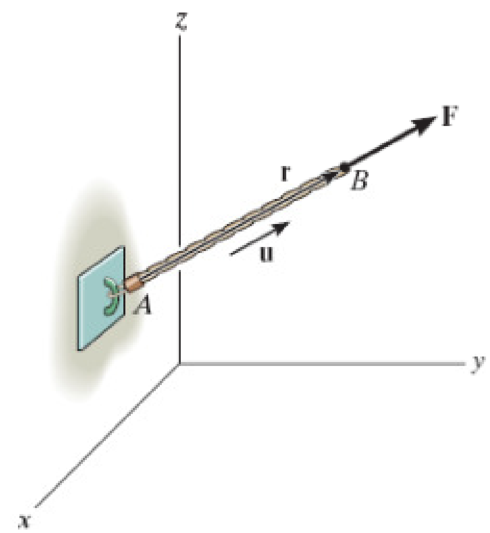
2.8 Force Vector Directed along a Line

- In 3D problems, direction of a force **F** is specified by 2 points, through which its line of action lies.

- **F** can be formulated as a Cartesian vector



$$\mathbf{F} = F\,\mathbf{u}$$

$$\mathbf{F} = F\left(\frac{\mathbf{r}}{r}\right)$$

$$\mathbf{F} = F\left(\frac{(x_B - x_A)\mathbf{i} + (y_B - y_A)\mathbf{j} + (z_B - z_A)\mathbf{k}}{\sqrt{(x_B - x_A)^2 + (y_B - y_A)^2 + (z_B - z_A)^2}}\right)$$

Application

Force F acting along the chain can be presented as a Cartesian vector as follows.

- Establish x, y, z axes
- Form a position vector **r** along length of chain
- Determine the unit vector **u= r**/r that defines the direction of both the chain and the force
- Finally, write

$$\mathbf{F} = F\,\mathbf{u}$$

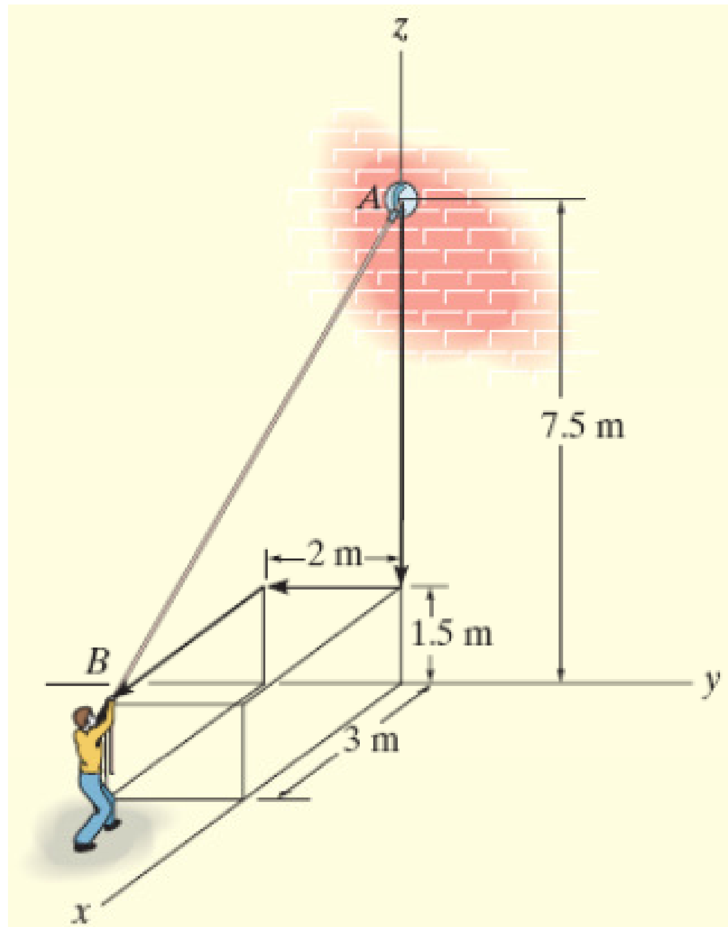
Example 2.13

Given :

The man pulls on the cord with a force of 350N.

Find :

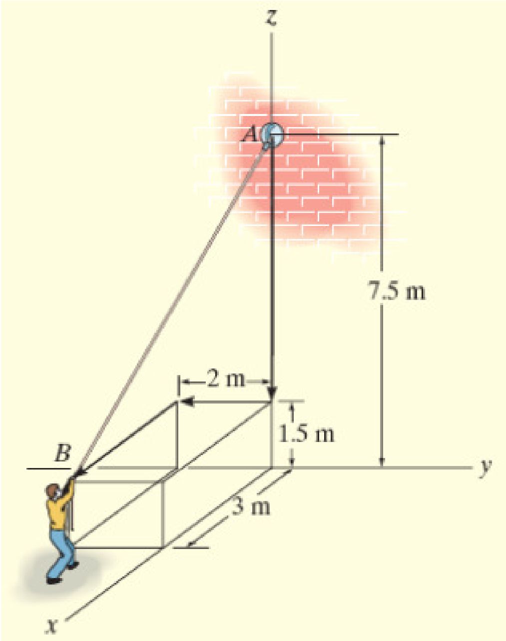
Represent this force acting on the support *A* as a Cartesian vector and determine its direction.



Solution

- Coordinates : A (0m, 0m, 7.5m)

  B (3 m, – 2 m, 1.5 m)



- Position vector from A to B:

$$\mathbf{r} = (x_B - x_A)\mathbf{i} + (y_B - y_A)\mathbf{j} + (z_B - z_A)\mathbf{k}$$

$$= (3\text{m} - 0\text{m})\mathbf{i} + (-2\text{m} - 0\text{m})\mathbf{j} + (1.5\text{m} - 7.5\text{m})\mathbf{k}$$

$$= \{3\mathbf{i} - 2\mathbf{j} - 6\mathbf{k}\}\text{ m}$$

- Length of cord AB:

$$r = \sqrt{(3)^2 + (-2)^2 + (-6)^2} = 7\text{ m}$$

- Unit vector in the direction of **r**.

$$\mathbf{u} = \frac{\mathbf{r}}{r} = \frac{3\mathbf{i} - 2\mathbf{j} - 6\mathbf{k}}{7} = \frac{3}{7}\mathbf{i} - \frac{2}{7}\mathbf{j} - \frac{6}{7}\mathbf{k}$$

- Force **F** has a magnitude of 350N & a direction specified by **u**.

$$\mathbf{F} = F\,\mathbf{u}$$

$$= (350\text{ N})\left(\frac{3}{7}\mathbf{i} - \frac{2}{7}\mathbf{j} - \frac{6}{7}\mathbf{k}\right)$$

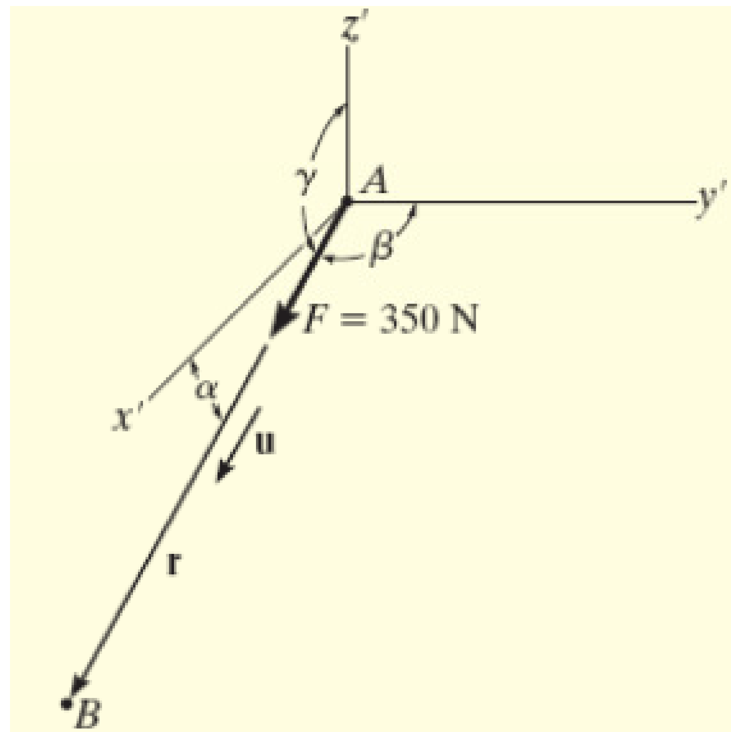
$$= \{150\mathbf{i} - 100\mathbf{j} - 300\mathbf{k}\}\ \text{N}$$

- The components of the unit vector **u** give the coordinate direction angles.

$$\alpha = \cos^{-1}\left(\frac{3}{7}\right) = 64.6^{\circ}$$

$$\beta = \cos^{-1}\left(\frac{-2}{7}\right) = 107^{\circ}$$

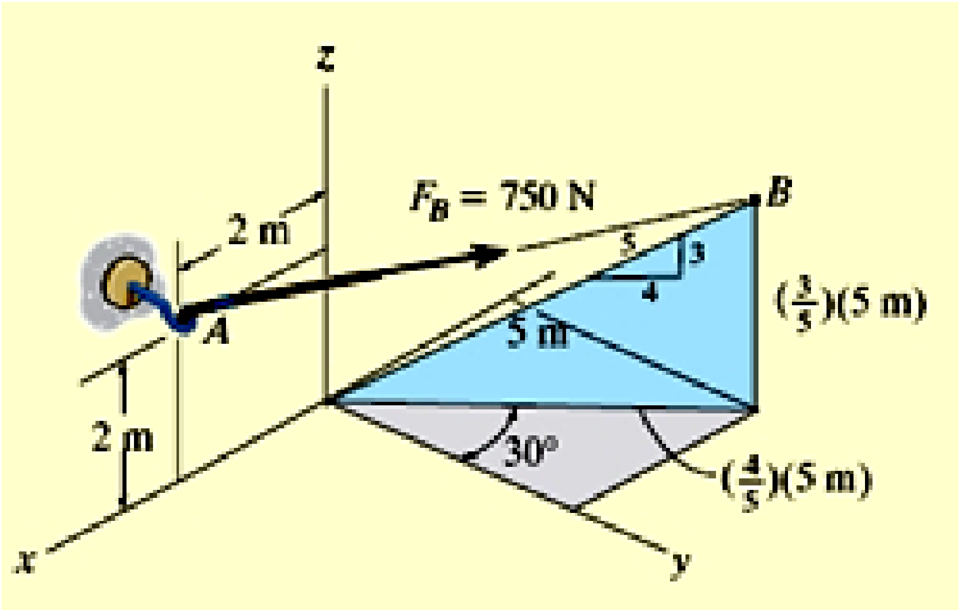
$$\gamma = \cos^{-1}\left(-\frac{6}{7}\right) = 149^{\circ}$$



Example 2.14

Given :

The force $F_B = 750$ N acts on the hook as shown.

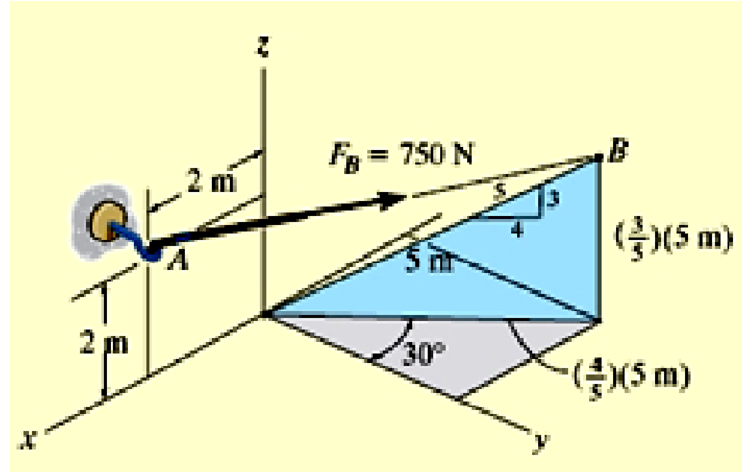


Find :

Express F_B as a Cartesian vector.

Solution

- Coordinates :



A (2 m, 0, 2 m)

$$B\left(-\left(\frac{4}{5}\right)(5)\sin 30^{\circ}\ \text{m}, \left(\frac{4}{5}\right)(5)\cos 30^{\circ}\ \text{m}, \left(\frac{3}{5}\right)(5)\ \text{m}\right)$$

$\Rightarrow$ B (– 2 m, 3.464 m, 3 m)

- Position vector:

$$\mathbf{r}_B = (x_B - x_A)\mathbf{i} + (y_B - y_A)\mathbf{j} + (z_B - z_A)\mathbf{k}$$

$$= (-2\ \text{m} - 2\ \text{m})\mathbf{i} + (3.464\ \text{m} - 0)\mathbf{j} + (3\ \text{m} - 2\ \text{m})\mathbf{k}$$

$$= \{-4\mathbf{i} + 3.464\mathbf{j} + 1\mathbf{k}\}\ \text{m}$$

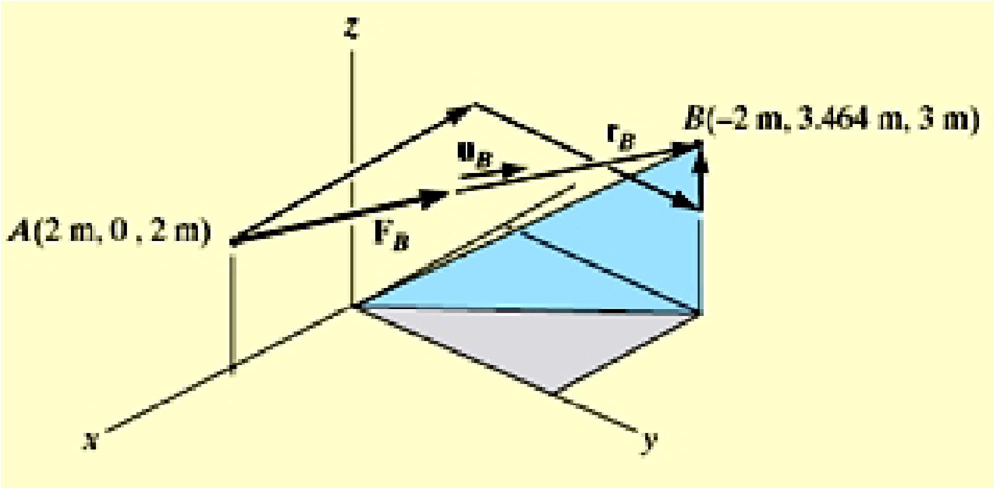
- Magnitude of $\mathbf{r}_B$

$$r_B = \sqrt{(-4)^2 + (3.464)^2 + (1)^2} \;=\; 5.385\ \text{m}$$

- Unit vector in the direction of $\mathbf{F}_B$

$$\mathbf{u}_B = \frac{\mathbf{r}_B}{r_B} = \frac{-4\mathbf{i} + 3.464\mathbf{j} + 1\mathbf{k}}{5.385}$$

$$= -0.7428\mathbf{i} \;+ 0.6433\mathbf{j} \;+ 0.1857\,\mathbf{k}$$



- Force $\mathbf{F}_B$ expressed as a Cartesian vector becomes

$$\mathbf{F}_B = F_B\,\mathbf{u}_B$$

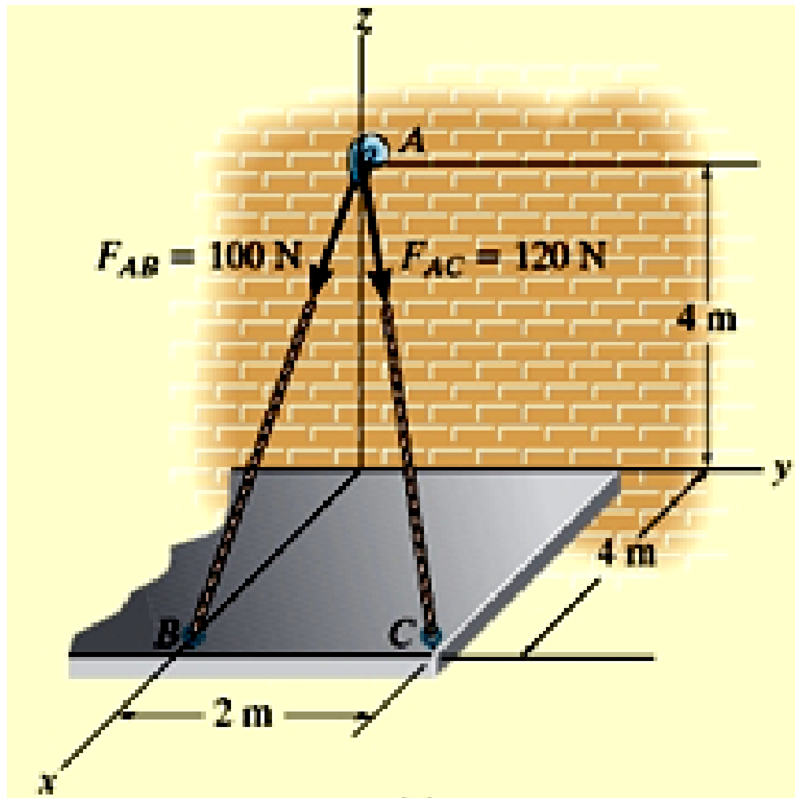
$$= (750\text{ N})\,(-0.7428\mathbf{i} + 0.6433\mathbf{j} + 0.1857\,\mathbf{k})$$

$$= \{-557\mathbf{i} + 482\mathbf{j} + 139\mathbf{k}\}\ \text{N}$$

Example 2.15

Given :

- The roof is supported by cables as shown.
- The cables exert forces $F_{AB} = 100\text{N}$ and $F_{AC} = 120\text{N}$ on the wall hook at *A*.



Find :

- Determine the resultant force acting at *A*.
- Express the resultant force as a Cartesian vector

Solution

- Coordinates : A (0, 0, 4 m)

  B (4 m, 0, 0)

  C (4 m, 2 m, 0)

- Express $\mathbf{F}_{AB}$ as a Cartesian vector

$$\mathbf{r}_{AB} = (x_B - x_A)\mathbf{i} + (y_B - y_A)\mathbf{j} + (z_B - z_A)\mathbf{k}$$

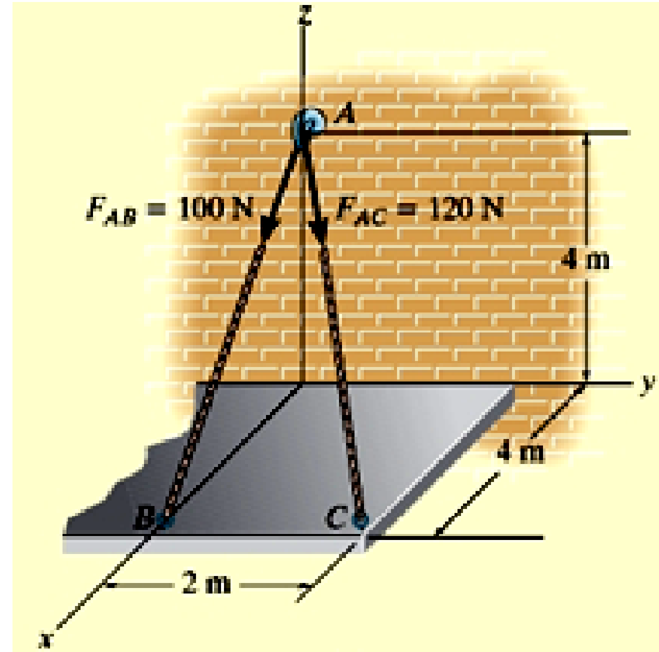
$$= (4\text{ m} - 0)\mathbf{i} + (0 - 0)\mathbf{j} + (0 - 4\text{ m})\mathbf{k}$$

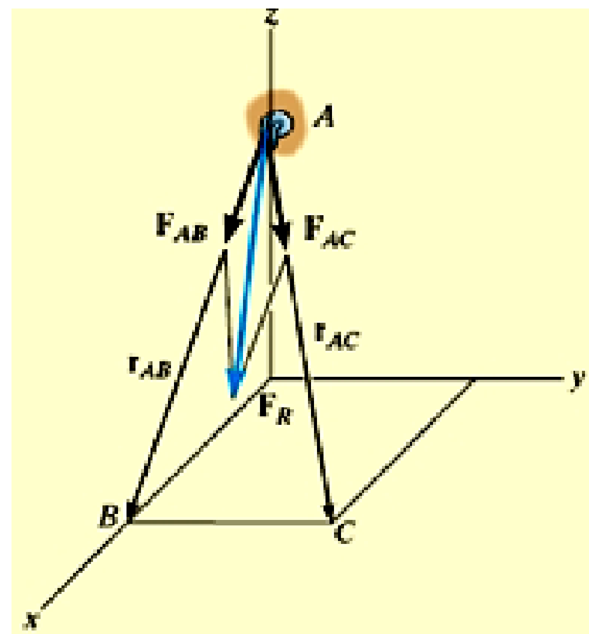
$$= \{4\mathbf{i} - 4\mathbf{k}\}\text{ m}$$

$$r_{AB} = \sqrt{(4)^2 + (-4)^2} = 5.66\text{ m}$$

$$\mathbf{F}_{AB} = F_{AB}\mathbf{u}_{AB} = F_{AB}\left(\frac{\mathbf{r}_{AB}}{r_{AB}}\right)$$

$$= (100\text{N})\left(\frac{4\mathbf{i} - 4\mathbf{k}}{5.66}\right) = \{70.7\mathbf{i} - 70.7\mathbf{k}\}\text{N}$$





- Express $\mathbf{F}_{AC}$ as a Cartesian vector

$$\mathbf{r}_{AC} = (x_C - x_A)\mathbf{i} + (y_C - y_A)\mathbf{j} + (z_C - z_A)\mathbf{k}$$

$$= (4\text{ m} - 0)\mathbf{i} + (2\text{ m} - 0)\mathbf{j} + (0 - 4\text{ m})\mathbf{k}$$

$$= \{4\mathbf{i} + 2\mathbf{j} - 4\mathbf{k}\}\text{ m}$$

$$r_{AC} = \sqrt{(4)^2 + (2)^2 + (-4)^2} = 6\text{ m}$$

$$\mathbf{F}_{AC} = F_{AC}\,\mathbf{u}_{AC} = F_{AC}\left(\frac{\mathbf{r}_{AC}}{r_{AC}}\right)$$

$$= (120\text{N})\left(\frac{4\mathbf{i} + 2\mathbf{j} - 4\mathbf{k}}{6}\right)$$

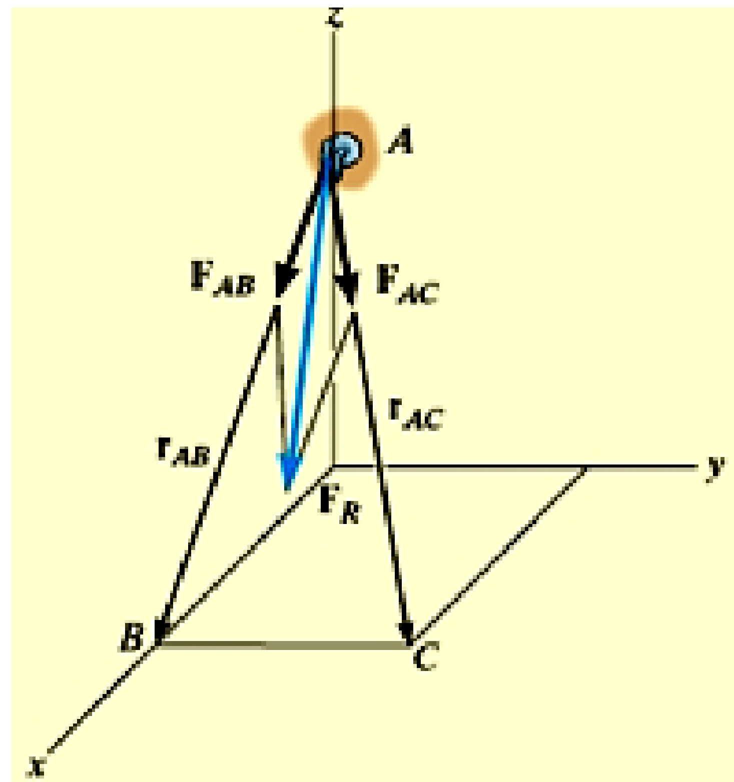
$$= \{80\mathbf{i} + 40\mathbf{j} - 80\mathbf{k}\}\text{ N}$$

- Resultant force

$$\mathbf{F}_R = \mathbf{F}_{AB} + \mathbf{F}_{AC}$$

$$= \{70.7\mathbf{i} - 70.7\mathbf{k}\}\ \text{N} + \{80\mathbf{i} + 40\mathbf{j} - 80\mathbf{k}\}\ \text{N}$$

$$= \{1501\mathbf{i} + 40\mathbf{j} - 151\mathbf{k}\}\ \text{N}$$



- The magnitude of $\mathbf{F}_R$

$$F_R = \sqrt{(151)^2 + (40)^2 + (-151)^2}$$

$$= 217\ \text{N}$$

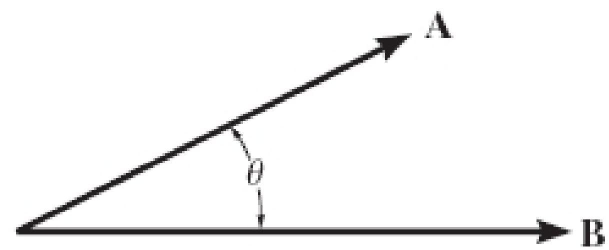
2.9 Dot Product

- The ***dot product*** of vectors **A** and **B** is written as **A·B** (read **A** dot **B**).

- It is defined as the products of the magnitudes of **A** and **B** and the angle between their tails.

$$\mathbf{A}\cdot\mathbf{B} = AB\cos\theta$$

where $0^\circ \le \theta \le 180^\circ$.



- It is often referred to as the ***scalar product*** of vectors as the result is a scalar.

❑ Laws of Operation

1. Commutative law

$$\mathbf{A} \cdot \mathbf{B} = \mathbf{B} \cdot \mathbf{A}$$

2. Multiplication by a scalar

$$a\,(\mathbf{A} \cdot \mathbf{B}) = (a\mathbf{A}) \cdot \mathbf{B} = \mathbf{A} \cdot (a\mathbf{B})$$

3. Distribution law

$$\mathbf{A} \cdot (\mathbf{B} + \mathbf{D}) = (\mathbf{A} \cdot \mathbf{B}) + (\mathbf{A} \cdot \mathbf{D})$$

Cartesian Vector Formulation

- Dot product of Cartesian unit vectors

$$\mathbf{i} \cdot \mathbf{i} = (1)\,(1)\ \cos 0^\circ = 1$$

$$\mathbf{i} \cdot \mathbf{j} = (1)\,(1)\ \cos 90^\circ = 0$$

$$\mathbf{i} \cdot \mathbf{k} = (1)\,(1) \cos 90^\circ = 0$$

- In summary,

$$\mathbf{i} \cdot \mathbf{i} = 1 \qquad \mathbf{j} \cdot \mathbf{j} = 1 \qquad \mathbf{k} \cdot \mathbf{k} = 1$$

$$\mathbf{i} \cdot \mathbf{j} = 0 \qquad \mathbf{i} \cdot \mathbf{k} = 0 \qquad \mathbf{j} \cdot \mathbf{k} = 0$$

- If $\mathbf{A} = A_x\mathbf{i} + A_y\mathbf{j} + A_z\mathbf{k}$ and $\mathbf{B} = B_x\mathbf{i} + B_y\mathbf{j} + B_z\mathbf{k}$, then

$$\mathbf{A}\cdot\mathbf{B} = (A_x\mathbf{i} + A_y\mathbf{j} + A_z\mathbf{k})\cdot(B_x\mathbf{i} + B_y\mathbf{j} + B_z\mathbf{k})$$

$$= A_x B_x(\mathbf{i}\cdot\mathbf{i}) + A_x B_y(\mathbf{i}\cdot\mathbf{j}) + A_x B_z(\mathbf{i}\cdot\mathbf{k})$$

$$+ A_y B_x(\mathbf{j}\cdot\mathbf{i}) + A_y B_y(\mathbf{j}\cdot\mathbf{j}) + A_y B_z(\mathbf{j}\cdot\mathbf{k})$$

$$+ A_z B_x(\mathbf{k}\cdot\mathbf{i}) + A_z B_y(\mathbf{k}\cdot\mathbf{j}) + A_z B_z(\mathbf{k}\cdot\mathbf{k})$$

- Thus, to determine the dot product of two Cartesian vectors, *multiply their corresponding x, y, z components and sum these products algebraically.*

$$\boxed{\mathbf{A}\cdot\mathbf{B} = A_x B_x + A_y B_y + A_z B_z}$$

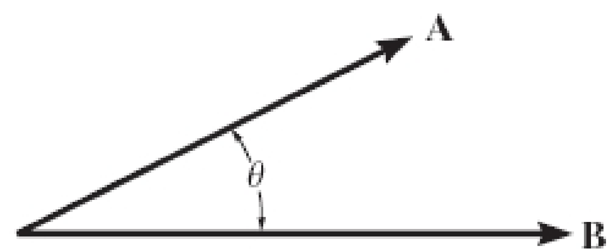
Note: The result will be either a positive or negative **scalar**.

❑ Applications

(1) To determine the angle formed between two vectors or intersecting lines

- The angle θ between the tails of vectors **A** and **B** is given by

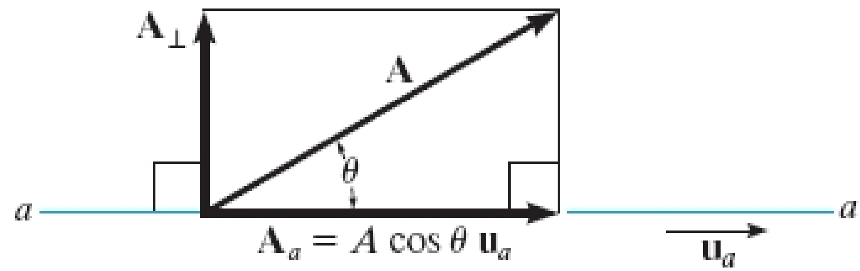
$$\theta = \cos^{-1}\left(\frac{\mathbf{A}\cdot\mathbf{B}}{AB}\right) \qquad 0^\circ \le \theta \le 180^\circ$$



- If $\mathbf{A}\cdot\mathbf{B} = 0$, $\cos^{-1}(0) = 90^\circ$.

$\Longrightarrow$ **A** is perpendicular to **B**.

(2) To determine the components of a vector parallel and perpendicular to a line.



(a) Components parallel to a line

- The component of **A** parallel to or collinear with the line *aa*

$$A_a = A \cos \theta$$

or

$$A_a = \mathbf{A} \cdot \mathbf{u}_a$$

- In vector form

$$\mathbf{A}_a = A_a \, \mathbf{u}_a$$

- A_a is also known as the ***scalar projection*** of **A** onto the line *aa*.

(b) Components perpendicular to a line

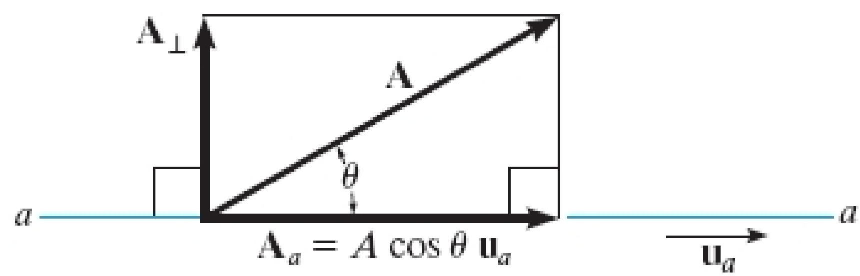
- The component of **A** that is perpendicular to line *aa* is given by

$$\mathbf{A}_{\perp} = \mathbf{A} - \mathbf{A}_a$$

- The magnitude of $\mathbf{A}_{\perp}$ can be determined from one of the following two ways.

(i) $$A_{\perp} = \sqrt{A^2 - A_a^2}$$

(ii) $$A_{\perp} = A \sin\theta$$



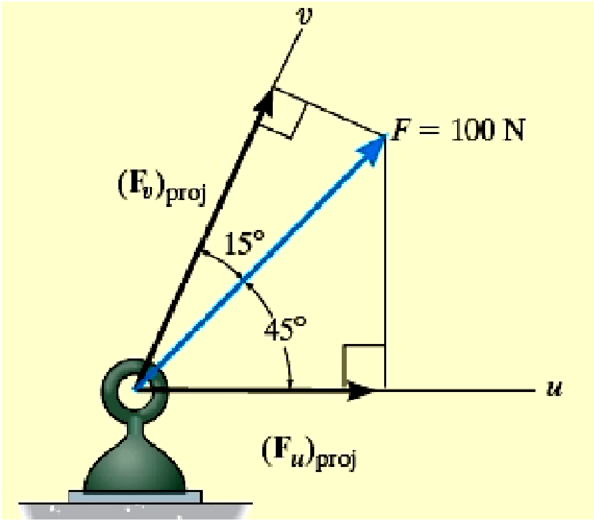
where

$$\theta = \cos^{-1}\left(\frac{\mathbf{A}\cdot\mathbf{u}_A}{A}\right)$$

Example 2.16

Given :

The force F = 100 N acts on a ring as shown.



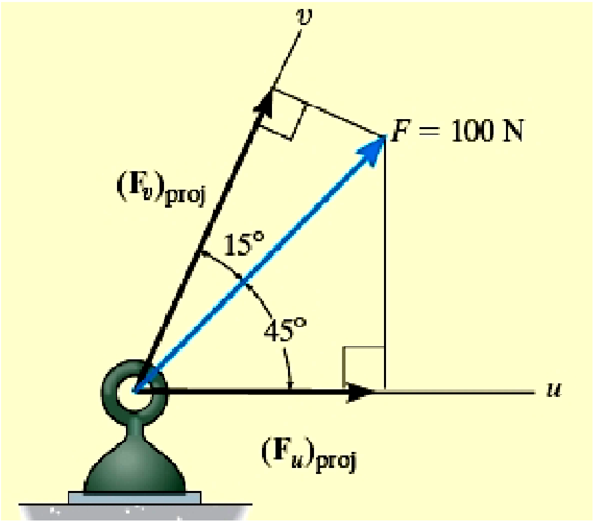
Find :

Determine the magnitudes of the projection of the force **F** onto the u and v axes.

Solution

- Projection of **F** onto the *u* axis

$$(F_u)_{\text{proj}} = F \cos 45^\circ$$
$$= (100 \text{ N}) \cos 45^\circ$$
$$= 70.7 \text{ N}$$



- Projection of **F** onto the *u* axis

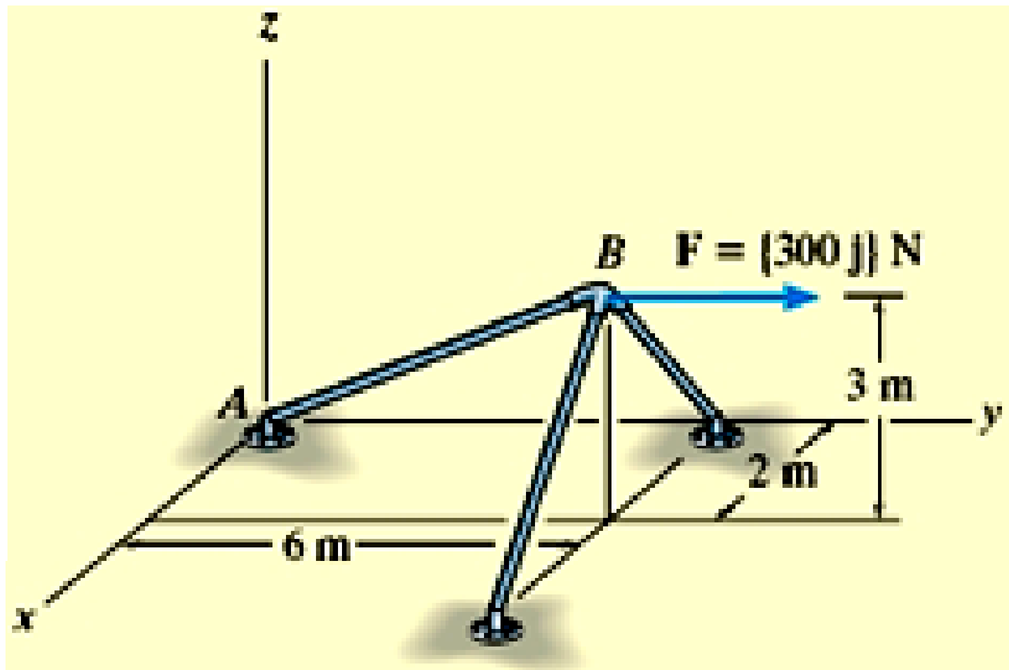
$$(F_v)_{\text{proj}} = F \cos 15^\circ$$
$$= (100 \text{ N}) \cos 15^\circ$$
$$= 96.6 \text{ N}$$

Note: These projections are not equal to the magnitudes of the components of force **F** along the *u* & *v* axes found from the *parallelogram law*. They will only be equal if the *u* & *v* axes are *perpendicular* to one another.

Example 2.17

Given :

The frame is subjected to a horizontal force **F** = {300j} N.



Find :

Determine the magnitude of the components of **F** parallel and perpendicular to member *AB*.

Solution

- Coordinates : B (2 m, 6 m, 3 m)

- Unit vector in the direction of AB.

$$\mathbf{u}_B = \frac{\mathbf{r}_B}{r_B} = \frac{2\mathbf{i} + 6\mathbf{j} + 3\mathbf{k}}{\sqrt{(2)^2 + (6)^2 + (3)^2}}$$

$$= 0.286\mathbf{i} + 0.857\mathbf{j} + 0.429\,\mathbf{k}$$

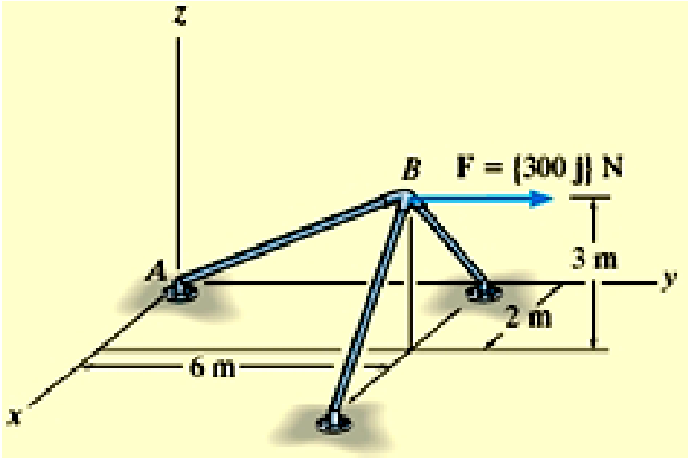
- Component of $\mathbf{F}$ parallel to AB

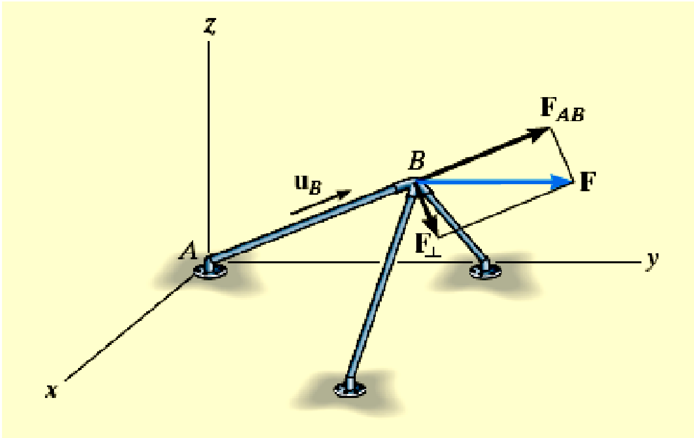
$$F_{AB} = \mathbf{F} \cdot \mathbf{u}_B$$

$$= (300\mathbf{j}) \cdot (0.286\mathbf{i} + 0.857\mathbf{j} + 0.429\,\mathbf{k})$$

$$= (0)(0.286) + (300)(0.857) + (0)(0.429)$$

$$= 257.1 \text{ N}$$



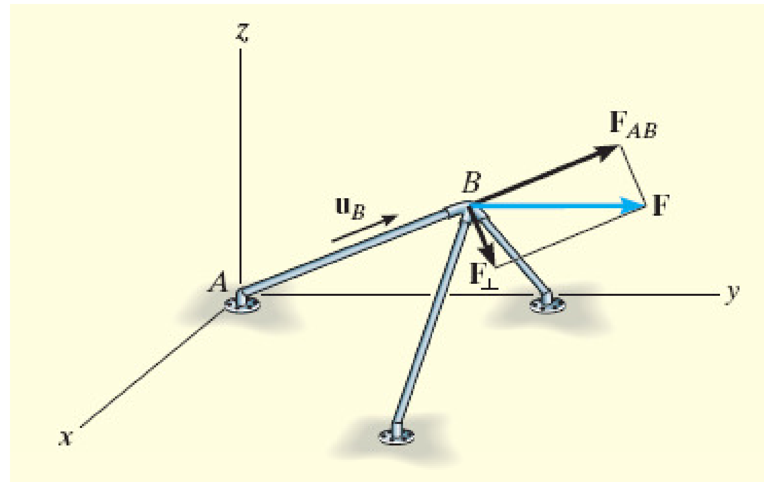


- Component of **F** perpendicular to *AB*

$$F_{\perp} = \sqrt{F^2 - F_{AB}^2}$$

$$= \sqrt{(300\ \text{N})^2 - (257.1\,\text{N})^2}$$

$$= 155\ \text{N}$$



Note : A second approach to find the component of F perpendicular to AB

- Express $\mathbf{F}_{AB}$ as a Cartesian vector

$$\mathbf{F}_{AB} = F_{AB}\,\mathbf{u}_B$$
$$= (257.1 \text{ N})\,(0.286\mathbf{i} + 0.857\mathbf{j} + 0.429\,\mathbf{k})$$
$$= \{73.5\mathbf{i} + 220\mathbf{j} + 110\,\mathbf{k}\} \text{ N}$$

- Component of **F** perpendicular to *AB*

$$\mathbf{F}_{\perp} = \mathbf{F} - \mathbf{F}_{AB}$$
$$= 300\,\mathbf{j} - (73.5\mathbf{i} + 220\mathbf{j} + 110\,\mathbf{k}) \text{ N}$$
$$= \{-73.5\mathbf{i} + 80\mathbf{j} - 110\,\mathbf{k}\} \text{ N}$$

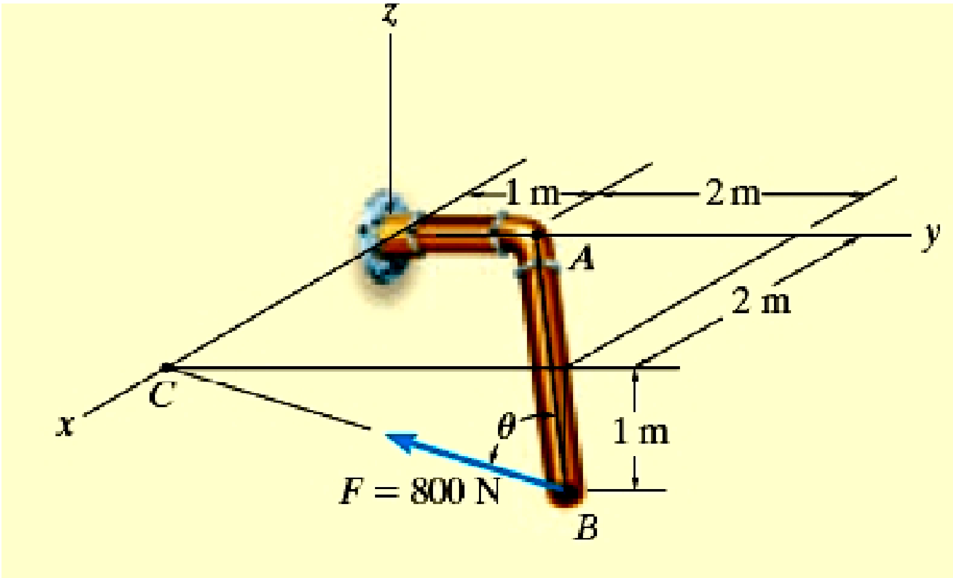
- The magnitude of $\mathbf{F}_{\perp}$

$$F_{\perp} = \sqrt{(-73.5)^2 + (80)^2 + (-110)^2} = 155 \text{ N}$$

Example 2.18

Given :

The pipe is subjected to the force of $F = 800$ N.



(a)

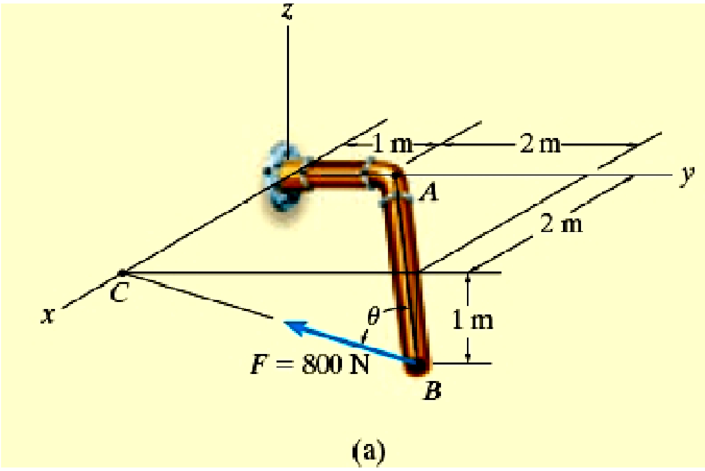
Find :

- Determine the angle θ between **F** and the pipe segment *BA*.
- The projection of **F** along the pipe segment *BA*.

Solution

- Find θ

- Coordinates : $A\ (0, 1\text{m}, 0)$
  $B\ (2\text{m}, 3\text{m}, -1\text{m})$
  $C\ (2\text{m}, 0, 0)$



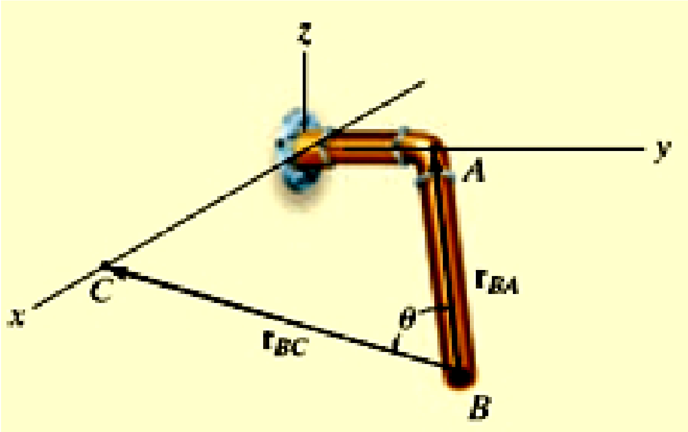
- Position vector of A relative to B

$$\mathbf{r}_{BA} = (x_A - x_B)\mathbf{i} + (y_A - y_B)\mathbf{j} + (z_A - z_B)\mathbf{k}$$

$$= (0 - 2)\mathbf{i} + (1 - 3)\mathbf{j} + (0 - [-1])\mathbf{k}$$

$$= \{-2\mathbf{i} - 2\mathbf{j} + 1\mathbf{k}\}\ \text{m}$$

$$r_{BA} = \sqrt{(-2)^2 + (-2)^2 + (1)^2} = 3\ \text{m}$$



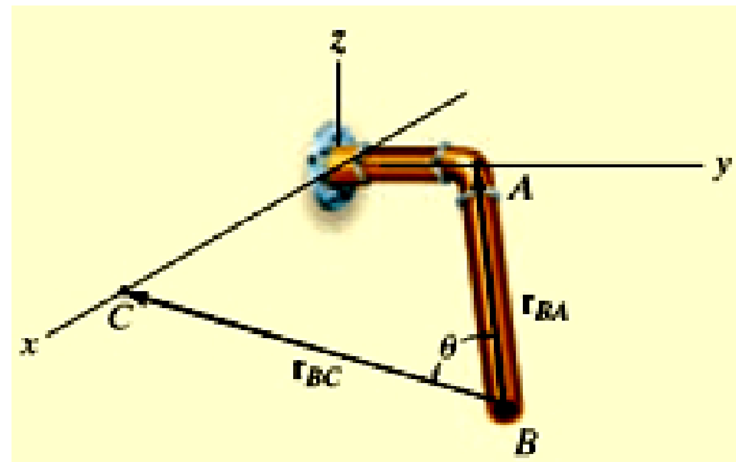
- Position vector of *C* relative to *B*

$$\mathbf{r}_{BC} = (x_C - x_B)\mathbf{i} + (y_C - y_B)\mathbf{j} + (z_C - z_B)\mathbf{k}$$

$$= (2 - 2)\mathbf{i} + (0 - 3)\mathbf{j} + (0 - [-1])\mathbf{k}$$

$$= \{-3\mathbf{j} + 1\mathbf{k}\}\ \text{m}$$

$$r_{BA} = \sqrt{(-3)^2 + (1)^2} = \sqrt{10}\ \text{m}$$



- Angle between **F** (or *BC*) and *BA*

$$\cos\theta = \frac{\mathbf{r}_{BA} \cdot \mathbf{r}_{BC}}{r_{BA}\ r_{BC}} = \frac{(-2)(0) + (-2)(-3) + (1)(1)}{(3)\sqrt{10}} = 0.7379$$

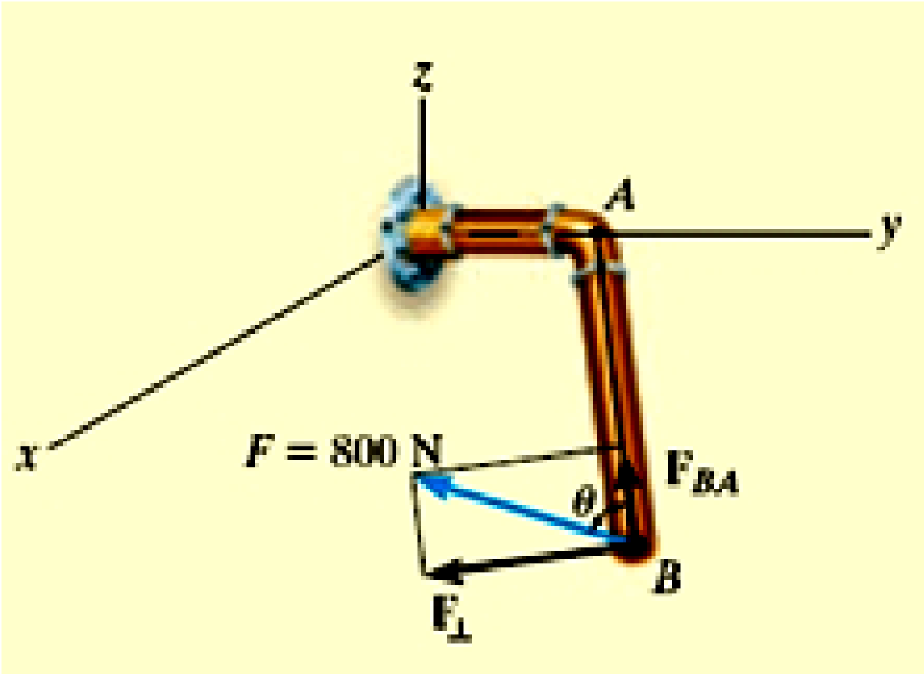
$$\therefore \quad \theta = 42.5^\circ$$

- Find the projection of **F** along *BA*

$$F_{BA} = F \cos \theta$$

$$= (800 \text{ N}) \cos 42.5^\circ$$

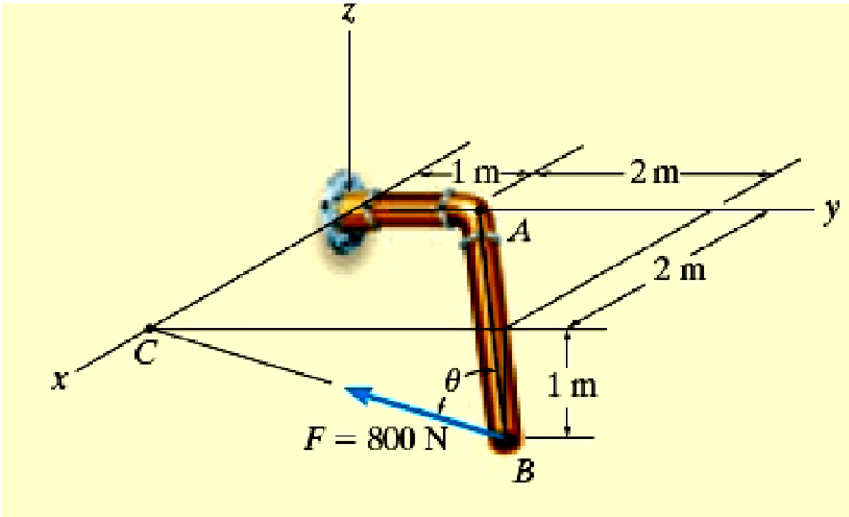
$$= 590 \text{ N}$$



Note : A second approach to find the projection of **F** *along BA*

- Express force **F** as a Cartesian vector

$$\mathbf{u}_{BC} = \frac{\mathbf{r}_{BC}}{r_{BC}} = \frac{-3\mathbf{j} + 1\mathbf{k}}{\sqrt{10}}$$



(a)

$$\mathbf{F} = F\,\mathbf{u}_{BC}$$

$$= (800\text{ N})\left(\frac{-3\mathbf{j} + 1\mathbf{k}}{\sqrt{10}}\right)$$

$$= (-758.9\mathbf{j} + 253.0\,\mathbf{k})\ \text{N}$$

- Unit vector along *BA*.

$$\mathbf{u}_{BA} = \frac{\mathbf{r}_{BA}}{r_{BA}} = \frac{-2\mathbf{i} - 2\mathbf{j} + 1\mathbf{k}}{3} = -\frac{2}{3}\mathbf{i} - \frac{2}{3}\mathbf{j} + \frac{1}{3}\mathbf{k}$$

- The projection of $\mathbf{F}$ along BA is given by

$$F_{BA} = \mathbf{F} \cdot \mathbf{u}_{BA}$$

$$= (-758.9\mathbf{j} + 253.0\,\mathbf{k}) \cdot \left(-\frac{2}{3}\mathbf{i} - \frac{2}{3}\mathbf{j} + \frac{1}{3}\mathbf{k}\right)$$

$$= (0)\left(-\frac{2}{3}\right) + (-758.9)\left(-\frac{2}{3}\right) + (253.0)\left(\frac{1}{3}\right)$$

$$= 590 \text{ N}$$