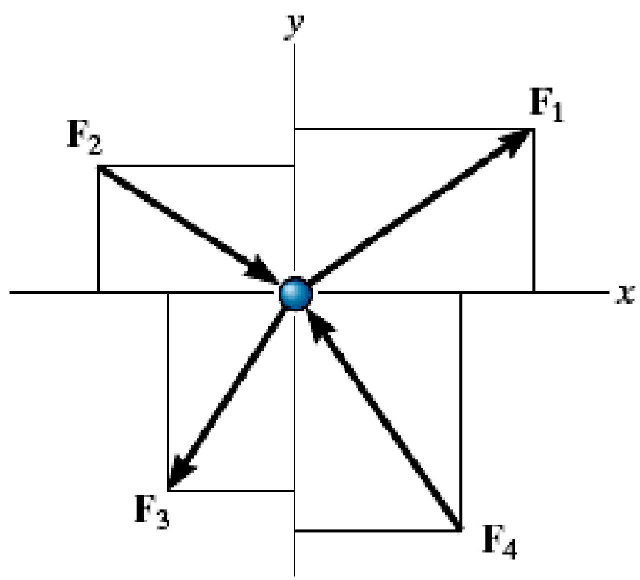
3.3 Coplanar Systems

- Consider a particle which is subjected to a system of coplanar forces in the *x*-*y* plane as shown in the figure.



- For equilibrium,

$$\sum \mathbf{F} = 0$$

- Resolve each force into its **i** and **j** components, we have

$$\sum F_x \mathbf{i} + \sum F_y \mathbf{j} = 0$$

- Hence,

$$\sum F_x = 0$$
$$\sum F_y = 0$$

❑ Procedure for Analysis

1. Free-Body Diagram

- Establish the *x*, *y* axes
- Label all the unknown and known forces

  - The sense of the unknown force can be assumed

2. Equations of Equilibrium

- Apply the equations of equilibrium

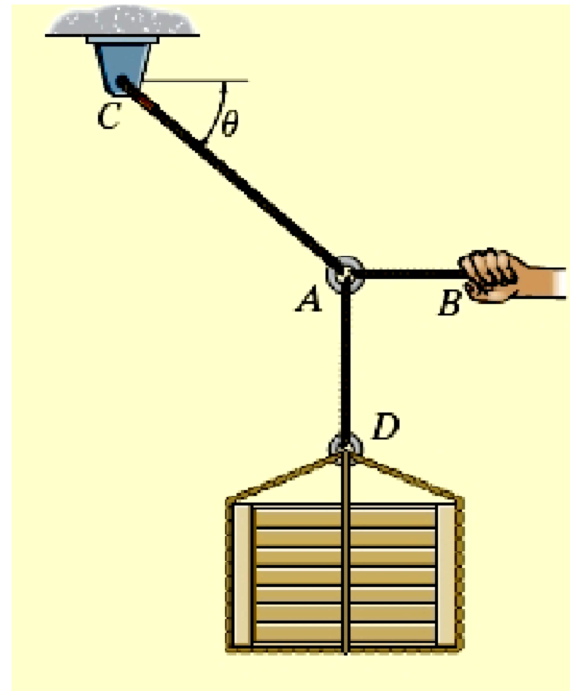
$$\sum F_x = 0, \qquad \sum F_y = 0$$

- If more than 2 unknowns exist and the problem involves a spring, apply $F = ks$ to relate the spring force to the deformation of the spring.

Example 3.3

Given :

- The 200-kg crate is suspended using the ropes *AB* & *AC*.
- Each ropes can withstand a maximum force of 10 kN before it breaks.
- *AB* always remains horizontal.



Find :

Determine the smallest angle θ to which the crate can be suspended before one of the ropes breaks.

Solution

FBD of ring *A*

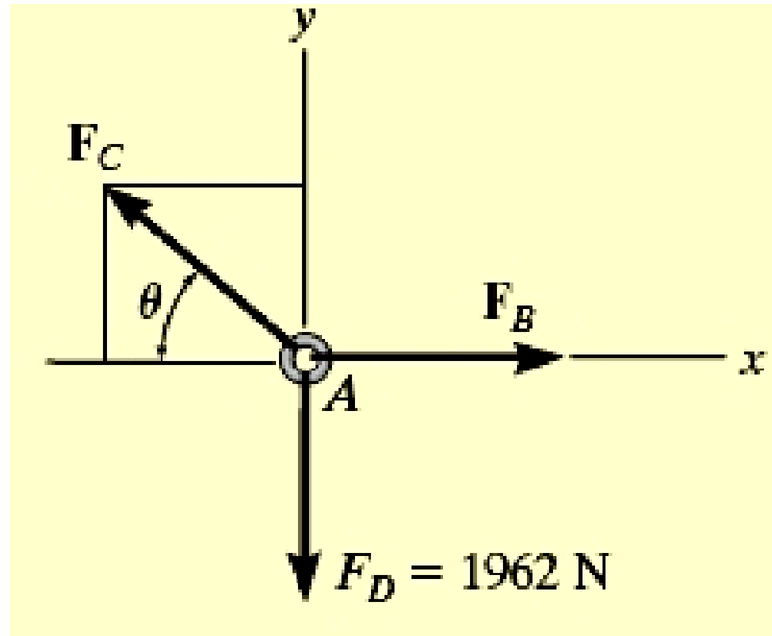
Three forces acting at *A*:

(1) Force by cable *AC*, $\mathbf{F}_C$

(2) Force by cable *AB*, $\mathbf{F}_B$

(3) Weight of the crate, $\mathbf{F}_D$

$F_D = (200)(9.81) = 1962$ N.



Equation of Equilibrium

$$\overset{+}{\rightarrow} \sum F_x = 0: \qquad -F_C \cos\theta + F_B = 0 \qquad (1)$$

$$+\uparrow \sum F_y = 0: \qquad F_C \sin\theta - 1962\text{ N} = 0 \qquad (2)$$

From Eq. (1),

$$F_C = \frac{F_B}{\cos\theta} \qquad (3)$$

- Since $\cos\theta \le 1$, Eq. (3) implies that

$$F_C > F_B$$

$\Rightarrow F_C$ will reach the maximum force of 10 kN before F_B.

- Substituting F_C = 10 kN into Eq. (2), we obtain

$$(10 \times 10^3\ \text{N})\ \sin\theta - 1962\ \text{N} = 0$$

$$\theta = \sin^{-1}(0.1962)$$

$$\theta = 11.31^\circ$$

- **Check:**

When F_C reaches maximum value, the force in rope *AB* is

$$F_B = F_C \cos\theta$$

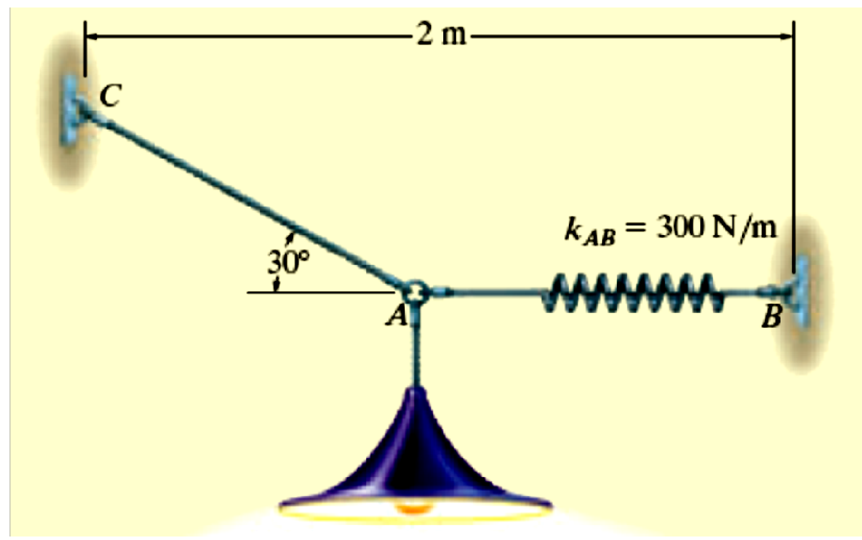
$$= (10\ \text{kN})\ \cos(11.31)\ \text{N} = 9.81\ \text{kN}$$

Example 3.4

Given :

- The 8-kg lamp is suspended as shown.

- The undeformed length of the spring *AB* is l'_{AB} = 0.4m, and the spring has a stiffness of k_{AB} = 300N/m



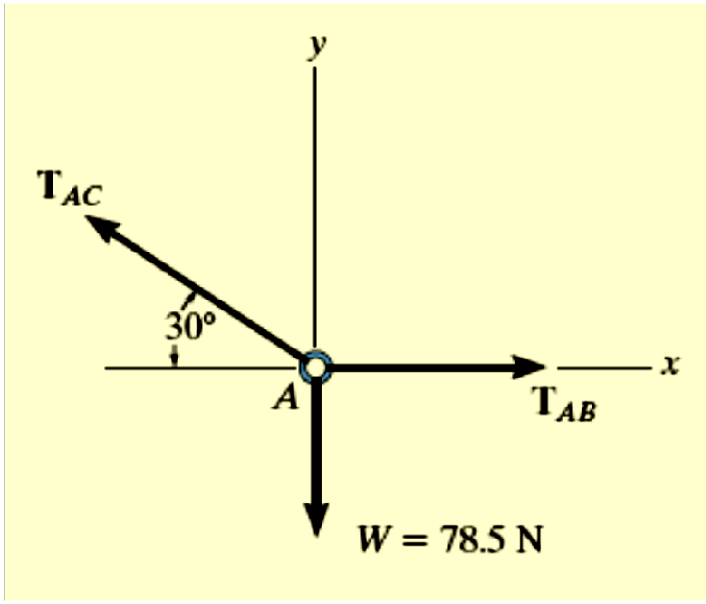
Find :

Determine the required length of the cord *AC* so that the 8kg lamp can be suspended in the position as shown.

Solution

FBD at Point *A*

Three forces acting at *A*:
(1) Force by cable *AC*
(2) Force in spring *AB*
(3) Weight of the lamp,
W=(8)(9.81) =78.5N.



Equation of Equilibrium

$$\overset{+}{\rightarrow} \sum F_x = 0: \qquad T_{AB} - T_{AC}\cos 30^\circ = 0$$

$$+\uparrow \sum F_y = 0: \qquad T_{AC}\sin 30^\circ - 78.5\text{N} = 0$$

Solving the above 2 equations yields

$$T_{AB} = 135.9\text{ N}$$

$$T_{AC} = 157.0\text{ N}$$

Spring

- The stretch (change in length) of spring *AB* is given by

$$T_{AB} = k_{AB}\, s_{AB}$$

$$135.9 \text{ N} = (300 \text{ N/m})\ s_{AB}$$

$$s_{AB} = 0.453 \text{ m}$$

- Therefore, the stretched length (i.e., final length) of spring *AB* is

$$l_{AB} = l'_{AB} + s_{AB}$$

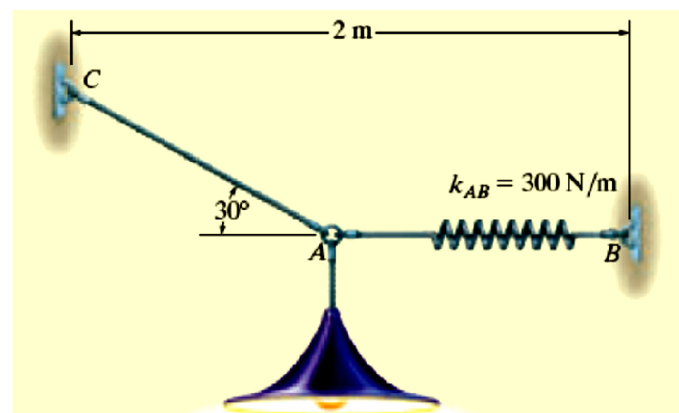
$$l_{AB} = 0.4 \text{ m} + 0.453 \text{ m} = 0.853 \text{ m}$$

Cord length

$$l_{AC} \cos 30^\circ + l_{AB} = 2 \text{ m}$$

$$l_{AC} \cos 30^\circ + 0.853 \text{ m} = 2 \text{ m}$$

$$l_{AC} = 1.32 \text{ m}$$



3.4 Three-Dimensional Force Systems

- For particle equilibrium,

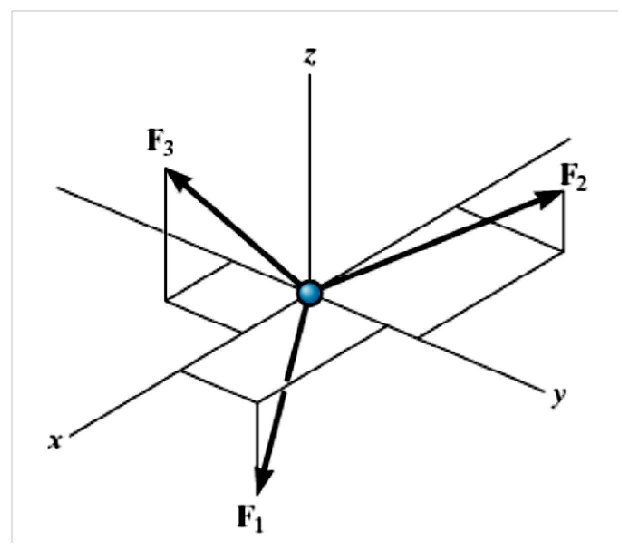
$$\sum \mathbf{F} = 0$$

- In a three-dimensional force system, the forces can be resolved into their respective **i**, **j**, **k** components, so that

$$\sum F_x \mathbf{i} + \sum F_y \mathbf{j} + \sum F_z \mathbf{k} = 0$$

- Hence,

$$\sum F_x = 0$$
$$\sum F_y = 0$$
$$\sum F_z = 0$$



❑ Procedure for Analysis

1. Free-Body Diagram

- Establish the *x, y, z* axes
- Label all the unknown and known forces
  - The sense of the unknown force can be assumed

2. Equations of Equilibrium

- Apply the equations of equilibrium

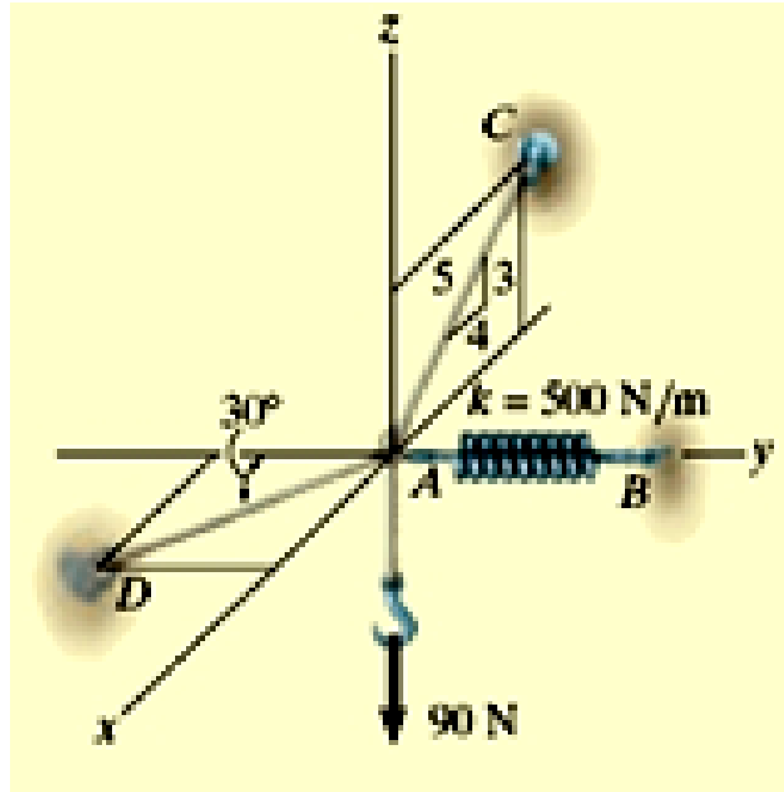
$$\sum F_x = 0, \qquad \sum F_y = 0, \qquad \sum F_z = 0$$

- For complicated geometry,
  (i) express each force as a Cartesian vector
  (ii) substitute force vectors into $\sum \mathbf{F} = 0$
  (iii) set the **i**, **j**, **k** components equal to zero.

Example 3.5

Given :

- A 90-N load is supported by two cables and a spring having a stiffness $k = 500$ N/m as shown.
- Cable *AD* lies in the *x*-*y* plane and cable *AC* lies in the *x*-*z* plane



Find :

Determine the force developed in the cables and the stretch of the spring for equilibrium.

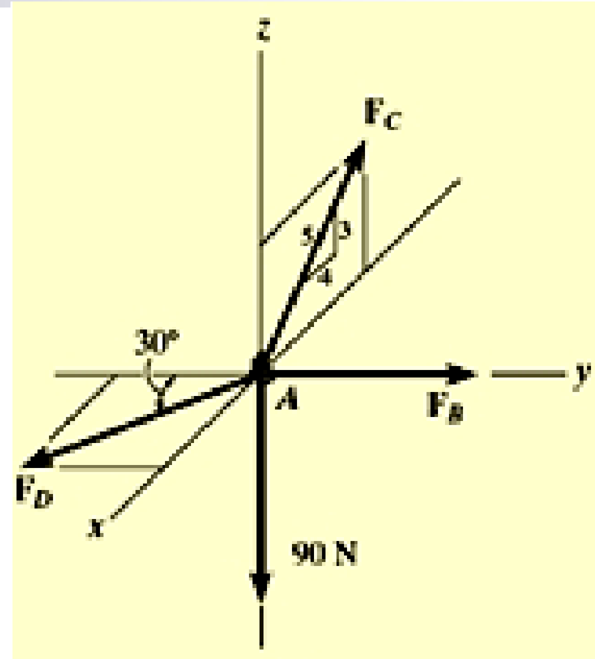
Solution

FBD at Point *A*

Four forces acting at *A*:

(1) 3 unknown forces: $\mathbf{F}_B$, $\mathbf{F}_C$, $\mathbf{F}_D$.

(2) 1 known force: 90-N load



Equation of Equilibrium

$$\sum F_x = 0: \qquad F_D \sin 30^\circ - F_C \left(\frac{4}{5}\right) = 0 \qquad (1)$$

$$\sum F_y = 0: \qquad -F_D \cos 30^\circ + F_B = 0 \qquad (2)$$

$$\sum F_z = 0: \qquad F_C \left(\frac{3}{5}\right) - 90\,\text{N} = 0 \qquad (3)$$

$$\Rightarrow \quad F_C = 150\ \text{N}$$

- Substituting F_C into Eq. (1), yields

$$F_D = 240 \text{ N}$$

- Substituting F_D into Eq. (2), yields

$$F_B = 207.8 \text{ N}$$

Spring

- The stretch of the spring is given by

$$F_B = k\, s_{AB}$$

$$207.8 \text{ N} = (500 \text{ N/m})\; s_{AB}$$

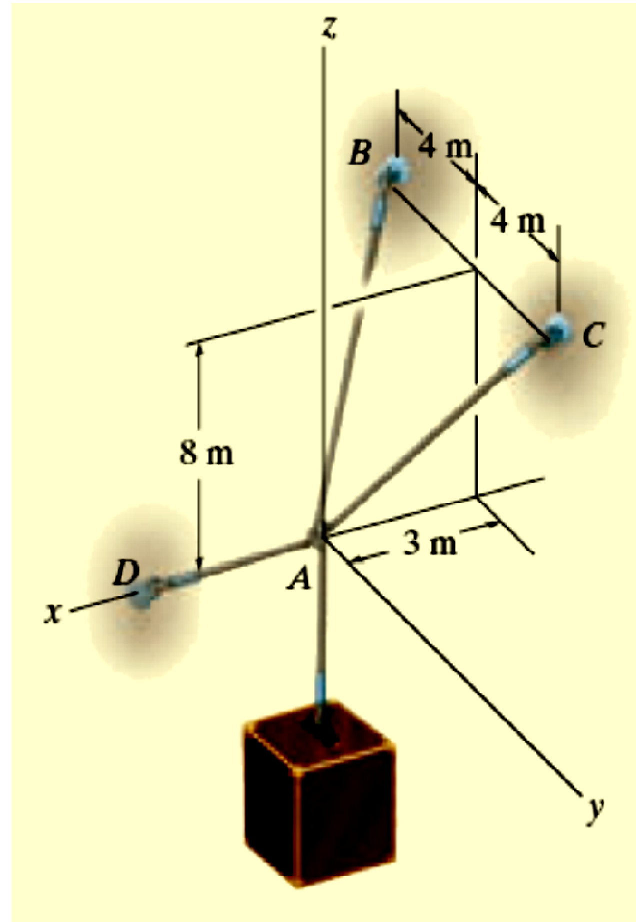
$$s_{AB} = 0.416 \text{ m}$$

Note: Since F_C & F_D are positive, the cables are in tension.

Example 3.7

Given :

- A 40-kN crate is supported by four cables as shown.



Find :

Determine the force developed in each cable.

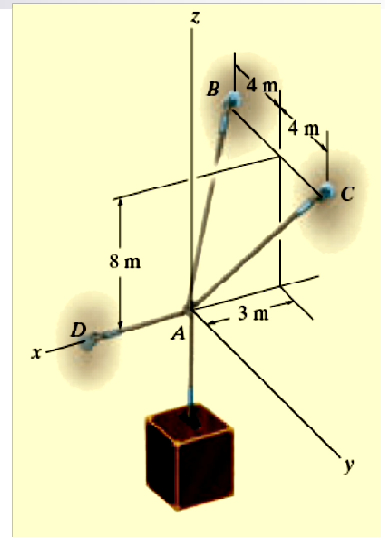
Solution

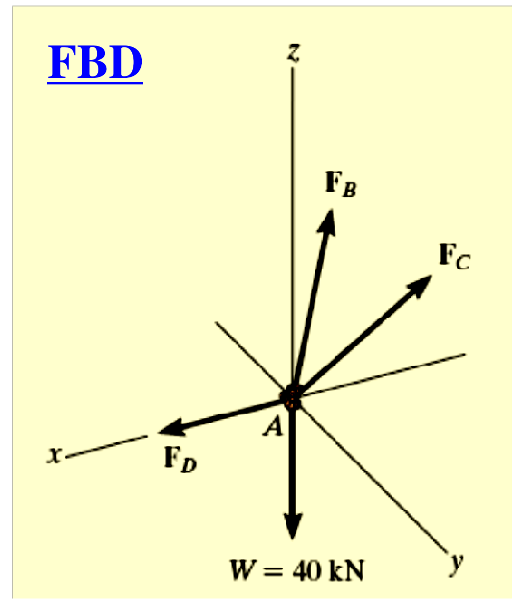
FBD at Point *A*

Four forces acting at *A*:

(1) 3 unknown forces: $\mathbf{F}_B, \mathbf{F}_C, \mathbf{F}_D$.

(2) 1 known force, i.e., the weight of the crate, $W = 40$ kN.





Express each force as a Cartesian vector

- Coordinates : *B* (–3m, –4m, 8m)

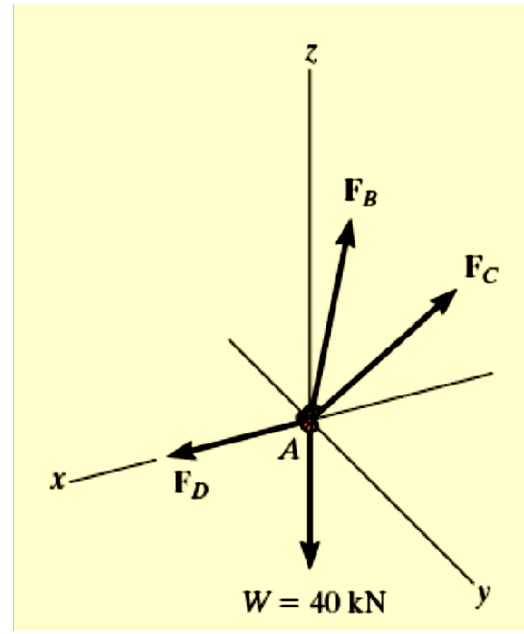
  C (–3m, 4m, 8m)

$$\mathbf{F}_B = F_B\,\mathbf{u}_B = F_B\left(\frac{\mathbf{r}_{AB}}{r_{AB}}\right)$$

$$= F_B\left(\frac{-3\mathbf{i}-4\mathbf{j}+8\mathbf{k}}{\sqrt{(-3)^2+(-4)^2+(8)^2}}\right) = -0.318F_B\,\mathbf{i} - 0.424F_B\,\mathbf{j} + 0.848F_B\,\mathbf{k}$$

$$\mathbf{F}_C = F_C\,\mathbf{u}_C = F_C\left(\frac{\mathbf{r}_{AC}}{r_{AC}}\right) = F_C\left(\frac{-3\mathbf{i}+4\mathbf{j}+8\mathbf{k}}{\sqrt{(-3)^2+(4)^2+(8)^2}}\right)$$

$$= -0.318\,F_C\,\mathbf{i} + 0.424F_C\,\mathbf{j} + 0.848\,F_C\,\mathbf{k}$$



$$\mathbf{F}_D = F_D\,\mathbf{i}$$

$$\mathbf{W} = \{-40\mathbf{k}\}\text{ kN}$$

Equation of Equilibrium

$$\sum \mathbf{F} = 0$$

$$\mathbf{F}_B + \mathbf{F}_C + \mathbf{F}_D + \mathbf{W} = \mathbf{0}$$

$$(-0.318F_B\,\mathbf{i} - 0.424F_B\,\mathbf{j} + 0.848\,F_B\,\mathbf{k})$$
$$+\ (-0.318F_C\,\mathbf{i} + 0.424F_C\,\mathbf{j} + 0.848\,F_C\,\mathbf{k}) + F_D\,\mathbf{i} - 40\mathbf{k} = \mathbf{0}$$

- Equating the respective **i**, **j**, **k** components, we have

i : $$-0.318F_B - 0.318F_C + F_D = 0 \quad (1)$$

j : $$-0.424F_B + 0.424F_C = 0 \quad (2)$$

k: $$0.848\,F_B + 0.848\,F_C - 40 = 0 \quad (3)$$

- From Eq. (2), $$F_C = F_B \quad (4)$$

- Substituting Eq. (4) into Eq. (3) yields,

$$0.848\,F_B + 0.848\,F_B - 40 = 0$$

$$\Rightarrow \quad F_B = 23.6 \quad \text{kN}$$

$$\Rightarrow \quad F_C = 23.6 \quad \text{kN}$$

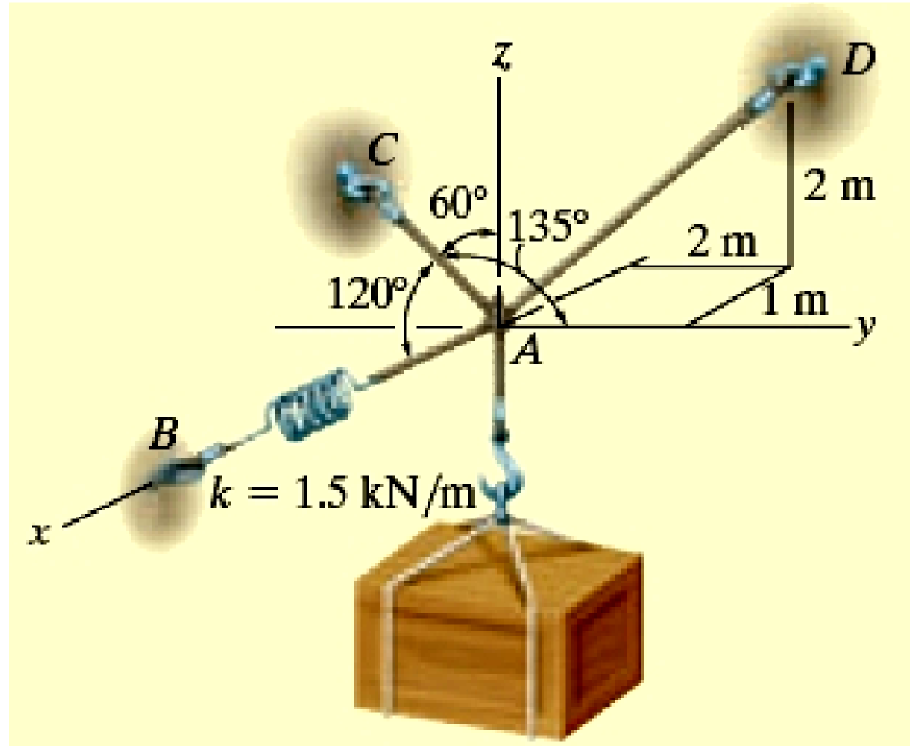
- From Eqs.(1),

$$F_D = 0.318(F_B + F_C) = 15.0\ \text{kN}$$

Example 3.8

Given :

- A 100-kg crate is supported by four cords as shown.



Find :

Determine the tension in each cord.

Solution

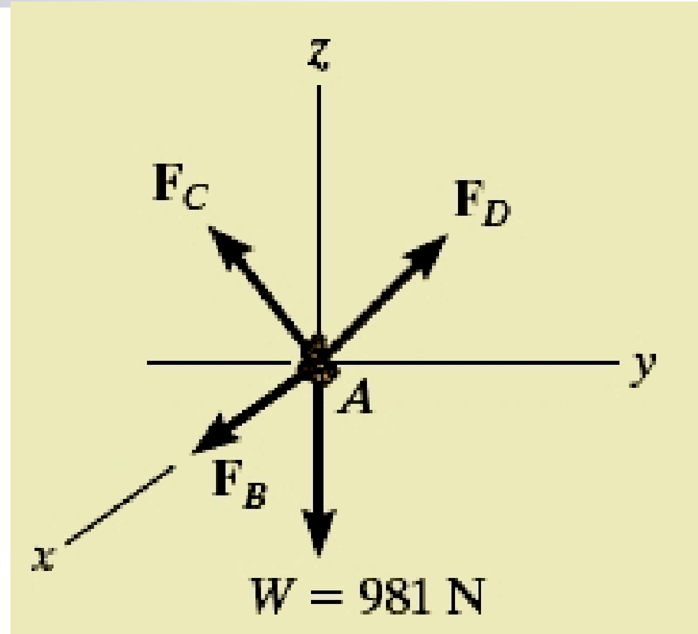
FBD at Point *A*

Four forces acting at *A*:

(1) 3 unknown forces: $\mathbf{F}_B, \mathbf{F}_C, \mathbf{F}_D$.

(2) 1 known force: weight of the crate,

$W = (100)(9.81) = 981\text{N}$.



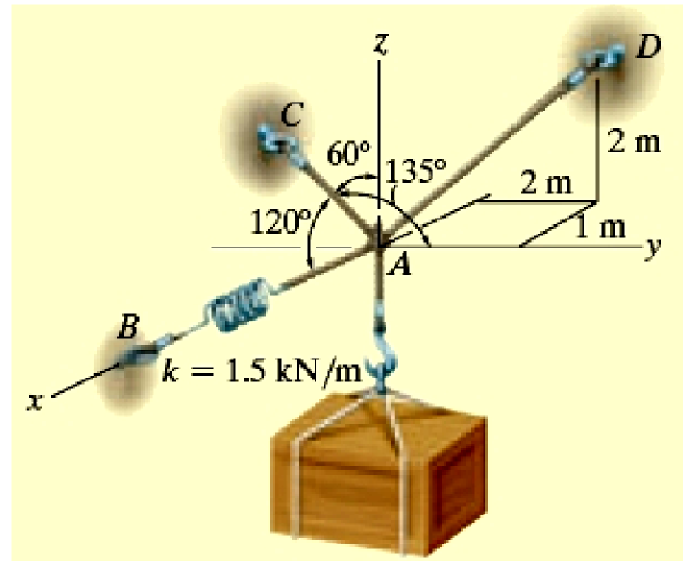
Express each force as a Cartesian vector

- Coordinates : $D\ (-1\text{m},\ 2\text{m},\ 2\text{m})$

$$\mathbf{F}_D = F_D\,\mathbf{u}_D = F_D\left(\frac{\mathbf{r}_{AD}}{r_{AD}}\right)$$

$$= F_D\left(\frac{-1\mathbf{i}+2\mathbf{j}+2\mathbf{k}}{\sqrt{(-1)^2+(2)^2+(2)^2}}\right) = -0.333F_D\,\mathbf{i}+0.667F_D\,\mathbf{j}+0.667F_D\,\mathbf{k}$$

$$\mathbf{F}_C = (F_C)_x\,\mathbf{i} + (F_C)_y\,\mathbf{j} + (F_C)_Z\,\mathbf{k}$$
$$= F_C \cos\alpha\ \mathbf{i} + F_C \cos\beta\ \mathbf{j} + F_C \cos\gamma\,\mathbf{k}$$
$$= F_C \cos 120°\ \mathbf{i} + F_C \cos 135°\ \mathbf{j} + F_C \cos 60°\ \mathbf{k}$$
$$= -0.5\, F_C\,\mathbf{i}\ -0.707\, F_C\,\mathbf{j} + 0.5\, F_C\,\mathbf{k}$$



$$\mathbf{F}_B = F_B\,\mathbf{i}$$

$$\mathbf{W} = \{-\,981\ \mathbf{k}\}\ \text{N}$$

Equation of Equilibrium

$$\sum \mathbf{F} = 0: \qquad \mathbf{F}_B + \mathbf{F}_C + \mathbf{F}_D + \mathbf{W} = \mathbf{0}$$

$$F_B\,\mathbf{i} + (\,-0.5\,F_C\,\mathbf{i}\ -0.707\,F_C\,\mathbf{j} + 0.5\,F_C\,\mathbf{k})$$
$$+ (-0.333F_D\mathbf{i} + 0.667F_D\mathbf{j} + 0.667\,F_D\,\mathbf{k}) + (-\,981\mathbf{k})\ =\ \mathbf{0}$$

- Equating the respective **i**, **j**, **k** components, we have

i : $$F_B - 0.5F_C - 0.333F_D = 0 \qquad (1)$$

j : $$-0.707F_C + 0.667F_D = 0 \qquad (2)$$

k: $$0.5F_C + 0.667F_D - 981 = 0 \qquad (3)$$

- From Eq. (2), $$F_D = 1.06\,F_C \qquad (4)$$

- Substituting Eq. (4) into Eq. (3) yields,

$$0.5\,F_C + 0.667(1.06F_C) - 981 = 0$$

$$\Rightarrow \quad F_C = 813 \text{ N}$$

- Using Eq.(4), $F_D = 1.06\,(813) = 862 \text{ N}$

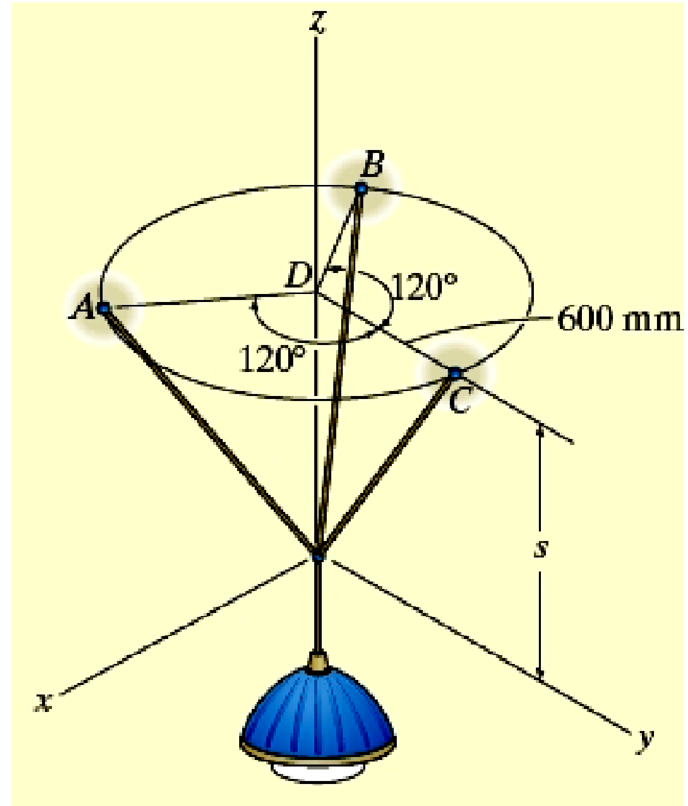
- From Eq. (1),

$$F_B = 0.5F_C + 0.333F_D = 0.5(813) + 0.333(862) = 694 \text{ N}$$

Example 3.6

Given :

- The 10-kg lamp is suspended from the three equal-length cords.
- The force developed in any cord is not allowed to exceed 50 N.



Find :

Determine the smallest vertical *s* from the ceiling.

Solution

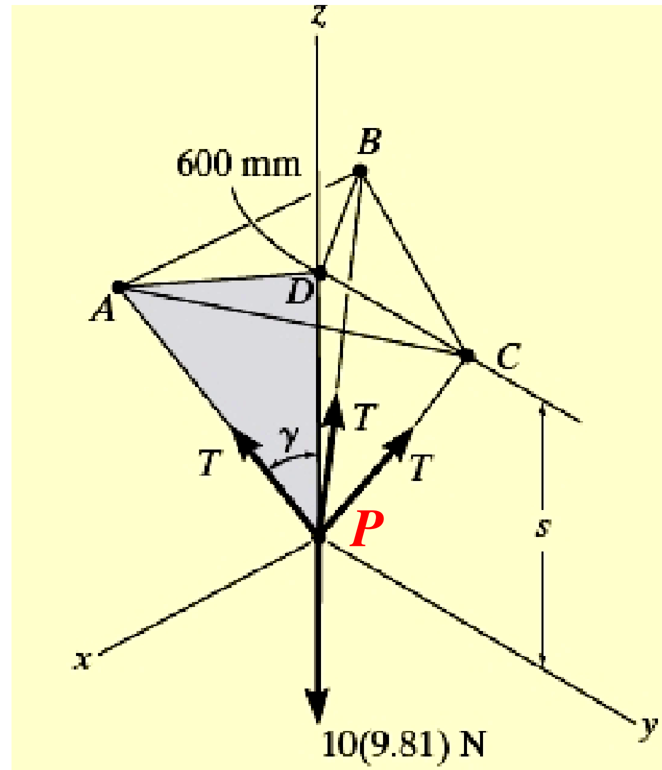
FBD at *P*

- Four forces acting at *P*:

  (1) Tension in each of the 3 cords.

  (2) Weight of the lamp:

  $W = (10)(9.81) = 98.1$ N.



- Due to symmetry,
  - $DA = DB = DC = 600$ mm
  - The angle between each cord and the z axis is γ.
  - The tension in each cord is the same.

Equation of Equilibrium

$$+\uparrow \sum \mathbf{F}_z = 0: \qquad 3\,T\cos\gamma - 98.1 = 0$$

$$\gamma = \cos^{-1}\left(\frac{98.1}{3T}\right) \qquad (1)$$

Vertical distance *s*

- The vertical distance s is given by

$$\tan\gamma = \frac{600}{s}$$

$$\Rightarrow s = \frac{600}{\tan\gamma}$$

- Therefore, s is the smallest if γ is maximum.

- From Eq. (1), for γ to be maximum, T must reach its maximum allowable value, i.e.,

$$T = T_{max} = 50\text{N}$$

- Therefore,

$$\gamma_{\max} = \cos^{-1}\left(\frac{98.1}{3T_{\max}}\right) = \cos^{-1}\left(\frac{98.1}{3(50)}\right) = 49.16^{\circ}$$

- Hence, the smallest vertical distance s from the ceiling is

$$s_{\min} = \frac{600}{\tan\gamma_{\max}} = \frac{600}{\tan 49.16^{\circ}} = 519\text{ mm}$$