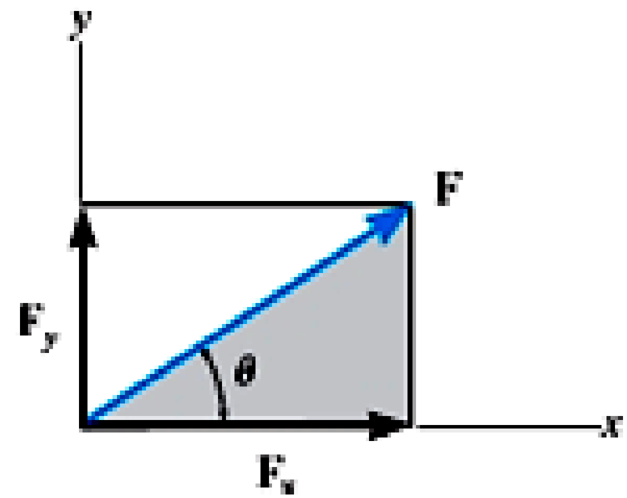
2.4 Addition of a System of Coplanar Forces

- A force can be resolved into 2 components along the *x* and *y* axes.

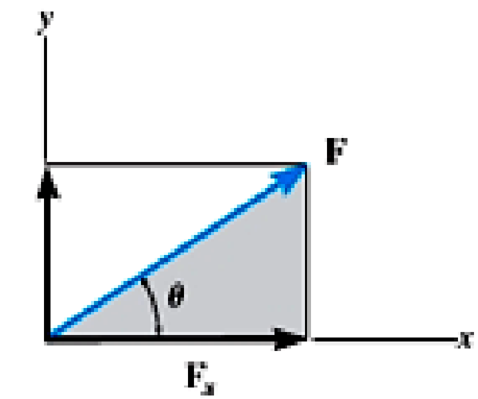
- These components are called ***rectangular components***.



- A force **F** can be represented by its rectangular components in 2 ways:
  - ❖ Scalar notation
  - ❖ Cartesian vector notation

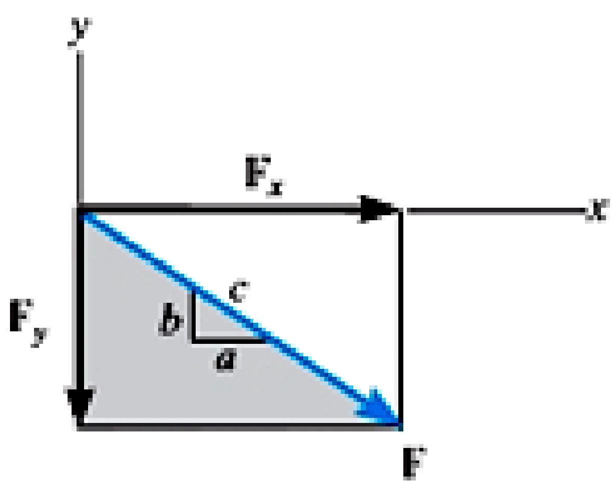
❑ Scalar Notation

- A force **F** is expressed in scalar form in terms of its rectangular components.



$$F_x = F \cos \theta$$

$$F_y = F \sin \theta$$



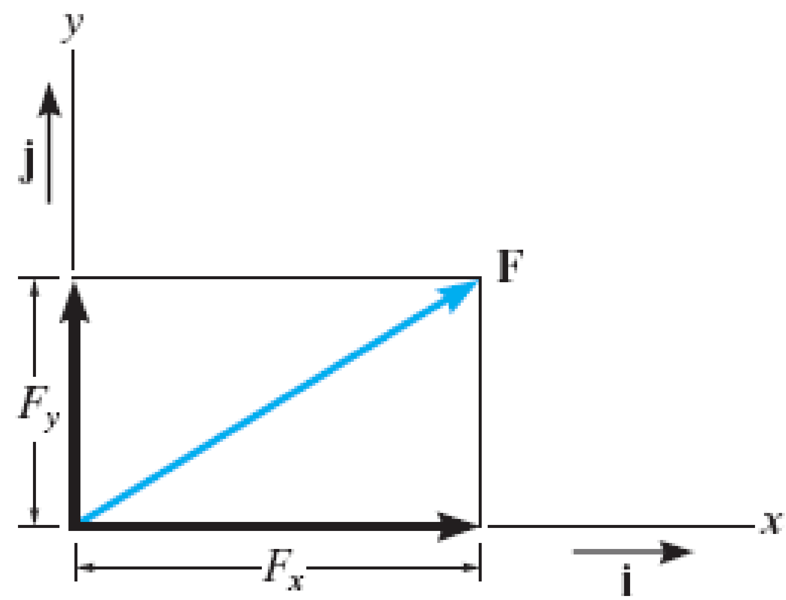
$$\frac{F_x}{F} = \frac{a}{c} \quad \Rightarrow \quad F_x = F\left(\frac{a}{c}\right)$$

$$\frac{F_y}{F} = \frac{b}{c} \quad \Rightarrow \quad F_y = F\left(\frac{b}{c}\right) \quad \downarrow$$

Cartesian Vector Notation

- A force **F** is expressed as a Cartesian vector in terms of unit vectors **i** and **j**

$$\mathbf{F} = F_x\,\mathbf{i} + F_y\,\mathbf{j}$$



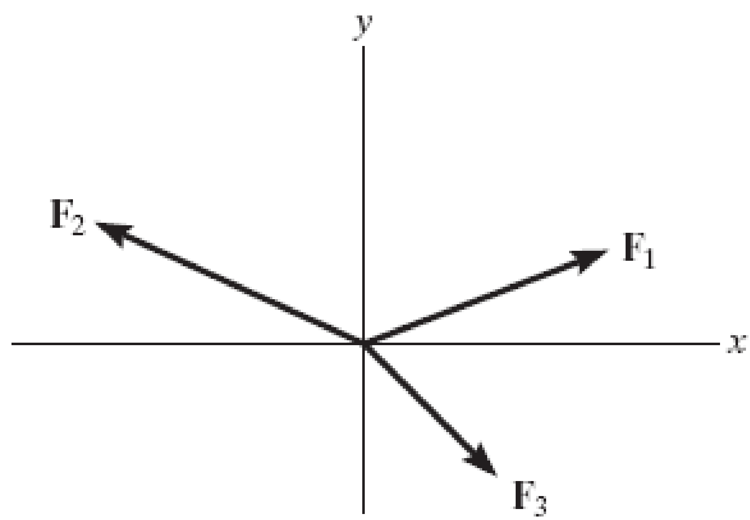
Note : Unit vectors **i** and **j** have dimensionless magnitude of unity (= 1)

❑ Coplanar Force Resultants

To determine resultant of several coplanar forces:

- Resolve each force into *x* and *y* components
- Add the respective components using scalar algebra
- Resultant force is found by adding the respective components of each force.

Example: Addition of 3 concurrent forces.

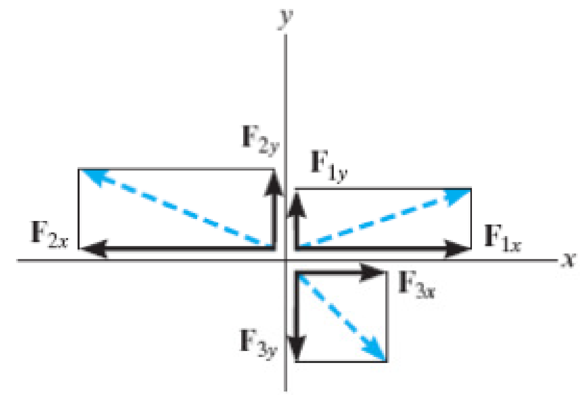


- Each force is first represened as a Cartesian vector

$$\mathbf{F}_1 = F_{1x}\,\mathbf{i} + F_{1y}\,\mathbf{j}$$

$$\mathbf{F}_2 = -F_{2x}\,\mathbf{i} + F_{2y}\,\mathbf{j}$$

$$\mathbf{F}_3 = F_{3x}\,\mathbf{i} - F_{3y}\,\mathbf{j}$$



- Find the resultant by adding the respective components
  - Vector notation

$$\begin{aligned}\mathbf{F}_R &= \mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3 \\ &= (F_{1x}\,\mathbf{i} + F_{1y}\,\mathbf{j}) + (-F_{2x}\,\mathbf{i} + F_{2y}\,\mathbf{j}) + (F_{3x}\,\mathbf{i} - F_{3y}\,\mathbf{j}) \\ &= (F_{1x} - F_{2x} + F_{3x})\,\mathbf{i} + (F_{1y} + F_{2y} - F_{3y})\,\mathbf{j} \\ &= F_{Rx}\,\mathbf{i} + F_{Ry}\,\mathbf{j}\end{aligned}$$

- Scalar notation

$(\overset{+}{\rightarrow}) \qquad F_{Rx} = F_{1x} - F_{2x} + F_{3x}$

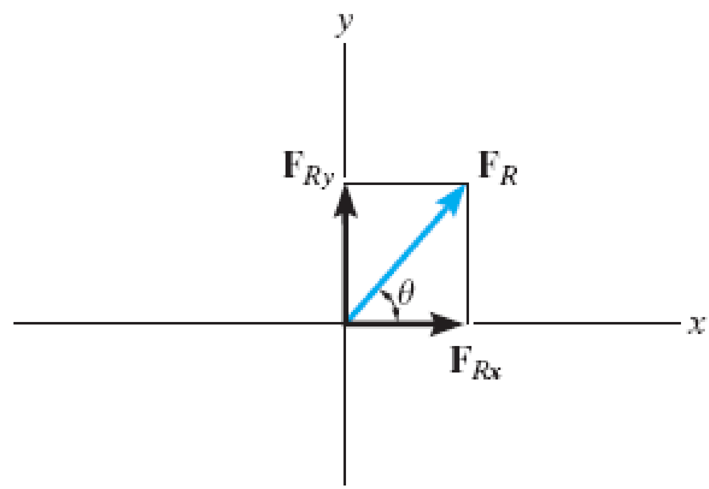
$(+\uparrow) \qquad F_{Ry} = F_{1y} + F_{2y} - F_{3y}$

Note : In both cases, $F_{Rx} = \Sigma F_x$, $F_{Ry} = \Sigma F_y$

- Magnitude of $\mathbf{F}_R$ can be found by Pythagorean Theorem

$$F_R = \sqrt{F_{Rx}^2 + F_{Ry}^2}$$

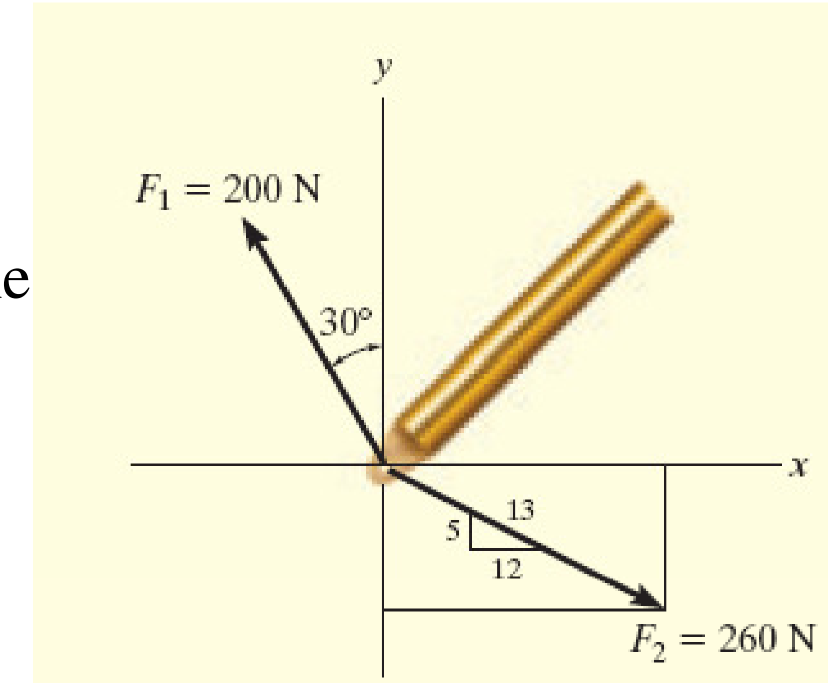
$$\theta = \tan^{-1}\left|\frac{F_{Ry}}{F_{Rx}}\right|$$



Example 2.5

Given :

Two forces $\mathbf{F}_1$ and $\mathbf{F}_2$ acting on the boom as shown.



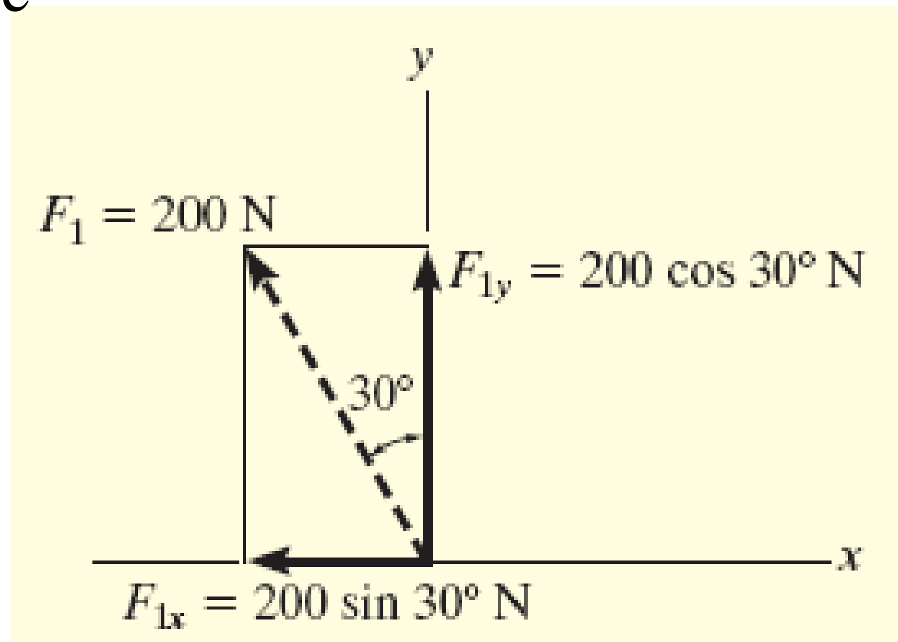
Find :

a) Determine *x* and *y* components of $\mathbf{F}_1$ and $\mathbf{F}_2$.
b) Express each force as a Cartesian vector.

Solution

a) Force Components

- Using parallelogram law to resolve $\mathbf{F}_1$ into x and y components.



$$(\overset{+}{\rightarrow}) \qquad F_{1x} = -200 \sin 30^\circ \text{ N}$$

$$= -100 \text{ N}$$

or $\quad F_{1x} = 100 \text{ N} \quad \leftarrow$

$$(+\uparrow) \qquad F_{1y} = 200 \cos 30^\circ \text{ N}$$

$$= 173 \text{ N}$$

or $\quad F_{1y} = 173 \text{ N} \quad \uparrow$

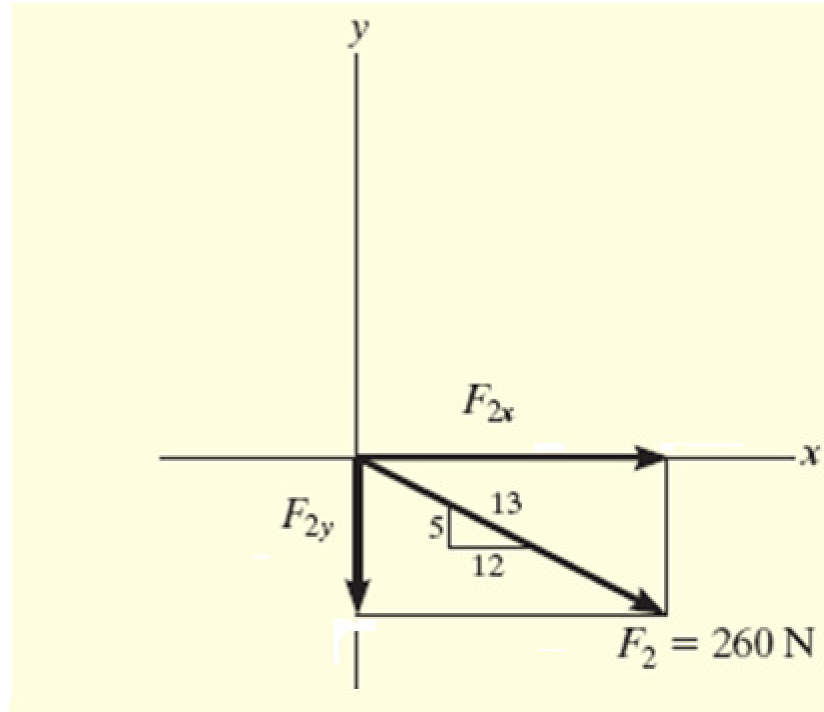
- Using parallelogram law to resolve $\mathbf{F}_2$ into x and y components.

$$(\overset{+}{\rightarrow})\quad F_{2x} = 260\left(\frac{12}{13}\right) = 240\text{N}$$

or $\quad F_{2x} = 240\ \text{N} \quad \rightarrow$

$$(+\uparrow)\quad F_{2y} = -260\left(\frac{5}{13}\right) = -100\text{N}$$

or $\quad F_{2y} = 100\ \text{N} \quad \downarrow$



b) Cartesian vector notation for each force

$$\mathbf{F}_1 = F_{1x}\,\mathbf{i} + F_{1y}\,\mathbf{j}$$

$$\Rightarrow \mathbf{F}_1 = \{ -100\,\mathbf{i} + 173\,\mathbf{j} \}\ \mathrm{N}$$

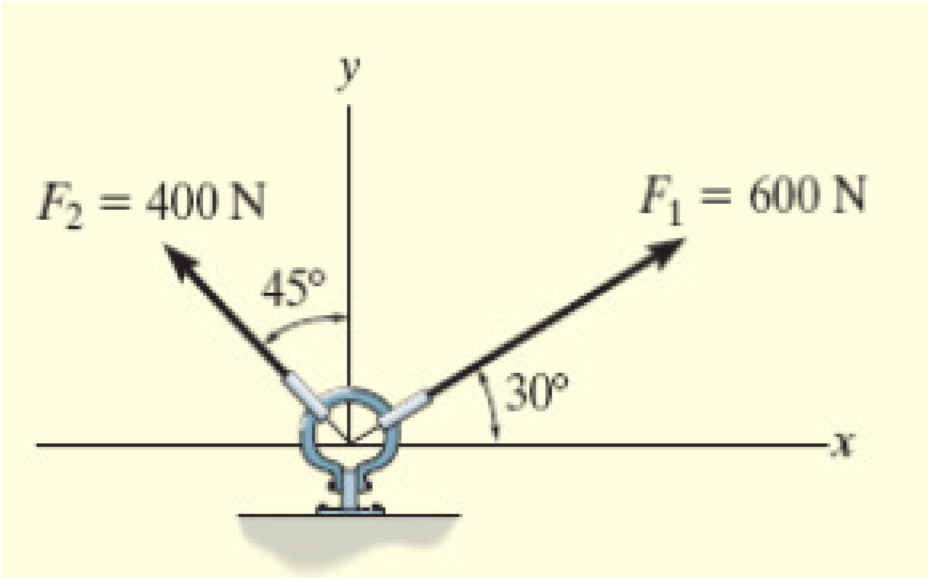
$$\mathbf{F}_2 = F_{2x}\,\mathbf{i} + F_{2y}\,\mathbf{j}$$

$$\Rightarrow \mathbf{F}_2 = \{ 240\,\mathbf{i} - 100\,\mathbf{j} \}\ \mathrm{N}$$

Example 2.6

Given :

The link is subjected to two forces $\mathbf{F}_1$ and $\mathbf{F}_2$.

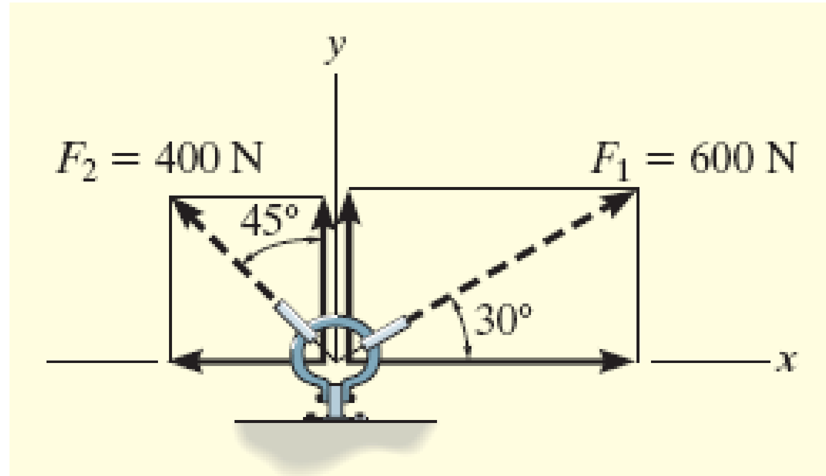


Find :

Determine the magnitude and direction of the resultant force.

Solution

I. Method 1: Scalar Notation



$$\overset{+}{\rightarrow} \quad F_{Rx} = \Sigma F_x : \qquad F_{Rx} = 600 \cos 30^\circ \ \text{N} - 400 \sin 45^\circ \ \text{N}$$

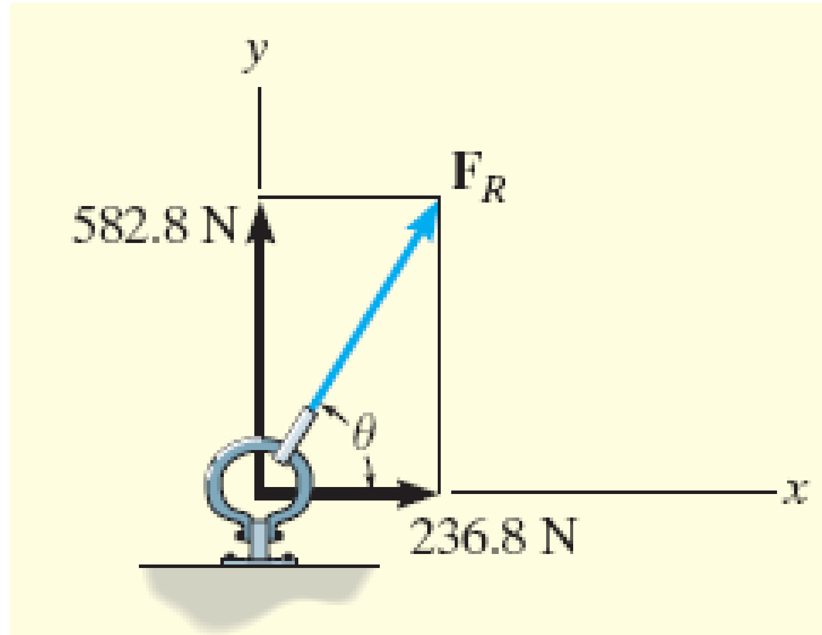
$$= 236.8 \text{ N}$$

$$+\uparrow \quad F_{Ry} = \Sigma F_y : \qquad F_{Ry} = 600 \sin 30^\circ \ \text{N} + 400 \cos 45^\circ \ \text{N}$$

$$= 582.8 \text{ N}$$

- Resultant Force

$$F_R = \sqrt{(236.8\text{ N})^2 + (582.8\text{ N})^2}$$
$$= 629\text{N}$$



- Direction

$$\theta = \tan^{-1}\left(\frac{582.8\text{N}}{236.8\text{N}}\right)$$
$$= 67.9^\circ$$

II. Method 2: Vector Notation

- ***Resolve each force into its x and y components :***

$$\mathbf{F}_1 = \{600 \cos 30^\circ \; \mathbf{i} + 600 \sin 30^\circ \; \mathbf{j}\} \text{ N}$$

$$\mathbf{F}_2 = \{-400 \sin 45^\circ \; \mathbf{i} + 400 \cos 45^\circ \; \mathbf{j}\} \text{ N}$$

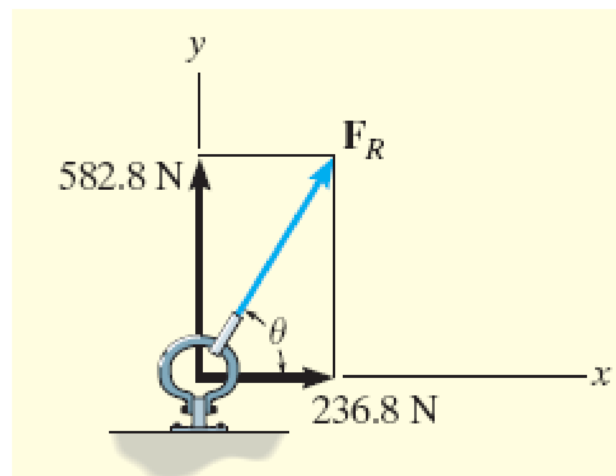
- ***Resultant force :***

$$\mathbf{F}_R = \mathbf{F}_1 + \mathbf{F}_2$$

$$= \{(600 \cos 30^\circ - 400 \sin 45^\circ)\mathbf{i} + (600 \sin 30^\circ + 400 \cos 45^\circ)\mathbf{j}\} \text{ N}$$

$$= \{236.8\mathbf{i} + 582.8\mathbf{j}\}\text{N}$$

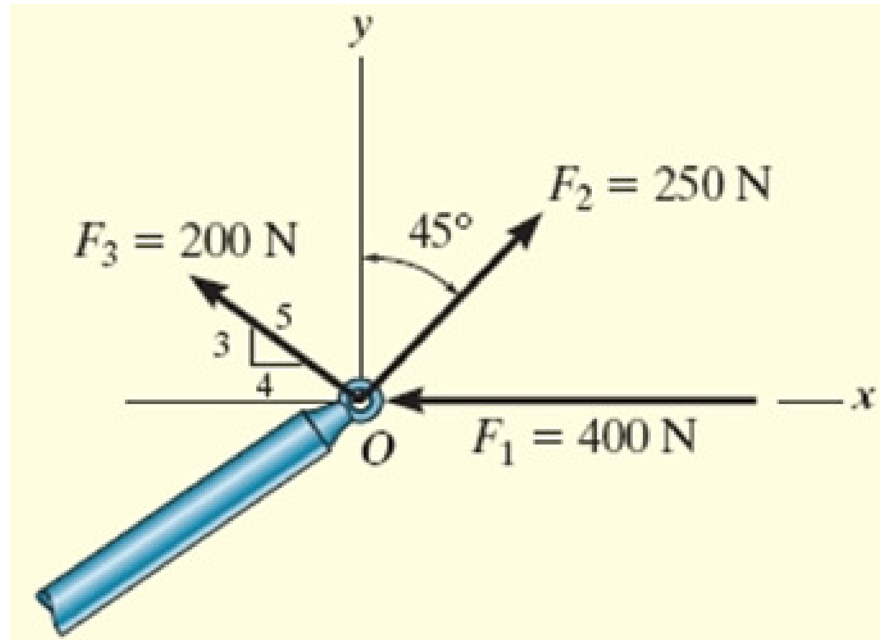
The magnitude and direction of $\mathbf{F}_R$ are determined in the same manner as before.



Example 2.7

Given :

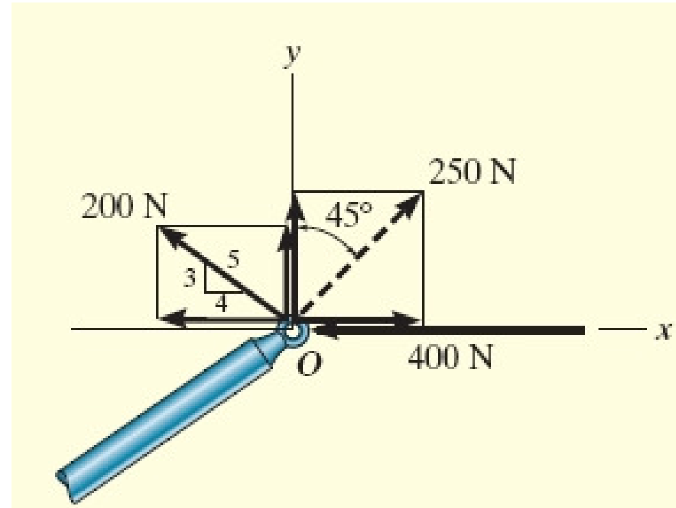
The end of the boom O is subjected to three concurrent and planar forces.



Find :

Determine the magnitude and direction of the resultant force.

Solution



$\overset{+}{\rightarrow}$ $F_{Rx} = \Sigma F_x :$ $\quad F_{Rx} = -400 + 250 \sin 45^\circ \text{ N} - 200(4/5) \text{ N}$

$$= -383.2 \text{ N}$$

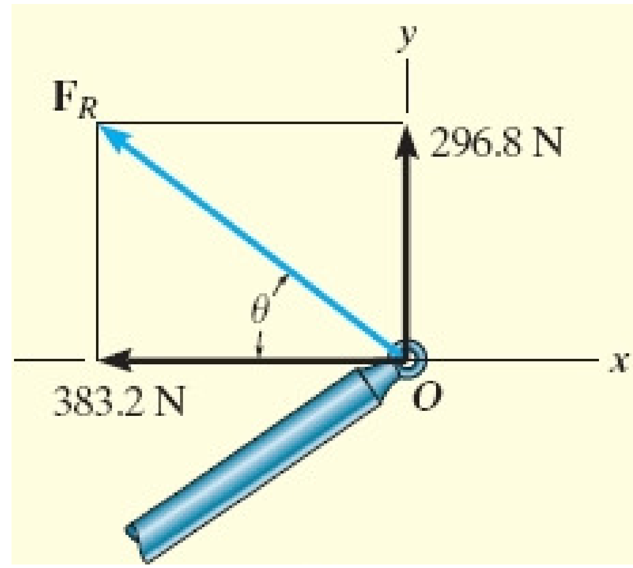
$$= 383.2 \text{ N} \leftarrow$$

$+\uparrow$ $F_{Ry} = \Sigma F_y :$ $\quad F_{Ry} = 250 \cos 45^\circ \text{ N} + 200\,(3/5) \text{ N}$

$$= 296.8 \text{ N} \uparrow$$

- Resultant Force

$$F_R = \sqrt{(-383.2\ \text{N})^2 + (296.8\ \text{N})^2}$$
$$= 485\ \text{N}$$



- Direction

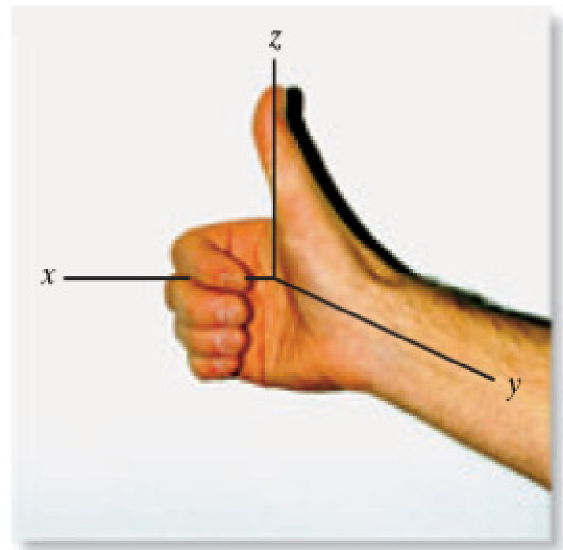
$$\theta = \tan^{-1}\left(\frac{296.8\text{N}}{383.2\text{N}}\right)$$
$$= 37.8^\circ$$

2.4 Cartesian Vectors

❑ Right-Handed Coordinate System

A rectangular or Cartesian coordinate system is said to be right-handed provided:

- Thumb of right hand points in the direction of the positive *z* axis when the right-hand fingers are curled about this axis and directed from the positive *x* toward the positive *y* axis.



Note : *z*-axis for the 2D problem would be perpendicular, directed out of the page.

Rectangular Components of a Vector

- A vector **A** may have one, two or three rectangular components along the *x, y* and *z* axes, depending on orientation.

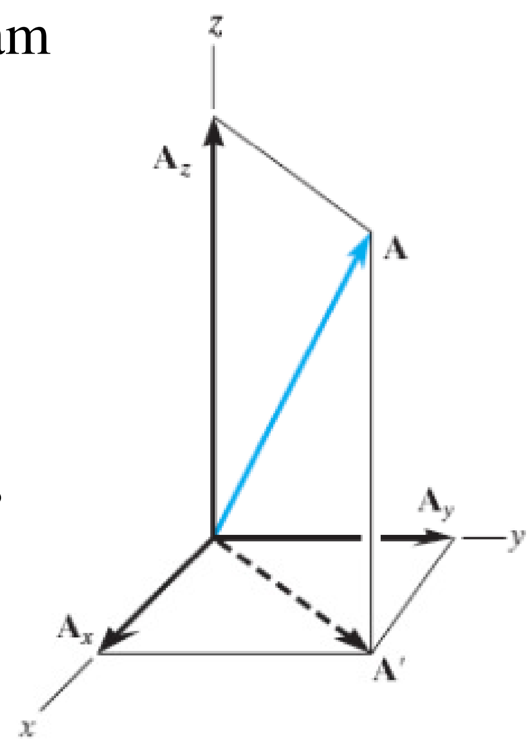
- By two successive application of the parallelogram law

$$\mathbf{A} = \mathbf{A}' + \mathbf{A}_z$$

$$\mathbf{A}' = \mathbf{A}_x + \mathbf{A}_y$$

- Combining the equations, **A** can be expressed as

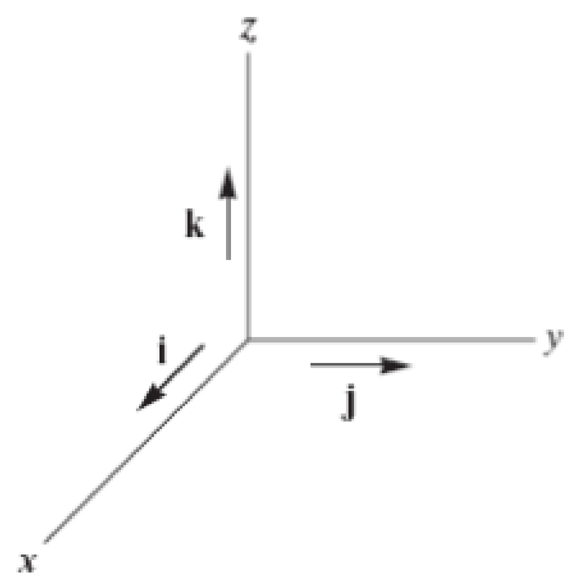
$$\mathbf{A} = \mathbf{A}_x + \mathbf{A}_y + \mathbf{A}_z$$



❑ Cartesian Unit Vectors

- In three dimensions, the set of Cartesian unit vectors **i**, **j**, **k** is used to designate the direction of the *x*, *y*, *z* axes, respectively.

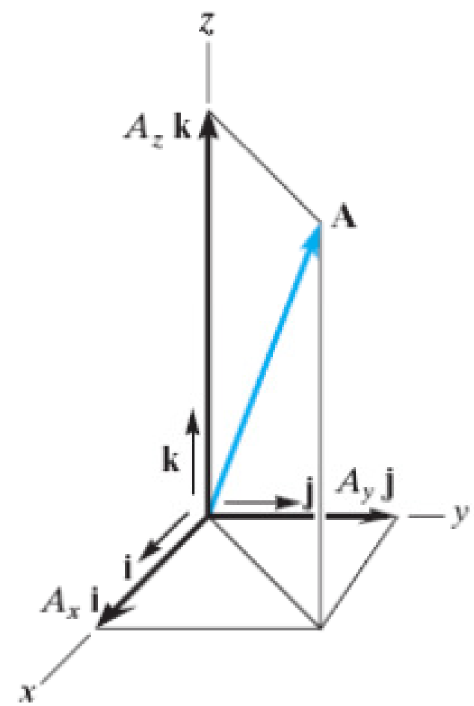
- The sense of these unit vectors is represented by a plus (+) sign or minus (–) sign.



❑ Cartesian Vector Representation

- A vector **A** with 3 components acting in the positive **i**, **j** and **k** directions can be written as

$$\mathbf{A} = A_x\,\mathbf{i} + A_y\,\mathbf{j} + A_Z\,\mathbf{k}$$



Note : The magnitude and direction of each components are separated to ease vector algebraic operations

❑ Magnitude of a Cartesian Vector

- From the blue right triangle,

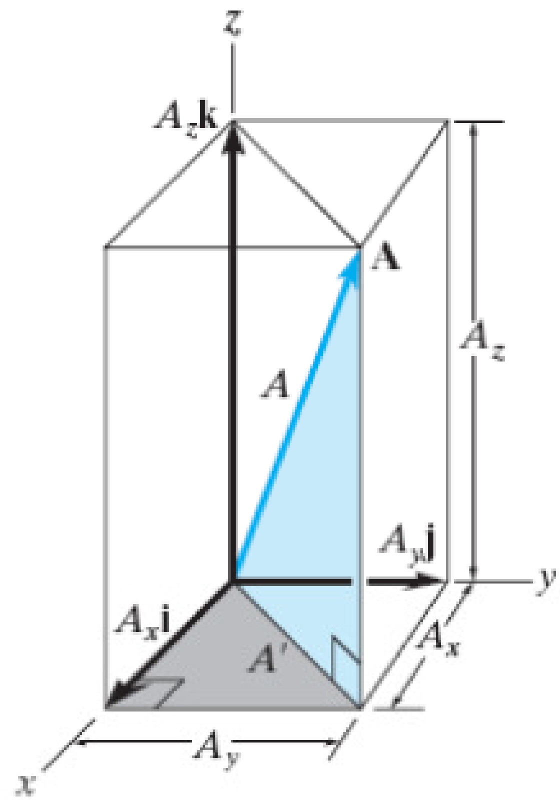
$$A = \sqrt{A'^2 + A_z^2}$$

- From the gray right triangle,

$$A' = \sqrt{A_x^2 + A_y^2}$$

- Combining these 2 equations yields

$$A = \sqrt{A_x^2 + A_y^2 + A_z^2}$$



Hence, the magnitude of **A** *is equal to the positive square root of the sum of the squares of its components.*

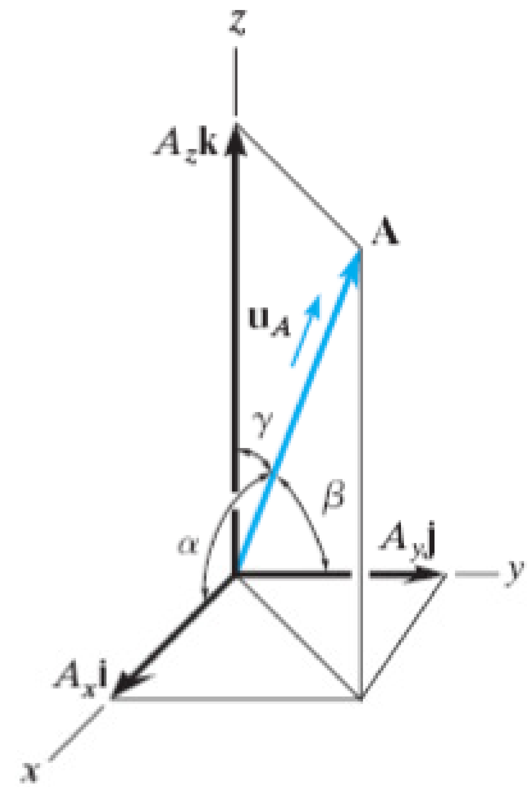
❑ Direction of a Cartesian Vector

- Direction of **A** is defined as the *coordinate direction angles* α, β and γ measured between the *tail* of **A** and the *positive* x, y and z axes.

- $0^\circ \le \alpha, \beta, \gamma \le 180^\circ$

- The direction cosines of **A** is

$$\left.\begin{aligned} \cos\alpha &= \frac{A_x}{A} \\ \cos\beta &= \frac{A_y}{A} \\ \cos\gamma &= \frac{A_z}{A} \end{aligned}\right\} \quad (1)$$



- The *coordinate direction angles* α, β and γ can be determined by the inverse cosines

❑ Uunit Vector in a Specified Direction

- A vector **A** can be expressed in terms of a unit vector $\mathbf{u}_A$ having the same direction as **A** as follows

$$\mathbf{A} = A\,\mathbf{u}_\mathrm{A}$$

where $\mathbf{u}_A = \dfrac{\mathbf{A}}{A}$

- If $\mathbf{A} = A_x\mathbf{i} + A_y\mathbf{j} + A_z\mathbf{k}$, then

$$\mathbf{u}_A = \frac{\mathbf{A}}{A} = \frac{A_x}{A}\mathbf{i} + \frac{A_y}{A}\mathbf{j} + \frac{A_z}{A}\mathbf{k} \qquad (2)$$

where $A = \sqrt{A_x^2 + A_y^2 + A_z^2}$

- From Eq.(1) & Eq. (2)

$$\mathbf{u}_A = \cos\alpha\ \mathbf{i} + \cos\beta\,\mathbf{j} + \cos\gamma\,\mathbf{k}$$

- Therefore, the magnitude of $\mathbf{u}_A$ is

$$u_A = \sqrt{\cos^2\alpha + \cos^2\beta + \cos^2\gamma} \qquad (3)$$

- Since $\mathbf{u}_A$ is a unit vector,

$$u_{\mathrm{A}} = 1 \qquad (4)$$

- Hence, from Eqs (3) & (4),

$$\boxed{\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1}$$

- In summary, if the magnitude and coordinate direction angles of vector **A** is known, then **A** may be expressed in Cartesian vector form as

$$\mathbf{A} = A\,\mathbf{u}_{\mathrm{A}}$$

$$= A\ (\cos\alpha\,\mathbf{i} + \cos\beta\,\mathbf{j} + \cos\gamma\,\mathbf{k})$$

$$= A\cos\alpha\,\mathbf{i} + A\cos\beta\,\mathbf{j} + A\cos\gamma\,\mathbf{k}$$

$$= A_x\,\mathbf{i} + A_y\,\mathbf{j} + A_Z\,\mathbf{k}$$

2.6 Addition and Subtraction of Cartesian Vectors

❑ Concurrent Force Systems

Procedures to find the resultant of a concurrent force system:

- Express each force as a Cartesian vector.
- Add the **i**, **j**, **k** components of all the forces in the system.
- Force resultant is the vector sum of all the forces in the system.

$$\mathbf{F}_R = \sum \mathbf{F} = \sum F_x \mathbf{i} + \sum F_y \mathbf{j} + \sum F_z \mathbf{k}$$

Example:

Find the resultant vector of **A** + **B**

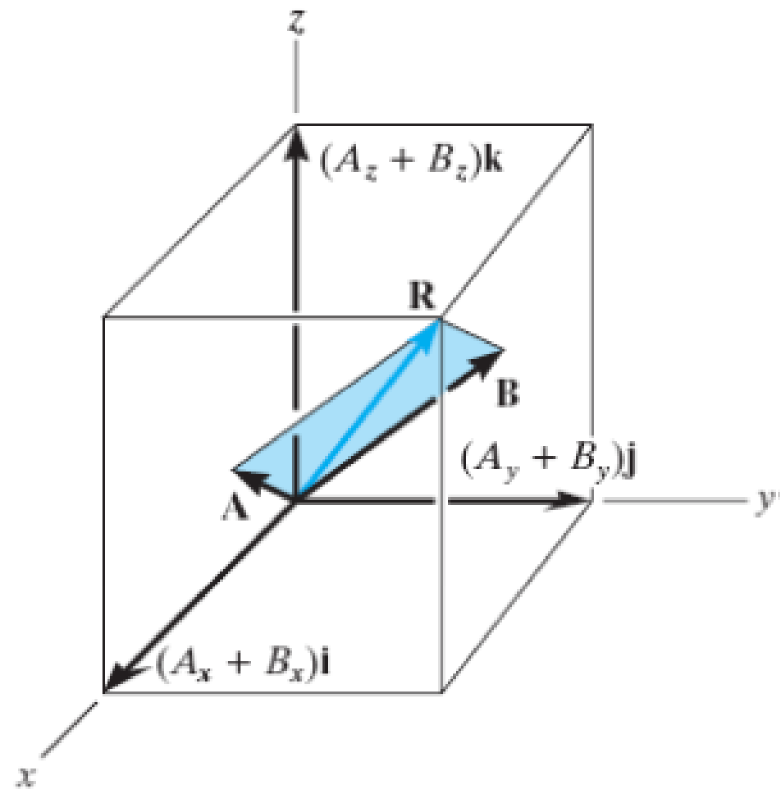
- Write $\mathbf{A} = A_x\,\mathbf{i} + A_y\mathbf{j} + A_Z\,\mathbf{k}$

  $\mathbf{B} = B_x\,\mathbf{i} + B_y\mathbf{j} + B_Z\,\mathbf{k}$

- The resultant is then given by

$$\mathbf{R} = \mathbf{A} + \mathbf{B}$$

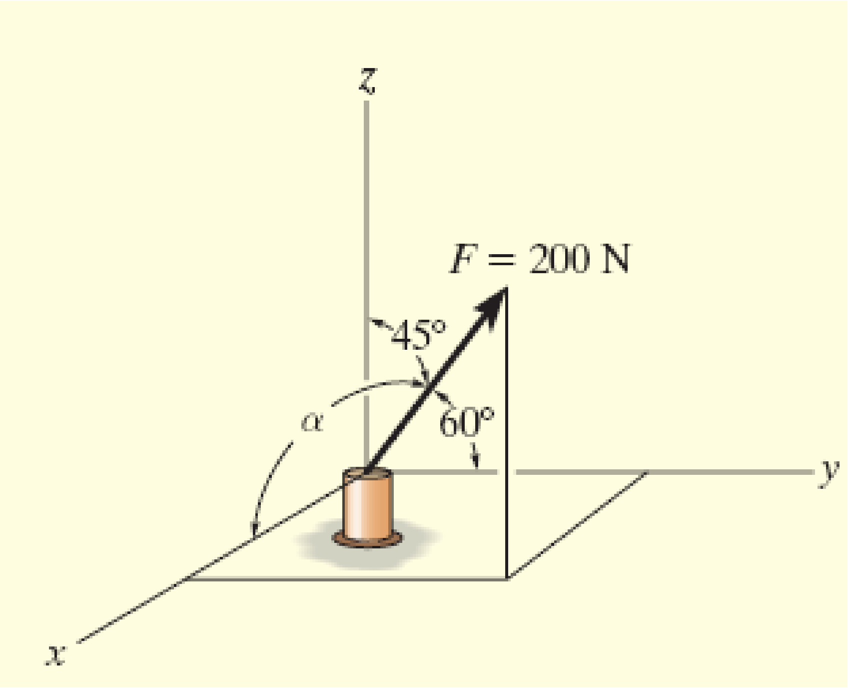
$$= (A_x + B_x)\,\mathbf{i} + (A_y + B_y)\,\mathbf{j} + (A_z + B_z)\,\mathbf{k}$$



Example 2.8

Given :

A force **F** of 200 N has a direction as shown.



Find :

Express the force **F** as a Cartesian vector.

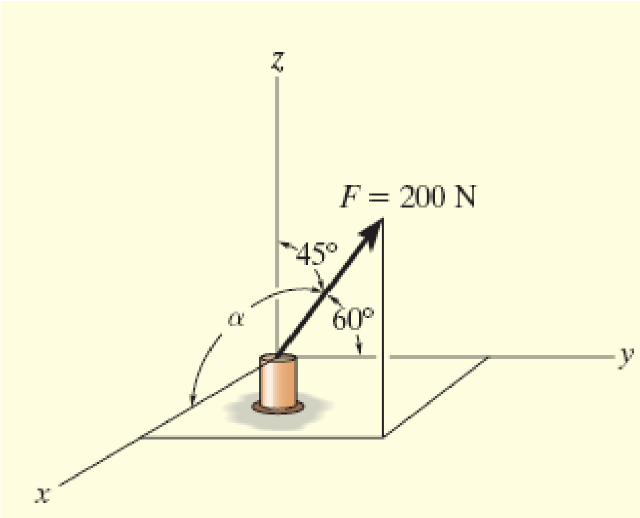
Solution

- Since two coordinate direction angles are specified, the third angle α is found by

$$\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = 1$$

$$\cos^2 \alpha + \cos^2 60° + \cos^2 45° = 1$$

$$\cos\alpha = \sqrt{1 - \cos^2 60^\circ - \cos^2 45^\circ} = \pm 0.5$$



- Thus,

$$\alpha = \cos^{-1}(0.5) = 60°$$

or

$$\alpha = \cos^{-1}(-0.5) = 120°$$

- From the figure, it is clear that F_x must be in the $+x$ direction.

  Therefore, $\alpha = 60°$

- Hence,

$$\mathbf{F} = F_x\,\mathbf{i} + F_y\,\mathbf{j} + F_Z\,\mathbf{k}$$

$$= F\cos\alpha\ \mathbf{i} + F\cos\beta\ \mathbf{j} + F\cos\gamma\,\mathbf{k}$$

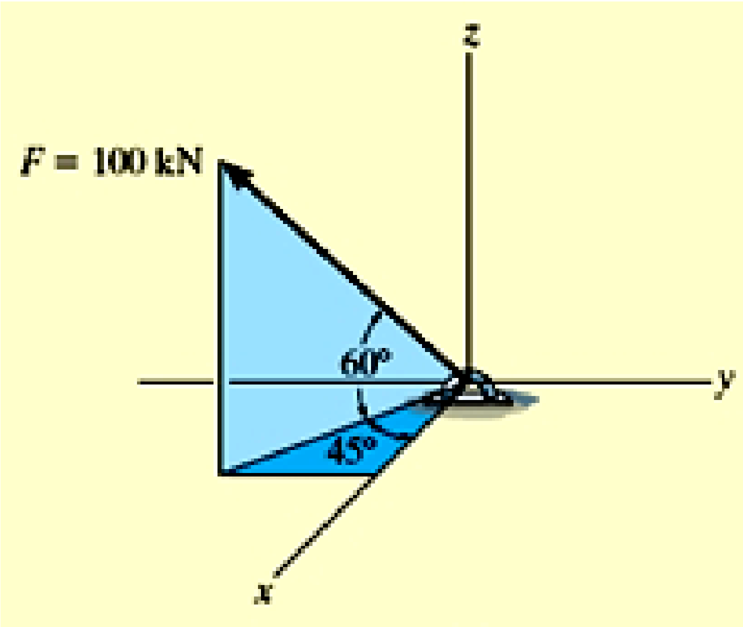
$$= 200\cos 60°\ \mathbf{i} + 200\cos 60°\ \mathbf{j} + 200\cos 45°\ \mathbf{k}$$

$$= \{100.0\ \mathbf{i} + 100.0\ \mathbf{j} + 141.4\ \mathbf{k}\}\ \text{N}$$

Example 2.10

Given :

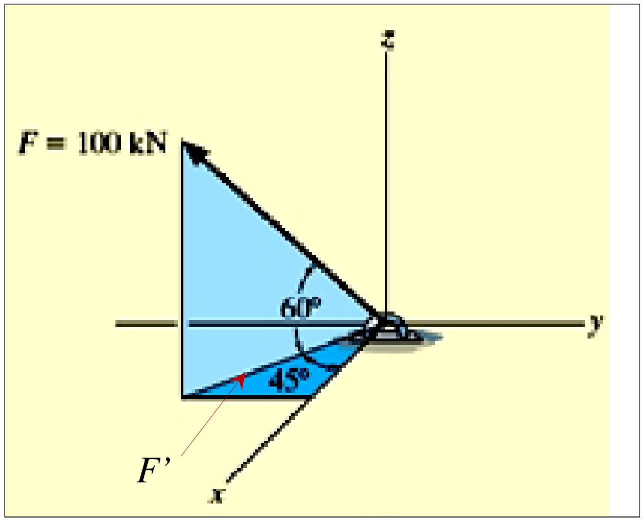
A force **F** of 100 kN has a direction as shown.



Find :

Express the force **F** as a Cartesian vector.

Solution



- Find the components of **F**:

$$F' = F \cos 60^\circ = 100 \ \cos 60^\circ \text{ kN}$$

$$F_x = F' \ \cos 45^\circ$$
$$= (100 \ \cos 60^\circ) \ \cos 45^\circ = 35.4 \ \text{kN}$$

$$F_y = F' \sin 45^\circ$$
$$= (100 \ \cos 60^\circ) \ \sin 45^\circ = 35.4 \ \text{kN}$$

$$F_z = F \sin 60^\circ = 100 \sin 60^\circ \text{ kN} = 86.6 \text{ kN}$$

- Thus,

$$\mathbf{F} = \{35.4 \ \mathbf{i} - -35.4 \ \mathbf{j} + 86.6 \ \mathbf{k}\} \text{ kN}$$

Comment: The following are not required by the problem.

- The magnitude of **F** is

$$F = \sqrt{(35.4)^2 + (-35.4)^2 + (86.6)^2} = 100\,\text{kN}$$

- Unit vector in the direction of **F:**

$$\mathbf{u}_F = \frac{\mathbf{F}}{F} = \frac{F_x}{F}\mathbf{i} + \frac{F_y}{F}\mathbf{j} + \frac{F_z}{F}\mathbf{k} \quad = \frac{35.4}{100}\mathbf{i} - \frac{35.4}{100}\mathbf{j} + \frac{86.6}{100}\mathbf{k}$$

$$= 0.354\ \mathbf{i} - 0.354\ \mathbf{j} + 0.866\ \mathbf{k}$$

- Coordinate direction angles of **F:**

$$\alpha = \cos^{-1}(0.354) = 69.3^\circ$$

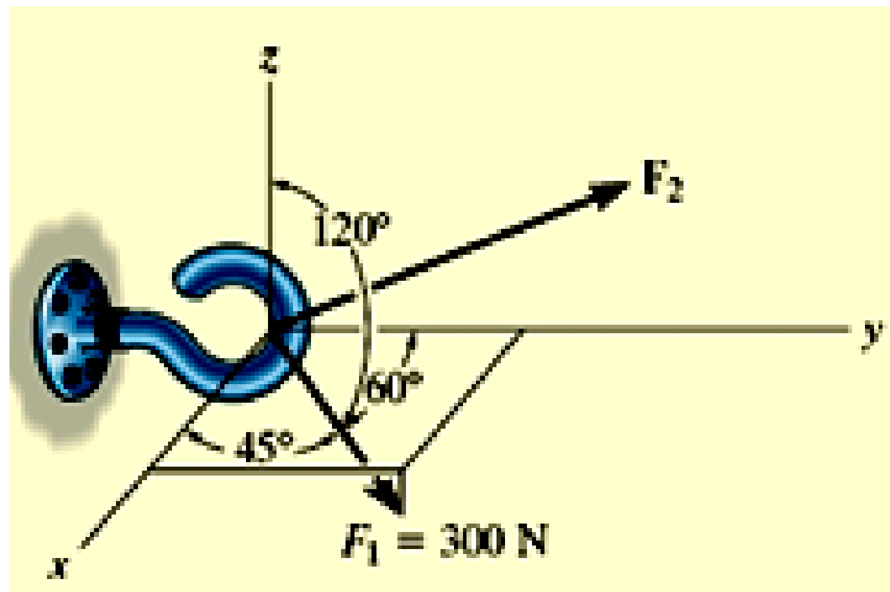
$$\beta = \cos^{-1}(-0.354) = 111^\circ$$

$$\gamma = \cos^{-1}(0.866) = 30^\circ$$

Example 2.11

Given :

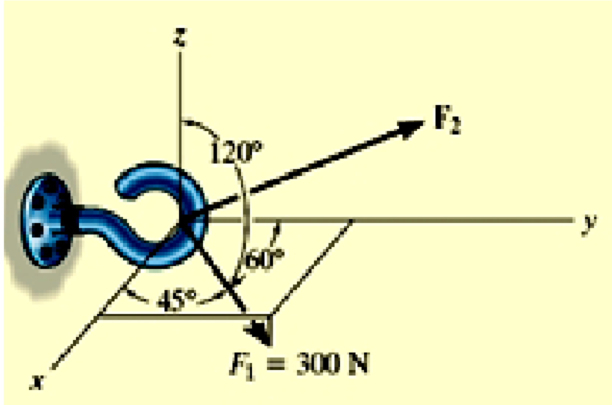
- Two forces act on the hook as shown.
- The resultant force $\mathbf{F}_R$ acts along the positive *y* axis & has a magnitude of 800N.



Find :

Deterrmine the magnitude and coordinate direction angles of $\mathbf{F}_2$.

Solution



- Express $\mathbf{F}_1$ as a Cartesian vector.

$$\mathbf{F}_1 = F_{1x}\mathbf{i} + F_{1y}\,\mathbf{j} + F_{1z}\,\mathbf{k}$$

$$= F_1 \cos\alpha_1\ \mathbf{i} + F_1 \cos\beta_1\,\mathbf{j} + F_1 \cos\gamma_1\,\mathbf{k}$$

$$= 300 \cos 45°\ \mathbf{i} + 300 \cos 60°\,\mathbf{j} + 300 \cos 120°\,\mathbf{k}$$

$$= \{212.1\mathbf{i} + 150\,\mathbf{j} - 150\,\mathbf{k}\}\ \text{N}$$

- Express $\mathbf{F}_2$ as a Cartesian vector**:**

$$\mathbf{F}_2 = F_{2x}\mathbf{i} + F_{2y}\,\mathbf{j} + F_{2z}\,\mathbf{k}$$

- Express the reusltant force $\mathbf{F}_R$ as a Cartesian vector**:**

$$\mathbf{F}_R = (800\text{N})\ (+\ \mathbf{j}) = \{800\mathbf{j}\}\ \text{N}$$

- We require

$$\mathbf{F}_R = \mathbf{F}_1 + \mathbf{F}_2$$

$$800\,\mathbf{j} = (212.1\mathbf{i} + 150\,\mathbf{j} - 150\,\mathbf{k}) + (F_{2x}\,\mathbf{i} + F_{2y}\,\mathbf{j} + F_{2z}\,\mathbf{k})$$

$$800\,\mathbf{j} = (212.1 + F_{2x})\,\mathbf{i} + (150 + F_{2y})\,\mathbf{j} + (-150 + F_{2z})\,\mathbf{k}$$

- Equating the **i**, **j**, **k** components, we have

i :

$$0 = 212.1 + F_{2x}$$

$$\Longrightarrow F_{2x} = -212.1 \text{ N}$$

j :

$$800 = 150 + F_{2y}$$

$$\Longrightarrow F_{2y} = 650 \text{ N}$$

k:

$$0 = -150 + F_{2z}$$

$$\Longrightarrow F_{2z} = 150 \text{ N}$$

- The magnitude of $\mathbf{F}_2$ is

$$F_2 = \sqrt{(-212.1\text{ N})^2 + (650\text{ N})^2 + (150\text{ N})^2} = 700\text{ N}$$

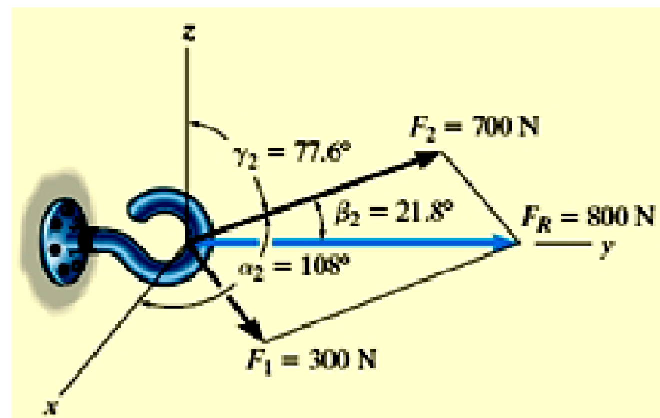
- Coordinate direction angles of $\mathbf{F}_2$.

$$F_{2x} = F_2 \cos\alpha \quad \Rightarrow \quad \cos\alpha = \frac{F_{2x}}{F_2} = \frac{-212.1}{700}$$

$$\therefore \quad \alpha = 108°$$

$$F_{2y} = F_2 \cos\beta \quad \Rightarrow \quad \cos\beta = \frac{F_{2y}}{F_2} = \frac{650}{700}$$

$$\therefore \quad \beta = 21.8°$$



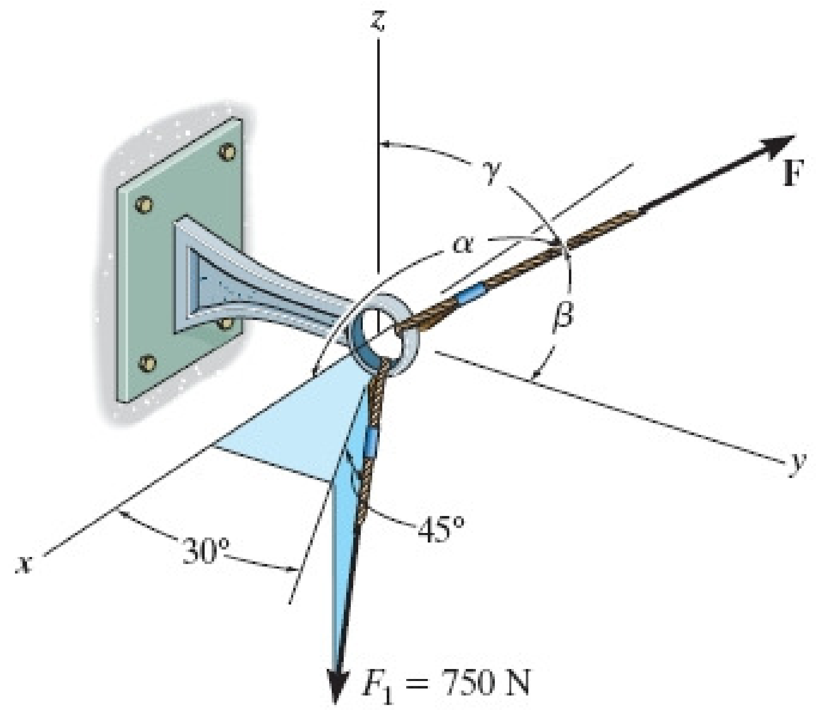
$$F_{2z} = F_2 \cos\gamma \quad \Rightarrow \quad \cos\gamma = \frac{F_{2z}}{F_2} = \frac{150}{700}$$

$$\therefore \quad \gamma = 77.6°$$

Problem 2-70

Given :

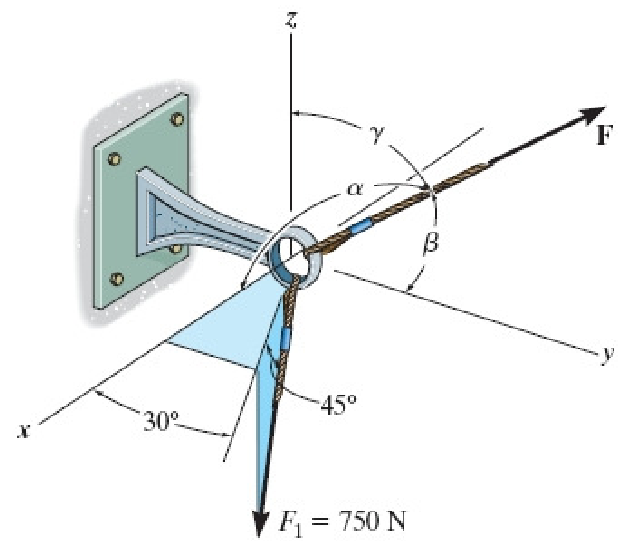
- Two forces act on the bracket as shown.
- The resultant force is
  $\mathbf{F}_R = \{800\,\mathbf{j}\}$N.



Find :

The magnitude & coordinate direction angles of **F**.

Solution



- Express $\mathbf{F}_1$ as a Cartesian vector.

$$F_{1x} = (750 \cos 45^\circ) \cos 30^\circ = 459.2798 \text{ N}$$

$$F_{1y} = (750 \cos 45^\circ) \sin 30^\circ = 265.1650 \text{ N}$$

$$F_{1z} = 750 \sin 45^\circ = 530.3301 \text{ N}$$

Thus,

$$\mathbf{F}_1 = \{459.2798\,\mathbf{i} + 265.1650\,\mathbf{j} - 530.3301\,\mathbf{k}\} \text{ N}$$

- Express $\mathbf{F}$ as a Cartesian vector**:**

$$\mathbf{F} = F_x\mathbf{i} + F_y\mathbf{j} + F_z\mathbf{k}$$

$$= F\cos\alpha\,\mathbf{i} + F\cos\beta\,\mathbf{j} + F\cos\gamma\,\mathbf{k}$$

- Resultant force

$$\mathbf{F}_R = \mathbf{F}_1 + \mathbf{F}$$

$$\begin{aligned} 800\,\mathbf{j} = & \;(\;\;459.2798 + F\cos\alpha)\,\mathbf{i} \\ & + (\;\;265.1650 + F\cos\beta)\,\mathbf{j} \\ & + (-530.3301 + F\cos\gamma)\,\mathbf{k} \end{aligned}$$

- Equating the **i**, **j**, **k** components, we have

i :

$$0 = 459.2798 + F\cos\alpha$$

$$\Longrightarrow \cos\alpha = -459.2798/F \qquad (1)$$

j :

$$800 = 265.1650 + F\cos\beta$$

$$\Longrightarrow \cos\beta = 534.8350/F \qquad (2)$$

k:

$$0 = -530.3301 + F\cos\gamma$$

$$\Longrightarrow \cos\gamma = 530.3301/F \qquad (3)$$

- Squaring Eqs.(1), (2) & (3) & then adding the results yields

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = \frac{(-459.2798)^2 + (534.8350)^2 + (530.3301)^2}{F^2}$$

$$1 = \frac{778236.4269}{F^2}$$

$$\Longrightarrow \quad F^2 = 778236.4269$$

$$\Longrightarrow \quad F = 882.1771 \text{ N} = 882.18 \text{ N} \qquad (4)$$

- From Eqs.(1) & (4),

$$\cos\alpha = \frac{-459.2798}{882.1771} \quad \Rightarrow \quad \alpha = 121.37^\circ$$

- From Eqs.(2) & (4),

$$\cos\beta = \frac{534.8350}{882.1771} \quad \Rightarrow \quad \beta = 52.68^\circ$$

- From Eqs.(3) & (4),

$$\cos\gamma = \frac{530.3301}{882.1771} \quad \Rightarrow \quad \gamma = 53.05^\circ$$